1. Introduction

This can be viewed as an English short version of my Master Degree thesis, written in Chinese, at Yau Mathematical Sciences Center, Tsinghua University, 2014.

To show that we need finitely many times of surgeries during discrete Yamabe flow, it is sufficient to prove the following theorem, which is the main theorem in my Master Degree thesis.

Theorem 1.1. *$\mathbb{R}^n = \cup_{i=1}^m D_i$ is an analytic cell decomposition. $f(x) \in C^1(\mathbb{R}^n)$ is strictly convex on $\mathbb{R}^n$ and analytic on each cell D_i, and has a unique minimum point. Then its gradient flow $\gamma(t)$, which satisfies $\gamma'(t) = -\nabla f(\gamma(t))$, intersects the cell faces $\cup_{i=1}^m \partial D_i$ finitely many times.*

We would separate this theorem into two parts: (1) Given any $t \geq 0$, there exists $\epsilon_t > 0$ such that $\gamma((t-\epsilon_t, t]) \subseteq D_i$ for some i and $\gamma([t, t+\epsilon_t)) \subseteq D_j$ for some j; (2) there exists $t_0 > 0$ such that $\gamma((t_0, \infty)) \subseteq D_i$ for some i.

1.1. Sketch of the Proof: Part 1.

It is enough to prove $\gamma([t, t+\epsilon)) \subseteq D_i$ and the other side is similar. We may assume $t = 0$ and $\gamma(0) = 0$.

If γ is an integral curve of a global analytic vector field V, it is also analytic (theorem 2.1). When V is just piecewise analytic, its integral curve is no longer analytic, but under proper conditions, its lower-dim component still has a principle term ct^m (c is a lower-dim vector), indicating the direction of γ in this lower-dim subspace (theorem 3.1). Using this key fact and some natural properties of analytic cell decomposition, we proved that $\gamma([t, t+\epsilon))$ is in a single cell D_i (theorem 4.1).

1.2. Sketch of the Proof: Part 2.

The outline of the proof of the part 2 is just the same as part 1, but the proof is much harder and full of techniques. First we also consider an analytic vector field V, we show that γ has an expansion $\sum_j P(t)e^{-a_j t}$ (a mimic to power series expansion in the analytic γ in part 1) when t is sufficiently large, where P_j are polynomials (theorem 5.1, proposition 5.5). When V is piecewise analytic, γ no longer has such expansion. But under proper conditions, its component has a principle term $cf(t)$ (a mimic to ct^m) indicating the direction of γ in this subspace (theorem 6.1, proposition 6.5). Using this key fact and the same proof of theorem 4.1 we are able to prove $\gamma((t_0, \epsilon))$ is in a single cell D_i (theorem 6.1).

The difficulties of part 2 include: (1) theorem 5.1, construction of the expansion $\sum_j P(t)e^{-\lambda_j t}$ and the proof to its convergence; (2) theorem 6.1, the existence of the principle term.

2. Theorem 2.1 (Cauchy Theorem)

2.1. Convention of Notations.

(1)$x \in \mathbb{R}^n$, then x_i (or sometimes x^i) denotes the i'the component of x, i.e., $x = (x_1, ..., x_n)$ (or $x = (x^1, ..., x^n)$);

(2)$x \in \mathbb{R}^n, I \in \mathbb{Z}^d$, $d \leq n$, then $x^I = x_1^{I_1} ... x_d^{I_d}$;

(3)$(x_i)_{i=1}^n$ denotes the vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$;

(4)$x \in \mathbb{R}^n$ (or $\mathbb{Z}^n$), then $|x| = ||x||_1 = |x_1| + ... + |x_n|$.

2.2. **Formulation and Proof of the Theorem.**

Theorem 2.1 (Cauchy Theorem)**.** *$V(x)$ is a real analytic vector field on a neighborhood of origin $0 \in \mathbb{R}^n$, locally there exists an analytic integral curve starting at 0, i.e., an analytic function $x = x(t)$ such that*

$$\begin{cases} x(0) = 0 \\ x'(t) = V(x(t)). \end{cases}$$

Proof. Formally assume

$$V(x) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} b_I x^I, \quad x(t) = \sum_{j=0}^{\infty} a_j t^j$$

for some $b_I \in \mathbb{R}^n$ and $a_j \in \mathbb{R}^n$. Substitute f and x by their power series in $x'(t) = V(x(t))$ we have

$$\sum_{j=0}^{\infty} (j+1) a_{j+1} t^j = \sum_{I \in \mathbb{Z}^n_{\geq 0}} b_I (\sum_{j=0}^{\infty} a_j t^j)^I.$$

The explicit formula is

$$(j+1) a^i_{j+1} = \sum_{I \in \mathbb{Z}^n_{\geq 0}} b^i_I \sum_{j_{k,l} : \sum_{k=1}^n \sum_{l=1}^{I_k} j_{k,l} = j} \prod_{k,l} a^k_{j_{k,l}}. \tag{2.1}$$

Since $x(0) = 0$, $a_0 = 0$ and by induction on j all a_j can be uniquely determined.

The remaining work is to prove these power series converge in a neighborhood of 0. Since f is analytic, there exists $M > 0$ such that $|b_I| \leq M^{|I|+1}$. Assume

$$\widetilde{V}(x) = \frac{M}{(1 - Mx_1)...(1 - Mx_n)} (1, ..., 1)^T = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \tilde{b}_I x^I$$

Then $|\tilde{b}_I| \geq M^{|I|+1} \geq |b_I|$. Assume $\tilde{a}_j$ are inductively defined by (2.1) using $\tilde{b}_I$ instead of b_I. Then it is straight forward to see $\tilde{a}_j \geq |a_j|$. To prove $\sum_{j=0}^{\infty} a_j t^j$ converge locally, it is enough to prove $\sum_{j=0}^{\infty} \tilde{a}_j t^j$ converge locally, i.e.,

$$\begin{cases} x(0) = 0 \\ x'(t) = \widetilde{V}(x(t)) \end{cases}$$

has an analytic solution locally. It is obvious that $x_1(t) = ... = x_n(t)$. It is equal to solve single valuable ODE $y'(t) = M/(1 - My)^n$ with initial value $y(0) = 0$. We solve it and get the analytic solution

$$y(t) = \frac{1}{M} (1 - (1 - (n+1) M^2 t)^{\frac{1}{n+1}}).$$

□

Remark 2.2. *This method is called the method of "dominating series". Cauchy first proved the similar asserts using the method of "dominating series" (1839-1842).*

3. Theorem 3.1

Theorem 3.1. *Let V be a vector field on a neighborhood U in $\mathbb{R}^n$. $0 \in \overline{U}$ and V is of form*

$$V(x) = \sum_{I \in \mathbb{Z}^d_{\geq 0}} b_I x^I + \sum_{i=d+1}^{n} O(|x_i|),$$

where $0 \leq d \leq n-1$ and $\sum_{I \in \mathbb{Z}^d_{\geq 0}} b_I x^I$ is an analytic vector field near 0.

