Digital Signal Processing

POSTECH

Department of Electrical Engineering

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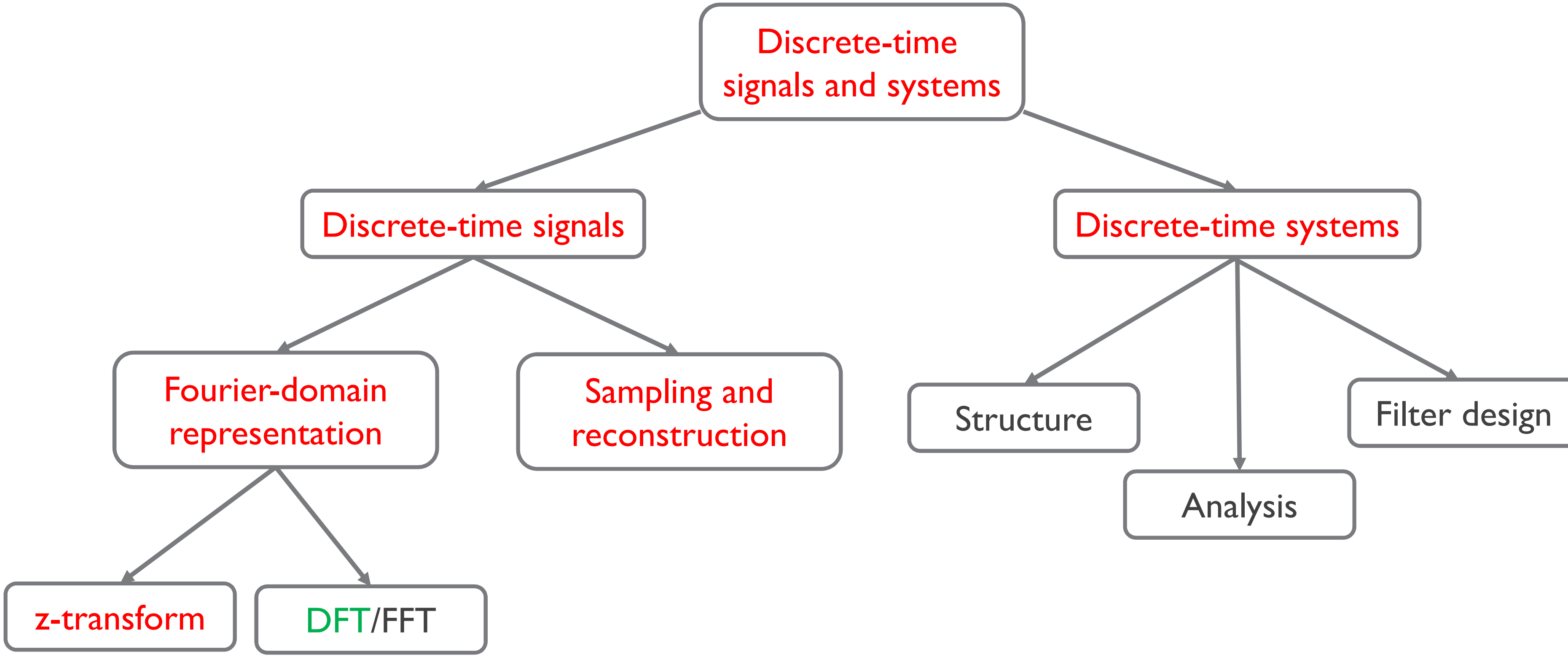
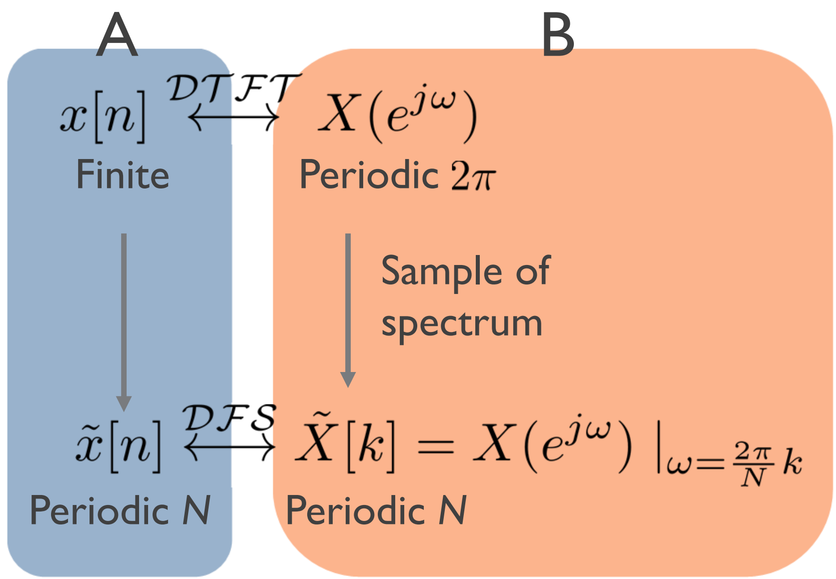
Course at glance