Assume $\gamma(t)$ is the integral curve of V starting at 0. To be specific, we mean γ is a curve such that $\gamma(0) = 0$, $\gamma((0,\epsilon)) \subseteq U$, and $\gamma'(t) = V(\gamma(t))$ for $t \in (0,\epsilon)$. Then $(\gamma_{d+1}(t), ..., \gamma_n(t)) \equiv 0$ or $(\gamma_{d+1}(t), ..., \gamma_n(t)) = ct^m + o(t^m)$ for some nonzero $c \in \mathbb{R}^{n-d}$ and $m \in \mathbb{Z}_{\geq 0}$, which both depend on the coefficients b_I.

Proof. We prove by induction. Of course we have $\gamma(t) = o(1)$. If we have that

$$\gamma(t) = \sum_{i=1}^{k} a_i t^i + o(t^k)$$

for some $k \geq 0$, $a_i \in \mathbb{R}^n$, and $(\gamma_{d+1}(t), ..., \gamma_n(t)) = o(t^k)$, then we have

$$\gamma'(t) = \sum_I b_I \gamma^I(t) + \sum_{i=d+1}^{n} O(|\gamma_i(t)|) = \sum_I b_I (\sum_{i=1}^{k} a_i t^i + o(t^k))^I + o(t^k) = \sum_{i=0}^{k} c_i t^i + o(t^k)$$

for some $c_i \in \mathbb{R}^n$. So $\gamma(t)$ is of form

$$\gamma(t) = \sum_{i=1}^{k+1} a_i t^i + o(t^{k+1}).$$

We continue this process, until for some k, $(\gamma_{d+1}(t), ..., \gamma_n(t)) \neq o(t^k)$ (which is equivalent to $(a_k^{d+1}, ..., a_k^n) \neq 0$) and then our proof is done.

If for any $k \geq 0$, $(\gamma_{d+1}(t), ..., \gamma_n(t)) = o(t^k)$, we get an expansion $\sum_{i=1}^{\infty} a_i t^i$ with the last $n-d$ components of a_i are all zero. Notice that now we are not able to say $\gamma(t) = \sum_{i=1}^{\infty} a_i t^i$ since the expansion may not equal to $\gamma(t)$ or even converge. But according to the computation of a_i, we can see that $\sum_{i=1}^{\infty} a_i t^i \in \mathbb{R}^d$ is the analytic integral curve of the analytic vector field $V_0(x) = \sum_{I \in \mathbb{Z}^d_{\geq 0}} b_I x^I$ (Cauchy Theorem).

Let $\gamma_0(t) = \sum_{i=1}^{\infty} a_i t^i$ and it is enough to prove $\gamma_0(t) = \gamma(t)$.

$$\gamma(t)' - \gamma_0'(t) = V(\gamma) - V_0(\gamma_0) = V_0(\gamma) - V_0(\gamma_0) + \sum_{i=d+1}^{n} O(\gamma_i) = O(\gamma(t) - \gamma_0(t))$$

Let $f(t) = \gamma(t) - \gamma_0(t)$, then $f(0) = 0$ and $f'(t) = O(f(t))$. It is not hard to find $f \equiv 0$. □

Remark 3.2. *The last part of proof ($\gamma(t) = \gamma_0(t)$) used the Lipschitz continuity indeed, just the same in the uniqueness part of proof in Picard theorem. But in our case we can not directly use the Picard theorem, due to the domain D may be quite exotic and it may be failed to extend V to be L continues function in the neighborhood of 0.*

4. Theorem 4.1

Theorem 4.1. *Assume U is an open ball in $\mathbb{R}^n$. $U = \cup_{i=1}^m D_i$ is an analytic cell decomposition. V is a continuous vector field on U and analytic on each cell D_i. Assume γ is an integral curve of V, then there exists $\epsilon > 0$ such that $\gamma([0,\epsilon)) \subseteq D_i$ for some i.*

Proof. We denote each face as $D_I = \cap_{i\in I} D_i$ and denote $int(X)$ as the interior of a subset X in U, and assume $\gamma(0) = 0$.

Assume $I_0 = \{i : 0 \in D_i\}$, then $\{0\} = D_{I_0}$ is a face, and of course we have $\gamma((0,\epsilon)) \subseteq int(\cup_{i\in I_0} D_i)$ for ϵ small enough.

Now we do the inductive procedure. Assume there exists a face D_I with dimension $d \leq n-1$, such that $\gamma((0,\epsilon)) \subseteq int(\cup_{i\in I} D_i)$. We assert that $\gamma([0,\epsilon')) \subseteq D_I$ for small ϵ', or there exists a face $D_{I'}$ with dimension $d' > d$ such that $\gamma((0,\epsilon') \subseteq int(\cup_{i\in I'} D_i)$.

If our assertion is true, we do the inductive procedure from $I = I_0$ and finally get an n-dim face (actually a cell) D_i satisfying $\gamma((0,\epsilon)) \subseteq D_i$.

Now we focus on proving the assertion. Choose an analytic coordinate chart such that $D_I \subseteq \mathbb{R}^d$.

In the area $int(\cup_{i\in I} D_i)$, since V is continuous and analytic in each D_i, V has the form of

$$V(x) = \sum_{I\in\mathbb{Z}^d_{\geq 0}} b_I x^I + \sum_{i=d+1}^{n} O(|x_i|)$$

and $\sum_I b_I x^I$ can extend to a neighborhood of 0. By theorem 3.1, $(\gamma_{d+1}, ..., \gamma_n) \equiv 0$ or $(\gamma_{d+1}, ..., \gamma_n) = ct^m + o(t^m)$.

If $(\gamma_{d+1}, ..., \gamma_n) \equiv 0$, $\gamma((0,\epsilon')) \in B(0,\delta) \cap \mathbb{R}^d \cap int(\cup_{i\in I} D_i) \subseteq D_I$. Here we used the fact that $B(0,\delta) \cap \mathbb{R}^d \cap int(\cup_{i\in I} D_i) \subseteq D_I$ for sufficiently small δ (lemma 4.2).