Sampling the Fourier Transform

A→B vs. B→A

- Until now, we have shown that if A, then B



- We want to prove that if B, then A

Proof of B→A

- Recall $X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}, \quad \tilde{X}[k] = X(e^{j(2\pi/N)k})$
- Synthesis equation

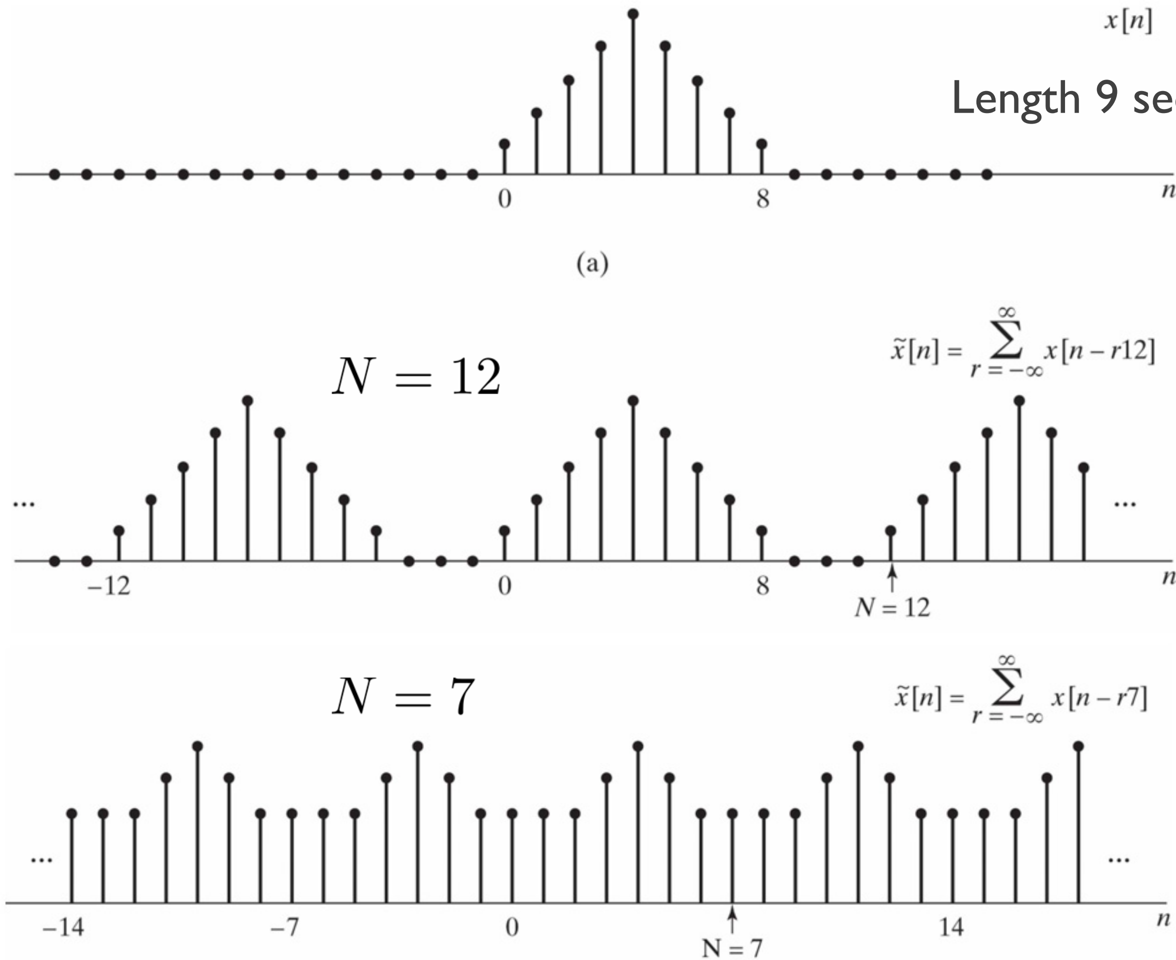
$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn} \qquad \tilde{p}[n] = \sum_{r=-\infty}^{\infty}\delta[n-rN]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}\right]W_N^{-kn}$$

$$= \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1}W_N^{-k(n-m)}\right] = \sum_{m=-\infty}^{\infty} x[m]\tilde{p}[n-m]$$

$$= \sum_{r=-\infty}^{\infty} x[n-rN]$$ ➡ Periodic replicas of $x[n]$

Examples of periodic replicas



Sampling in the frequency-domain

- Recovering $x[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$

  does not require to know its DTFT at all frequencies
  ➔ Sufficient to know only the N-points of $\tilde{X}[k]$