If $(\gamma_{d+1}, ..., \gamma_n) = ct^m + o(t^m)$ for nonzero c, add first d zero components to c we get n-vector $\tilde{c}$. Assume $I' = \{i \in I : (\mathbb{R}^d + \tilde{c}\mathbb{R}_{>0}) \cap L_i \neq \emptyset\}$, where L_i is the linear space angle with vertex at 0 of D_i. Then,

$$\gamma((0,\epsilon')) \cap (\cup_{i\in I-I'} D_i) = \emptyset$$

and $\gamma((0,\epsilon')) \subseteq int(\cup_{i\in I'} D_i)$. $(\mathbb{R}^d + \tilde{c}\mathbb{R}_{>0}) \cap (\cap_{i\in I'} L_i) \neq \emptyset$, $\mathbb{R}^d \subseteq (\cap_{i\in I'} L_i)$, so $L_{I'} = (\cap_{i\in I'} L_i)$ is an at least $d+1$-dim surface of $\mathbb{R}^n$, and thus $D_{I'}$ has dimension at least $d+1$. □

Lemma 4.2. *In the proof above, there exists $\delta > 0$, such that*

$$B_\delta \cap \mathbb{R}^d \cap int(\cup_{i\in I} D_i) \subseteq D_I.$$

Proof. Without loss of generality we may assume $I = \{i : D_I \subseteq D_i\}$. Assume the $(n-1)$-dim faces of D_i containing D_I are $F^i_j \subseteq ker(f^i_j)$, where f^i_j are defined on B_δ, analytic, $\mathbb{R}^d \cap B_\delta \subseteq ker(f^i_j)$ and $D_i \subseteq \{x : f^i_j(x) \geq 0\}$.

We can prove

$$B_\delta = \cup_i (\cap_j \{x : f^i_j(x) \geq 0\} \cap B_\delta)$$

is a cell decomposition of B_δ, and thus $\mathbb{R}^d \cap B_\delta \subseteq \cap_j \{x : f^i_j(x) \geq 0\}$ for any $i \in I$.

Assume $x_0 \overline{\in} D_I$, but $x_0 \in B_\delta \cap \mathbb{R}^d \cap int(\cup_{i\in I} D_i)$, then there exists $i_0 \in I$ such that $x_0 \overline{\in} D_{i_0}$. So

$$\begin{aligned} x_0 &\in int(\cup_{i\in I} D_i) - D_{i_0} \subseteq int(\cup_{i\in I\setminus\{i_0\}} D_i) \\ &\subseteq int(\cup_{i\in I\setminus\{i_0\}}(\cap_j \{x : f_j^i(x) \geq 0\} \cap B_\delta)) \\ &= B_\delta - \cap_j \{x : f_j^{i_0}(x) \geq 0\} \subseteq B_\delta - \mathbb{R}^d \end{aligned}$$

It is contradictory. □

5. Theorem 5.1

5.1. Convention and Formulation of the Theorem.

Given an $n \times n$ matrix B with real eigenvalues, we always assume its eigenvalues are $\lambda_1 < \lambda_2 < ... < \lambda_m$. $\mathbb{R}^n = V_1 \bigoplus \cdots \bigoplus V_m$ where V_i is the invariant subspace of B related to λ_i. $B = B_1 + ... + B_m$, where $B_i|_{V_i} = B|_{V_i}$ and $B_i|_{V_j} = 0$ for $j \neq i$. Given a vector $x \in \mathbb{R}^n$, assume $x = x^{(1)} + ... + x^{(m)}$ where $x^{(i)} \in V_i$.

If $P(x) = \sum_I a_I x^I$, we define

$$||P||(x) = \sum_I |a_I x^I|.$$

Theorem 5.1. *V is an analytic vector field on origin's neighborhood $U \subseteq \mathbb{R}^n$, and*

$$V(x) = -Bx + O(x^2)$$

where $0 < \lambda_1 < ... < \lambda_m$ are distinct eigenvalues of B. $\gamma : [0, \infty) \to \mathbb{R}^n$ is an integral curve of V and assume $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$. Then γ is an analytic curve (by Cauchy Theorem) and its i'th component locally has expansion

$$\gamma(t) = \sum_{J\in\mathbb{Z}^m_{\geq 0}} P_J(t) e^{-\lambda\cdot Jt},$$

where $P_J(t)$ is a polynomial vector and the expansion converges absolutely when t is sufficiently large.

Remark 5.2. *This theorem can be easily generalized to $Re(\lambda_i) > 0$.*

Remark 5.3. *If B is diagonalizable and $\lambda_i \overline{\in} \mathbb{Z}_{\geq 0}\lambda_1 + ... + \mathbb{Z}_{\geq 0}\lambda_{i-1}$ for any $i = 2, ..., m$, we have the polynomial vectors $P_J(t)$ are all constants. As i remembered, this condition should be the same as there exists an analytical transform to make the vector field like $V_i(x) = -b_i x_i$.*

5.2. Proof of the Theorem.

Proof. Assume

$$V(x) = \sum_{I\in\mathbb{Z}^n_{\geq 0}} b_I x^I$$

and let us construct a family of integral curves which are of the form

$$\gamma(t) = \sum_{J\in\mathbb{Z}^m_{\geq 0}} P_J(t) e^{-\lambda\cdot Jt}.$$

Substitute $\gamma(t)$ by above formula in $\gamma'(t) = V(\gamma(t))$, we have

$$\sum_{J\in\mathbb{Z}^m_{\geq 0}} (P'_J(t) - \lambda \cdot JP_J(t)) e^{-\lambda\cdot Jt} = \sum_{I\in\mathbb{Z}^n_{\geq 0}} b_I (\sum_{J\in\mathbb{Z}^m_{\geq 0}} P_J(t) e^{-\lambda\cdot Jt})^I$$

To ensure the above equality, we force the two sides' coefficients of $e^{-\lambda\cdot Jt}$ be the same, i.e.,

$$P_J^{i\,\prime}(t)-\lambda\cdot JP_J^i(t)=-\sum_{j=1}^n b_j^iP_J^j(t)+\sum_{I:|I|\geq 2}b_I^i\sum_{\sum_{k=1}^n\sum_{l=1}^{I_k}J_{k,l}=J}\prod_{k,l}P_{J_{k,l}}^k(t),$$

here $(b_j^i)_{ij}=B$. Assume

$$Q_J(t)=\left(\sum_{I:|I|\geq 2}b_I^i\sum_{\sum_{k=1}^n\sum_{l=1}^{I_k}J_{k,l}=J}\prod_{k,l}P_{J_{k,l}}^k(t)\right)_{i=1}^n$$

and we have

$$P_J'(t)-(\lambda\cdot J-B)P_J(t)=Q_J(t) \tag{5.1}$$

and for $|J|=1$, $Q_J(t)=0$.

Given $\mathbb{R}^n\ni c=c^{(1)}+...+c^{(m)}$, where $c^{(i)}\in V_i$, let

$$P_{e_i}(t,c)=e^{(\lambda_i-B_i)t}c^{(i)},$$

where $c_i\in V_i$. Then $P_{e_i}(t)$ satisfy (5.1) and is linear on c and $\deg_tP_{e_i}(t,c)<n$.