- Discrete Fourier Transform (DFT)
  ➔ Represent a finite length sequence by using DFS coefficients $\tilde{X}[k]$

Discrete Fourier Transform (DFT)

Finite-length sequence

- Consider a finite-length sequence $x[n]$ of length N samples
  - If the length is smaller than N, appending zeros to make length N

- Construct a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN] = x[(n \text{ modulo } N)] = x[((n))_N]$$

- The finite-length sequence $x[n]$ can be recovered as

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$
$$= \tilde{x}[n]\,(u[n] - u[n-N])$$

DFT of the finite-length sequence

- DFS coefficients, $\tilde{X}[k]$, of the periodic sequence $\tilde{x}[n]$ with period of *N*
  ➔ Also periodic with period of *N*

- To maintain a duality between time and frequency domains, choose one period of $\tilde{X}[k]$ as DFT

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

- Possible to recover DFS coefficients as

$$\tilde{X}[k] = X[(k \text{ modulo } N)] = X[((k))_N]$$

DFT vs. DFS pairs

- Analysis equations

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

- Synthesis equations

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

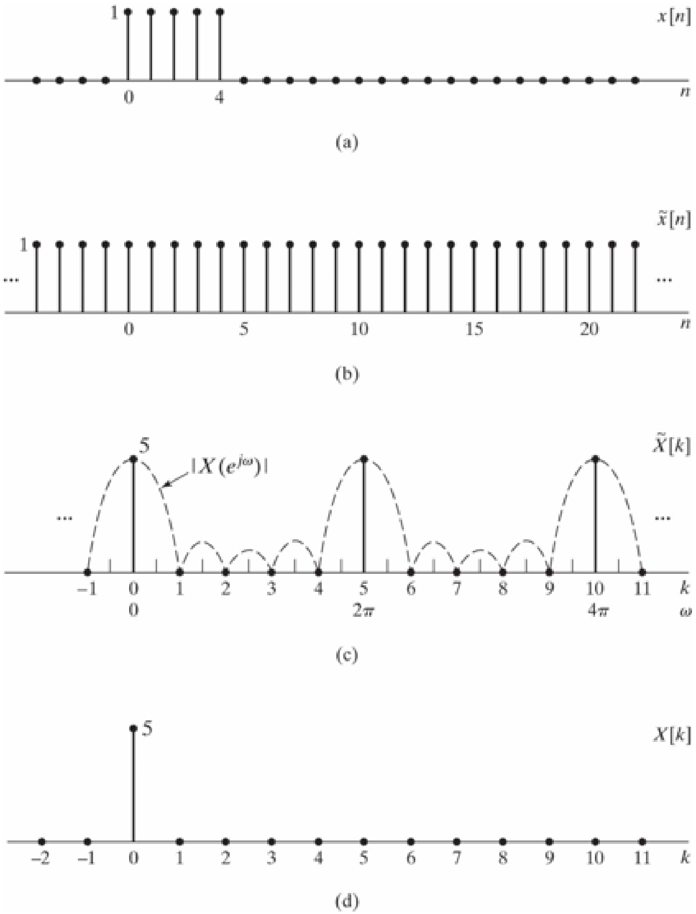
$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

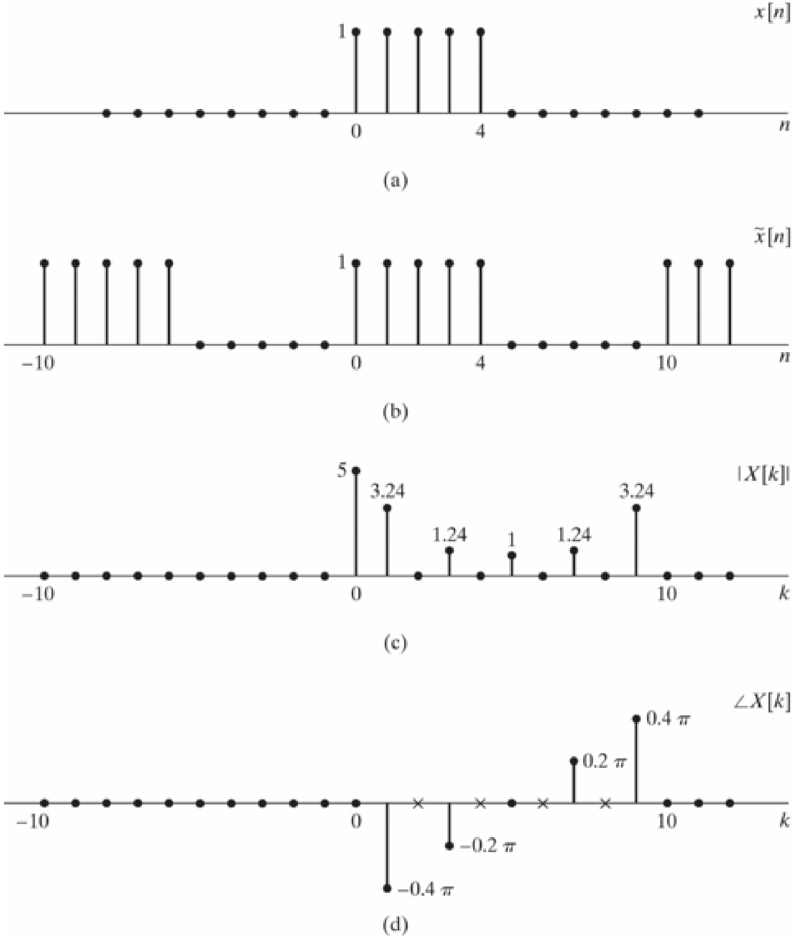
Zeros outside the range of [0,*N*]