Now we define general $P_J(t,c)$ by induction on $|J|$. Assume we have defined $P_J(t,c)$ for $|J|<s$ and they are $|J|$-homogeneous on c, $\deg_tP_J<(2|J|-1)n$.

For $|J|=s$, we can define $P_J(t,c)$ roughly by solving equation (5.1). We choose a solution as

$$P_J^{(i)}(t,c)=\begin{cases}\int_{-\infty}^t e^{(\lambda\cdot J-B_i)(t-\tau)}Q_J^{(i)}(\tau,c)d\tau, & \lambda\cdot J<\lambda_i\\ \int_0^t e^{(\lambda\cdot J-B_i)(t-\tau)}Q_J^{(i)}(\tau,c)d\tau, & \lambda\cdot J=\lambda_i\\ -\int_t^{+\infty} e^{(\lambda\cdot J-B_i)(t-\tau)}Q_J^{(i)}(\tau,c)d\tau, & \lambda\cdot J>\lambda_i\end{cases}$$

Then P_J is a polynomial vector s-homogeneous on c, and $\deg_tP_J(t,c)<(2s-1)n$.

(calculate t-degree by induction:

$$\begin{aligned}deg_tP_J&\leq(deg_te^{(B-\lambda\cdot J)t}+deg_tQ_J)+1\leq n+deg_tQ_{J_0}+deg_tQ_{J-J_0}\\&<n+(2|J_0|-1)n+(2(|J|-|J_0|)-1)n=(2|J|-1)n\end{aligned}$$

)

By this induction method we defined $P_J(t,c)$ and gave a family of series:

$$\gamma_c(t)\sim\sum_J P_J(t,c)e^{-\lambda\cdot Jt}$$

To prove these are integral curves indeed we need to show the series will absolutely converge (for sufficient large t). It is sufficient to prove the following statement:

Given c, there exists polynomial $P_0(t,c)$ such that $\forall t\geq 8n/\lambda_1$,

$$\sum_{J:|J|=s}||P_J||(t,c)\leq P_0^s(t,c).$$

When $|J|$ large enough (say $\geq m_0\geq 4/\lambda_1$) we have

$$||(\lambda\cdot J-B)^{-1}||_1\leq\frac{2}{|J|\lambda_1}.$$

Then for $t\geq 8n/\lambda_1$, $|J|\geq m_0$ we claim $||P_J||(t,c)\leq||Q_J||(t,c)$ (a).

Assume $|b_I| \leq M^{|I|}$. We make a second claim that for any t, J,

$$\sum_{J:|J|=s} ||Q_J|| \leq \sum_{i=2}^{s} ((n+1)M)^i \sum_{s_j:\sum_{j=1}^{i} s_j=s} \prod_{j=1}^{s} \sum_{J:|J|=s_j} ||P_J|| \quad (b).$$

Notice here we take t, c both as variables of polynomial $P_J(t,c)$ and $||P_J||(t,c) = \sum_{I\in\mathbb{Z}^{2n}_{\geq 0}} |b_I(t,c)^I|$.

The proofs of these two claims are in next subsection and we now just assume they are true.

Let $M_1 = (n+1)M$, and combine the two claims, we see that for $t \geq 8n/\lambda_1, |J| \geq m_0$,

$$\sum_{J:|J|=s} ||P_J|| \leq \sum_{i=2}^{s} M_1^i \sum_{s_j:\sum_{j=1}^{i} s_j=s} \prod_{j=1}^{s} \sum_{|J|=s_j} ||P_J||.$$

Choose a polynomial $P_1(t)$ with positive coefficients such that for any $s \leq m_0, t \geq 8n/\lambda_1$,

$$\sum_{J:|J|=s} ||P_J||(t,c) \leq P_1^s(t,c).$$

Define $P_s(t)$ $(s \geq 2)$ by induction on s,

$$P_s = \sum_{i=2}^{s} M_1^i \sum_{s_j:\sum_{j=1}^{i} s_j=s} \prod_{j=1}^{i} P_{s_j}.$$

We assume $M_1 > 1$ without loss of generality, and easy to verify that for $t \geq 8n/\lambda_1$ and $s \leq m_0$,

$$P_s(t,c) \geq P_1^s(t,c) \geq \sum_{J:|J|=s} ||P_J||(t,c).$$

By induction, it is also easy to prove for $t \geq 8n/\lambda_1$ and $s \geq m_0$,

$$P_s(t,c) \geq \sum_{J:|J|=s} ||P_J||(t,c).$$

Like the case in theorem 1, we have gotten a dominating polynomial P_s. Now it remain to prove that there exists a polynomial $P_0(t)$ such that for any $t \geq 8n/\lambda_1$, $P_s(t) \leq P_0^s(t)$.

By induction it is easy to see that $P_s(t,c) = a_s P_1^s(t,c)$, where $a_1 = 1$ and

$$a_s = \sum_{i=2}^{s} M_1^i \sum_{s_j:\sum_{j=1}^{i} s_j=s} \prod_{j=1}^{i} a_{s_j}. \tag{5.2}$$

It is enough to prove that there exists $M_2 > 0$, such that $a_s \leq M_2^s$, i.e.,

$$f(x) = \sum_{s=1}^{\infty} a_s x^s,$$

is analytic function locally. By equation (5.2) it should satisfy that

$$f(x) - x = \sum_{i=2}^{\infty} (M_1 f(x))^i = \frac{(M_1 f(x))^2}{1 - M_1 f(x)}.$$

Solve the equation and choose a solution

$$f(x) = \frac{1 + M_1 x - \sqrt{(1+M_1x)^2 - 4(M_1+1)M_1 x}}{2M_1^2 + M_1}.$$

Then f is well defined analytic function in the neighborhood of 0, and has the Taylor expansion $f(x) = \sum_{s=1}^{\infty} a_s x^s$.

In a summary, now we have constructed a family (c-parameterized) of integral curves

$$\gamma(t,c) = \sum_{J \in \mathbb{Z}^n_{\geq 0}} P_J(t,c) e^{-\lambda \cdot Jt},$$

where t large enough.

Now we prove this family of integral curves contains all the integral curves locally. For $|c_0| \leq 1$,

$$\begin{aligned} \sum_{J:|J|=s} ||P_J||(c_0,t) &\leq \sum_{J:|J|=s} \sum_{c:c_i=\pm 1} ||P_J||(c,t) \\ \leq \sum_{c:c_i=\pm 1} P_0^s(t,c) &\leq (\sum_{c:c_i=\pm 1} P_0(t,c))^s \end{aligned}$$

Let $P(t) = \sum_{c:c_i=\pm 1} P_0(t,c)$, we can see that there exists t_0 such that for $t \geq t_0$ we have $P(t) < e^{\frac{1}{2}\lambda_1 t}$ and $\gamma(t,c)$ is well defined for any $|c| \leq 1$. Now we show $c \mapsto \gamma(t_0,c)$ is a local homeomorphism at origin. Since

$$\sum_J ||P_J||(t_0,c) e^{-\lambda \cdot Jt_0} < \infty$$

for $|c| \leq 1$, $\gamma(t_0,c)$ is locally analytic. Since

$$\begin{aligned} \gamma(t_0,c) &= \sum_{i=1}^m P_{e_i}(t_0,c) e^{-\lambda_i t_0} c^{(i)} + O(|c|^2) = \sum_{i=1}^m e^{(\lambda_i - B_i)t_0} e^{-\lambda_i t_0} c^{(i)} + O(|c|^2) \\ &= \sum_{i=1}^m e^{-B_i t_0} c^{(i)} + O(|c|^2) = \sum_{i=1}^m e^{-Bt_0} c^{(i)} + O(|c|^2) = e^{-Bt_0} c + O(|c|^2), \end{aligned}$$

$$\left| \frac{\partial \gamma(t_0,c)}{\partial c} \right| (0) = |e^{-Bt_0}| > 0,$$

and $\gamma(t_0,c)$ is a local homeomorphism and $\gamma(t,c)$ contains all the local integral curves. □

From the proof we get following proposition,

Proposition 5.4. *The derivative of $\gamma(t)$ can be calculated term by term in the expansion, i.e.,*

$$\gamma'(t) = \sum_{J \in \mathbb{Z}^m_{\geq 0}} (P_J(t) e^{-\lambda \cdot Jt})' = \sum_{J \in \mathbb{Z}^m_{\geq 0}} (P_J'(t) - \lambda \cdot J P_J(t)) e^{-\lambda \cdot Jt}.$$

5.3. **Proof of Claim (a).**

According to equation (5.1), We have

$$(\frac{d}{dt}-(\lambda\cdot J-B))P_J=Q_J.$$

So for $J\geq m_0$ (when $(\lambda\cdot J-B)^{-1}$ exists),

$$P_J=-(\lambda\cdot J-B)^{-1}(1+(\lambda\cdot J-B)^{-1}\frac{d}{dt}+(\lambda\cdot J-B)^{-2}\frac{d^2}{dt^2}+...)Q_J$$

(Notice that Q_J is a polynomial. So in the above equation, the sum is of finite non-zero terms and well defined)

We denote

$$\Phi_J=-(\lambda\cdot J-B)^{-1}(1+(\lambda\cdot J-B)^{-1}\frac{d}{dt}+(\lambda\cdot J-B)^{-2}\frac{d^2}{dt^2}+...)$$

is a linear operator. For $t_0=8n/\lambda_1$, for $|J|>m_0$, $k\leq 2n|J|$,

$$\begin{aligned}&||\Phi_J(t^ke_i)||\\ =&||(\lambda\cdot J-B)^{-1}||\ ||t^ke_i+k(\lambda\cdot J-B)^{-1}t^{k-1}e_i+k(k-1)(\lambda\cdot J-B)^{-2}t^{k-2}e_i+...||\\ \leq&\frac{2}{|J|\lambda_1}t^k(1+\frac{2k}{t|J|\lambda_1}+\frac{4k(k-1)}{(t|J|\lambda_1)^2}+...)\leq\frac{1}{2}t^k(1+\frac{1}{2}+\frac{1}{2^2}+...)=||t^ke_i||\end{aligned}$$

Sum all the components of monomials of Q_J up we get

$$||P_J||(t,c)\leq\Phi_J||Q_J||(t,c)\leq||Q_J||(t,c).$$

5.4. **Proof of claim (b).**

$$\begin{aligned}\sum_{J:|J|=s}||Q_J^i||\leq&\sum_{J:|J|=s}\sum_{I:|I|\geq 2}|b_I^i|\sum_{J_{k,l}:\sum_{k=1}^n\sum_{l=1}^{I_k}J_{k,l}=J}\prod_{k,l}||P^k_{J_{k,l}}||\\ =&\sum_{I:|I|\geq 2}|b_I^i|\sum_{s_{k,l}:\sum_{k=1}^n\sum_{l=1}^{I_k}s_{k,l}=s}\sum_{(J_{k,l})_{k,l}:|J_{k,l}|=s_{k,l}}\prod_{k,l}||P^k_{J_{k,l}}||\\ =&\sum_{I:|I|\geq 2}|b_I^i|\sum_{s_{k,l}:\sum_{k=1}^n\sum_{l=1}^{I_k}s_{k,l}=s}\prod_{k,l}\sum_{J_{k,l}:|J_{k,l}|=s_{k,l}}||P^k_{J_{k,l}}||\\ =&\sum_{i=2}^sM^{|I|}\sum_{I:|I|=i}\sum_{s_{k,l}:\sum_{k=1}^n\sum_{l=1}^{I_k}s_{k,l}=s}\prod_{k,l}\sum_{J:|J|=s_{k,l}}||P^k_{J_{k,l}}||\\ =&\sum_{i=2}^sM^{|I|}C_{n+i-1}^{i-1}\sum_{s_j:\sum_{j=1}^i s_j=s}\prod_{j=1}^s(\sum_{J:|J|=s_j}||P_J||).\end{aligned}$$

So

$$\begin{aligned}\sum_{J:|J|=s}||Q_J||\leq&n\sum_{i=2}^sM^iC_{n+i-1}^{i-1}\sum_{s_j:\sum_{j=1}^i s_j=s}\prod_{j=1}^s(\sum_{|J|=s_j}||P_J||)\\ \leq&\sum_{i=2}^s((n+1)M)^i\sum_{s_j:\sum_{j=1}^i s_j=s}\prod_{j=1}^s(\sum_{|J|=s_j}||P_J||).\end{aligned}$$

5.5. **Further Discussion.** Let $\{a_i\}_{i=0}^{\infty} = \{\lambda \cdot J : J \in \mathbb{R}^m\}$, and $0 = a_0 < a_1 < a_2 ....$ Since every integral curve is of form

$$\gamma(t) = \sum_{J \in \mathbb{Z}^m_{\geq 0}} P_J(t) e^{-\lambda \cdot Jt},$$

and this series absolutely converge. So we can use the new form of

$$\gamma(t) = \sum_{j=1}^{\infty} P_j(t) e^{-a_j t},$$

where each $P_j(t) = \sum_{J:\lambda \cdot J = a_j} P_J(t)$ is a polynomial vector and the series on the right side converges. We already have

$$P_J^{i\,\prime}(t) - \lambda \cdot J P_J^i(t) = -\sum_{j=1}^{n} b_j^i P_J^j(t) + \sum_{I:|I|\geq 2} b_I^i \sum_{\sum_{k=1}^n \sum_{l=1}^{I_k} J_{k,l} = J} \prod_{k,l} P_{J_{k,l}}^k(t).$$

Sum all the above equations indexed by $\{J : \lambda \cdot J = a_j\}$ we have

$$P_j^{i\,\prime}(t) - a_j P_j^i(t) = -\sum_{l=1}^{n} b_l^i P_j^l(t) + \sum_{I:|I|\geq 2} b_I^i \sum_{\sum_{k=1}^n \sum_{l=1}^{I_k} a_{j_{k,l}} = a_j} \prod_{k,l} P_{j_{k,l}}^k(t).$$

Let

$$Q_j(t) = \left(\sum_{I:|I|\geq 2} b_I^i \sum_{\sum_{k=1}^n \sum_{l=1}^{I_k} a_{j_{k,l}} = a_j} \prod_{k,l} P_{j_{k,l}}^k(t) \right)_{i=1}^{n} = \sum_{J:\lambda \cdot J = a_j} Q_J(t), \tag{5.3}$$

and we have

$$P_j'(t) - (a_j - B) P_j(t) = Q_j(t). \tag{5.4}$$

Now from an integral curve $\gamma(t)$ we find a set $\{P_j(t)\}_{j=1}^{\infty}$ satisfying equation (5.4). We want to get the inverse fact:

Proposition 5.5. *Given any $\{P_j(t)\}_{j=1}^{\infty}$ satisfying equation (5.4),*

$$\gamma(t) = \sum_{j=1}^{\infty} P_j(t) e^{-a_j t}$$

is a local integral curve of V, i.e., there is a bijection between $\mathcal{A} = \{\{P_J(t,c)\}_J : c \in \mathbb{R}^n\}$ and $\mathcal{B} = \{\{P_j(t)\}_{j=1}^{\infty} :$ satisfy (5.4)$\}$.

Proof. First, let us parameterize $\mathcal{B}$ by $C \in \mathbb{R}^n$.

Assume $a_{j_i} = \lambda_i$. Given $P_1, ..., P_{j-1}$, $P_j^{(i)}$ is uniquely determined by (5.4), unless $j = j_i$. When $j = j_i$,

$$P_j^{(i)}(t) = \int_0^t e^{(\lambda_i - B_i)(t-\tau)} Q_j(\tau) d\tau + e^{(\lambda_i - B_i)t} C^{(i)},$$

where $C^{(i)} \in V_i$. Then given $P_1, ..., P_{j_i - 1}$, $\{P_{j_i} :$ satisfy (5.4)$\}$ can be parameterized by $C^{(i)} \in V_i$. So $\mathcal{B}$ can be parameterized by $(C^{(1)}, ..., C^{(m)}) \in V_1 \bigoplus \cdots \bigoplus V_m = \mathbb{R}^n$. To prove the lemma, it is enough to show the mapping

$$\mathbb{R}^n \ni c \mapsto \{P_J(t,c)\}_J \mapsto \{P_j(t,c)\}_j \mapsto (C^{(1)}, ..., C^{(m)}) \in \mathbb{R}^n$$

is indeed an identity. For $P_{e_i}(t) = e^{(\lambda_i - B_i)t}c^{(i)}$.

$$\begin{aligned}
C^{(i)} =& e^{(B_i-\lambda_i)t}(P^{(i)}_{j_i}(t) - \int_0^t e^{(\lambda_i - B_i)(t-\tau)}Q^{(i)}_{j_i}(\tau)d\tau) \\
=& e^{(B_i-\lambda_i)t}(P^{(i)}_{e_i}(t) + \sum_{J\neq e_i:\lambda\cdot J=a_{j_i}} P^{(i)}_J(t) - \int_0^t e^{(\lambda_i - B_i)(t-\tau)}Q^{(i)}_{j_i}(\tau)d\tau) \\
=& c^{(i)} + e^{(B_i-\lambda_i)t}(\sum_{J:\lambda\cdot J=a_{j_i}} \int_0^t e^{(\lambda_i - B_i)(t-\tau)}Q^{(i)}_J(\tau)d\tau - \int_0^t e^{(\lambda_i - B_i)(t-\tau)}Q^{(i)}_{j_i}(\tau)d\tau) \\
=& c^{(i)}
\end{aligned}$$

□

6. Theorem 6.1

6.1. Convention and Formulation of Theorem.

We define a notation similar to $o(f(x))$: we say $g(x) = \tilde{o}(f(x))$ if $P(x)g(x) = o(f(x))$ for any polynomial $P(x)$.

There are two basic facts: $O(f)\tilde{o}(g) = \tilde{o}(fg)$, $e^{-\lambda t} = \tilde{o}(1)$ if $\lambda > 0$.

Theorem 6.1. *B is an n-dim square matrix, $0 < \lambda_1 < ... < \lambda_m$ are the eigenvalues of B. $0 \le d \le n-1$, V is a vector field on $\mathbb{R}^n$ of form*

$$V(x) = -Bx + \sum_{I\in\mathbb{Z}^d_{\ge 0}:|I|\ge 2} b_I x^I + \sum_{i=d+1}^{n} O(|x||x_i|)$$

where $\sum_I b_I x^I$ is analytic locally, then for any integral curve $\gamma(t)$ near origin, we have $(\gamma_{d+1}(t), ..., \gamma_n(t)) = 0$, or $(\gamma_{d+1}(t), ..., \gamma_n(t)) = P(t)e^{-at} + o(e^{-at})$ for some positive a and $n-d$ dim nonzero polynomial vector $P(t)$. What's more, $P(t)$ and a are determined by B and b_I

6.2. Proof of the Theorem.

Proof. Since $\gamma'(t) = V(\gamma(t)) = -B\gamma(t) + O(|\gamma(t)|^2)$. By lemma 6.2 (which will be proved in next subsection), we know that $\gamma(t) = \tilde{o}(e^{-\lambda_1 t/2}) = \tilde{o}(1)$. (This fact is natural because we should have $\gamma(t) \approx e^{-Bt}\gamma(0) = O(t^{n-1}e^{-\lambda_1 t}) = \tilde{o}(e^{-\lambda_1 t/2})$ locally.)

Let $\{a_i\}_{i=0}^{\infty} = \{\lambda\cdot J : J \in \mathbb{Z}^m\}$ and $0 = a_0 < a_1 < a_2....$ We will prove the following inductive assertion: if we have

$$\gamma(t) = \sum_{j=1}^{s-1} P_j(t)e^{-a_j t} + \tilde{o}(e^{-a_{s-1}t})$$

where P_j are polynomial vectors, equation (5.4) is true for $j = 1, ..., s-1$, and $\gamma_i(t) = o(e^{-a_{s-1}t})$ for $d+1 \le i \le n$, then

$$\gamma(t) = \sum_{j=1}^{s} P_j(t)e^{-a_j t} + \tilde{o}(e^{-a_s t}).$$

where P_s is also a polynomial vector, and equation (5.4) is true for $j = s$.

Proof of the assertion: Let

$$\Delta\gamma(t) = \gamma(t) - \sum_{j=1}^{s-1} P_j(t)e^{-a_j t} = \tilde{o}(e^{-a_{s-1}t})$$

$$\gamma'(t) = V(\gamma(t)) = -B\gamma(t) + \sum_{I:|I|\geq 2} b_I\gamma^I(t) + \sum_{i=d+1}^{n} O(|\gamma(t)||\gamma_i(t)|).$$

When $s = 1, i \geq d+1$, $O(|\gamma(t)||\gamma_i(t)|) = O(|\gamma(t)|^2) = \tilde{o}(e^{-\lambda_1 t}) = \tilde{o}(e^{-a_s t})$. When $s \geq 2, i \geq d+1$, $O(|\gamma(t)||\gamma_i(t)|) = O(|P_1(t)e^{-a_1 t}|)\tilde{o}(|e^{-a_{s-1}t}|) = \tilde{o}(P_1 e^{-(a_1+a_{s-1})t}) = \tilde{o}(e^{-a_s t})$. So

$$\begin{aligned}
&(\sum_{j=1}^{s-1} P_j(t)e^{-a_j t})' + \Delta\gamma'(t) \\
=& -B\sum_{j=1}^{s-1} P_j(t)e^{-a_j t} - B\Delta\gamma(t) + \sum_{I:|I|\geq 2} b_I(\sum_{j=1}^{s-1} P_j(t)e^{-a_j t} + \tilde{o}(e^{a_{s-1}t}))^I + \tilde{o}(e^{-a_s t}) \\
=& -B\sum_{j=1}^{s-1} P_j(t)e^{-a_j t} - B\Delta\gamma(t) + \sum_{j=1}^{s} Q_j(t)e^{-a_j t} + \tilde{o}(e^{-a_s t})
\end{aligned}$$

Since (5.4) is true for $j = 1, ..., s-1$, we can simplify the above equation to

$$\Delta\gamma'(t) + B\Delta\gamma(t) = Q_s(t)e^{-a_s t} + \tilde{o}(e^{-a_s t}).$$

Restrict this ODE in V_i we get

$$\Delta\gamma^{(i)'}(t) + B_i\Delta\gamma^{(i)}(t) = Q_s^{(i)}(t)e^{-a_s t} + \tilde{o}(e^{-a_s t}).$$

From lemma 6.3 (proved in later subsection) we know that for any $\lambda \in \mathbb{R}$, $\int_0^t \tilde{o}(e^{\lambda\tau})d\tau = C + \tilde{o}(e^{\lambda t})$. So

$$e^{B_i t}\Delta\gamma^{(i)}(t) = \int_0^t e^{(B_i - a_s)\tau} Q_s^{(i)} d\tau + C + \tilde{o}(e^{(\lambda_i - a_s)t}),$$

Since $\int_0^t e^{(B_i-a_s)\tau}Q_s^{(i)}d\tau = R(t)e^{(\lambda_i-a_s)t} + C$ for some polynomial vector $R(t)$ and constant C, we may write

$$\Delta\gamma^{(i)}(t) = e^{-B_i t}(R(t)e^{(\lambda_i-a_s)t}+C)+\tilde{o}(e^{-a_s t}) = e^{(\lambda_i-B_i)t}R(t)e^{-a_s t}+Ce^{-\lambda_i t}+\tilde{o}(e^{-a_s t})$$

Let

$$P_s^{(i)} = \begin{cases} e^{(\lambda_i-B_i)t}R(t) & \lambda_i \neq a_s \\ e^{(\lambda_i-B_i)t}R(t) + e^{(a_s-B_i)t}C & \lambda_i = a_s \end{cases}$$

be a polynomial. If $\lambda_i > a_s$, $Ce^{-\lambda_i t} = \tilde{o}(e^{-a_s t})$ and $\Delta\gamma^{(i)}(t) = P_s^{(i)}e^{-a_s t} + \tilde{o}(e^{-a_s t})$. If $\lambda_i = a_s$, $\Delta\gamma^{(i)}(t) = P_s^{(i)}e^{-a_s t} + \tilde{o}(e^{-a_s t})$. If $\lambda_i < a_s$, since $\Delta\gamma^{(i)}(t) = \tilde{o}(e^{-a_{s-1}t})$, $C = 0$ and $\Delta\gamma^{(i)}(t) = P_s^{(i)}e^{-a_s t} + \tilde{o}(e^{-a_s t})$.

It remains to prove P_s satisfy (5.4). From the definition of $P_s^{(i)}$ we see that

$$(e^{(B_i-a_s)t}P_s^{(i)})' = (R(t)e^{(\lambda_i-a_s)t})' = e^{(B_i-a_s)t}Q_s^{(i)}.$$

Then $(e^{(B-a_s)t}P_s^{(i)})' = e^{(B-a_s)t}Q_s^{(i)}$ and $(e^{(B-a_s)t}P_s)' = e^{(B-a_s)t}Q_s$, i.e., $P_s' - (a_s - B)P_s = Q_s$.

Since we already have $\gamma(t) = \tilde{o}(1) = \tilde{o}(e^{-a_0 t})$. By the inductive assertion we can calculate $P_j(t), t = 1, 2, ...,$ until there exists P_j such that $(P_j^{d+1}, ..., P_j^n) \neq 0$ and

our proof is done. If for all the P_j $(j = 1, 2, ...)$ we got, $(P_j^{d+1}, ..., P_j^n) = 0$. Since $\{(P_j^i)_{i=1}^d\}_{j=1}^\infty$ satisfy (5.4), by proposition 5.5 $(\gamma_0(t))_{i=1}^d = \sum_{j=1}^\infty (P_j^i(t))_{i=1}^d e^{-a_j t}$ is locally an integral curve of $V_0(x) = -B_d x + \sum_{I\in\mathbb{Z}^d_{\geq 0}:|I|\geq 2} b_I x^I$, and in plane $\mathbb{R}^d$. So $\gamma_0(t)$ is an integral curve of V, and $\gamma(t) - \gamma_0(t) = o(e^{-\lambda t}), \forall\lambda > 0$. By lemma 6.4, $\gamma(t) = \gamma_0(t)$ (It is not hard to verify that the conditions in lemma 6.4 are satisfied). □

6.3. **Lemma 6.2.**

Lemma 6.2. *Assume $V(x) = -Bx + o(|x|)$. B is a square matrix and $0 < \lambda_1 < ... < \lambda_m$ are the eigenvalues of B. γ is an integral curve of the vector field V near origin. Then for any $\lambda < \lambda_1$, $|\gamma(t)| = o(e^{-\lambda t})$ locally.*

Proof. Choose $0 < \epsilon < \lambda_1 - \lambda$, there exists P such that $sym(P^{-1}BP)$'s eigenvalues are all larger than $\lambda_1 - \epsilon$. Let $||x|| = ||P^{-1}x||_2$.

$$\begin{aligned}
&(||\gamma(t)||^2)' = (||P^{-1}\gamma(t)||_2^2)' = 2(P^{-1}\gamma(t))^T P^{-1}\gamma(t)' \\
=&2(P^{-1}\gamma(t))^T P^{-1}(-B\gamma(t) + o(|\gamma(t)|)) \\
=&-2(P^{-1}\gamma(t))^T(P^{-1}BP)(P^{-1}\gamma(t)) + o(|\gamma(t)|^2) \\
\leq&-2(\lambda_1 - \epsilon)||\gamma(t)||^2
\end{aligned}$$

for sufficient large t (say $\geq t_0$). So $||\gamma(t)||^2 \leq e^{-2(\lambda_1-\epsilon)(t-t_0)}||\gamma(t_0)||^2 = o(e^{-2\lambda t})$ and $\gamma(t) = o(e^{-\lambda t})$. □

6.4. **lemma 6.3.**

Lemma 6.3. *$f \in C[0, \infty)$ and $f(t) = \tilde{o}(e^{\lambda t})$ then there exists $C \in \mathbb{R}$ such that*

$$\int_0^t f(t) = C + \tilde{o}(e^{\lambda t}) \quad t \to +\infty.$$

Proof. (1) If $\lambda \leq 0$, $f(t) = \tilde{o}(1)$ and $\int_0^\infty |f(t)| < \infty$. Let $C = \int_0^\infty f(t)dt$ and it is enough to prove $C - \int_0^t f(\tau)d\tau = \int_t^\infty f(\tau)d\tau = \tilde{o}(e^{\lambda t})$, i.e., for any $n \geq 0$, $t^n \int_t^\infty f(\tau)d\tau = o(e^{\lambda t})$. Assume $t^n f(t) = \epsilon(t)e^{\lambda t}$ where $\epsilon(t) \to 0$ and

$$\begin{aligned}
&|t^n \int_t^\infty f(\tau)d\tau| \leq \int_t^\infty |\tau^n f(\tau)|d\tau \\
=&\int_t^\infty |\epsilon(\tau)e^{\lambda\tau}|d\tau \leq \sup_{\tau\geq t}|\epsilon(\tau)| \int_t^\infty e^{\lambda\tau}d\tau = \sup_{\tau\geq t}|\epsilon(\tau)|\frac{1}{\lambda}e^{\lambda t} = o(e^{\lambda t}).
\end{aligned}$$

(2)If $\lambda > 0$, it is enough to prove for any $n \geq 0$, $t^n \int_0^t f(\tau)d\tau = o(e^{\lambda t})$. Assume $(1 + 2t)^n f(t) = \epsilon(t)e^{\lambda t}$, where $\epsilon(t) \to 0$.

$$\begin{aligned}
&|t^n \int_0^t f(\tau)d\tau| \leq t^n|\int_0^{t/2} f(\tau)d\tau| + t^n|\int_{t/2}^t f(\tau)d\tau| \\
\leq&t^n|\int_0^{t/2} \epsilon(\tau)e^{\lambda\tau}d\tau| + t^n|\int_{t/2}^t \frac{\epsilon(\tau)}{(1+2\tau)^n}e^{\lambda\tau}d\tau| \\
\leq&t^n \sup_{\tau\geq 0}|\epsilon(\tau)| \int_0^{t/2} e^{\lambda\tau}d\tau + \sup_{t/2\leq\tau\leq t}|\epsilon(\tau)||\int_{t/2}^t e^{\lambda\tau}d\tau| \\
\leq&O(t^n e^{\lambda t/2}) + o(1)O(e^{\lambda t}) = o(e^{\lambda t}).
\end{aligned}$$

□

6.5. **lemma 6.4.**

Lemma 6.4. *Assume $V(x) = -Bx + \Delta V(x)$. B is a square matrix and $0 < \lambda_1 < ... < \lambda_m$ are the eigenvalues of B. γ and $\gamma + \Delta\gamma$ are two integral curves of the vector field V near origin. $|\Delta V(\gamma + \Delta\gamma) - \Delta V(\gamma)| = O(|\gamma||\Delta\gamma|) + O(|\Delta\gamma|^2)$. If $|\Delta\gamma(t)| = o(e^{-\lambda t})$ for some $\lambda > \lambda_m$, than we have $\Delta\gamma(t) = 0$.*

Proof. We have

$$\gamma'(t) = V(\gamma(t)) = -B\gamma(t) + \Delta V(\gamma(t)),$$
$$\gamma'(t) + \Delta\gamma'(t) = -B\gamma(t) - B\Delta\gamma(t) + \Delta V(\gamma(t) + \Delta\gamma(t)).$$

So

$$\begin{aligned}\Delta\gamma'(t) =& - B\Delta\gamma(t) + \Delta V(\gamma(t) + \Delta\gamma(t)) - \Delta V(\gamma(t))\\ =& - B\Delta\gamma(t) + O(|\gamma(t)||\Delta\gamma(t)|) + O(|\Delta\gamma(t)|^2)\\ =& - B\Delta\gamma(t) + o(|\Delta\gamma(t)|).\end{aligned}$$

Choose $0 < \epsilon < \lambda - \lambda_m$, and there exists P such that $||sym(P^{-1}BP)||_2 < \lambda_m + \epsilon$. Let $||x|| = ||P^{-1}x||_2$.

$$\begin{aligned}&(||\Delta\gamma(t)||^2)' = (||P^{-1}\Delta\gamma(t)||_2^2)' = 2(P^{-1}\Delta\gamma(t))^T P^{-1}\Delta\gamma(t)'\\ =&2(P^{-1}\Delta\gamma(t))^T P^{-1}(-B\Delta\gamma(t) + o(|\Delta\gamma(t)|))\\ =& - 2(P^{-1}\Delta\gamma(t))^T(P^{-1}BP)(P^{-1}\Delta\gamma(t)) + o(|\Delta\gamma(t)|^2)\\ \geq& - 2(\lambda_n + \epsilon)||\Delta\gamma(t)||^2 \geq -2\lambda||\Delta\gamma(t)||^2\end{aligned}$$

for t sufficient large (say $\geq t_0$). Then $||\Delta\gamma(t)||^2 \geq e^{-2\lambda(t-t_0)}||\Delta\gamma(t_0)||$. Since $\Delta\gamma(t) = o(e^{-\lambda t})$, $||\Delta\gamma(t)|| = ||\Delta\gamma(t_0)|| = 0$. □