Periodic with period *N*

- If we evaluate the values of DFT pairs outside of [0,*N*], they are not zeros, but a rather a periodic extension of *x*[*n*] and *X*[*k*]
  ➔ Assume they are zeros because…

Example (8.7 in the textbook)





Reconstruction of DTFT from DFT

- Note $x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

$$= \tilde{x}[n]w[n] = \left\{ \sum_{r=-\infty}^{\infty} x[n-rN] \right\} w[n]$$

where $w[n] = u[n] - u[n-N]$ ➔ Rectangular window

Periodic convolution

- DTFT of $x[n] = \tilde{x}[n]w[n] \overset{\mathcal{DTFT}}{\longleftrightarrow} X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{X}(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$

where $w[n] \overset{\mathcal{DTFT}}{\longleftrightarrow} W(e^{j\omega}) = \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(N-1)}{2}\omega}$

Reconstruction of DTFT from DFT

- Note $$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{X}(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(X(e^{j\theta})\sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\delta\left(\theta-\frac{2\pi k}{N}\right)\right)W(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi}\left(X(e^{j\theta})\sum_{k=0}^{N-1}\frac{2\pi}{N}\delta\left(\theta-\frac{2\pi k}{N}\right)\right)W(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]W(e^{j(\omega-2\pi k/N)})$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\frac{\sin\left(\frac{N}{2}(\omega-2\pi k/N)\right)}{\sin\left(\frac{1}{2}(\omega-2\pi k/N)\right)}e^{-j\frac{(N-1)}{2}(\omega-2\pi k/N)}$$

Properties of the DFT

Difference of DFT properties

- Many properties similar to the properties of DTFT and z-transform

- Need careful derivations
  - Due to the finite-length assumption and implicit periodicity

Linearity

- With two finite-length sequences $x_1[n]$ and $x_2[n]$, if

$$x_3[n] = ax_1[n] + bx_2[n]$$

  then $X_3[k] = aX_1[k] + bX_2[k]$
- The lengths of $x_1[n]$ and $x_2[n]$ may be different!

  $\text{Length } N_1$ $\qquad$ $\text{Length } N_2$

- The length of $x_3[n]$ should be $N_3 = \max(N_1, N_2)$
- DFTs $X_1[k]$ and $X_2[k]$ should be computed with the same length $N \geq N_3$
  ➔ Zero-padding for shorter sequence to have length *N* sequence

DFT with zero-padding

- If $N_1 < N_2$, make the sequence

$$x_1[n] = \{x_1[n], \underbrace{0, 0, \ldots, 0}_{N_2 - N_1 \text{ zeros}}\}$$

- DFTs become

Can be N_2 instead

$$X_1[k] = \sum_{n=0}^{N_1 - 1} x_1[n] W_{N_2}^{kn}, \quad 0 \le k \le N_2 - 1$$

$$X_2[k] = \sum_{n=0}^{N_2 - 1} x_2[n] W_{N_2}^{kn}, \quad 0 \le k \le N_2 - 1$$

Circular shift

- For DTFT, if $x[n] \overset{\mathcal{DTFT}}{\longleftrightarrow} X(e^{j\omega})$, then $x[n-m] \overset{\mathcal{DTFT}}{\longleftrightarrow} e^{-j\omega m} X(e^{j\omega})$
  ➔ Delay in time corresponds to change in phase

- For DFT with finite-length sequence, if $x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]$, then

$$X_1[k] = e^{-j(2\pi k/N)m} X[k] = W_N^{km} X[k] \overset{\mathcal{DFT}}{\longleftrightarrow} x_1[n] \;???$$

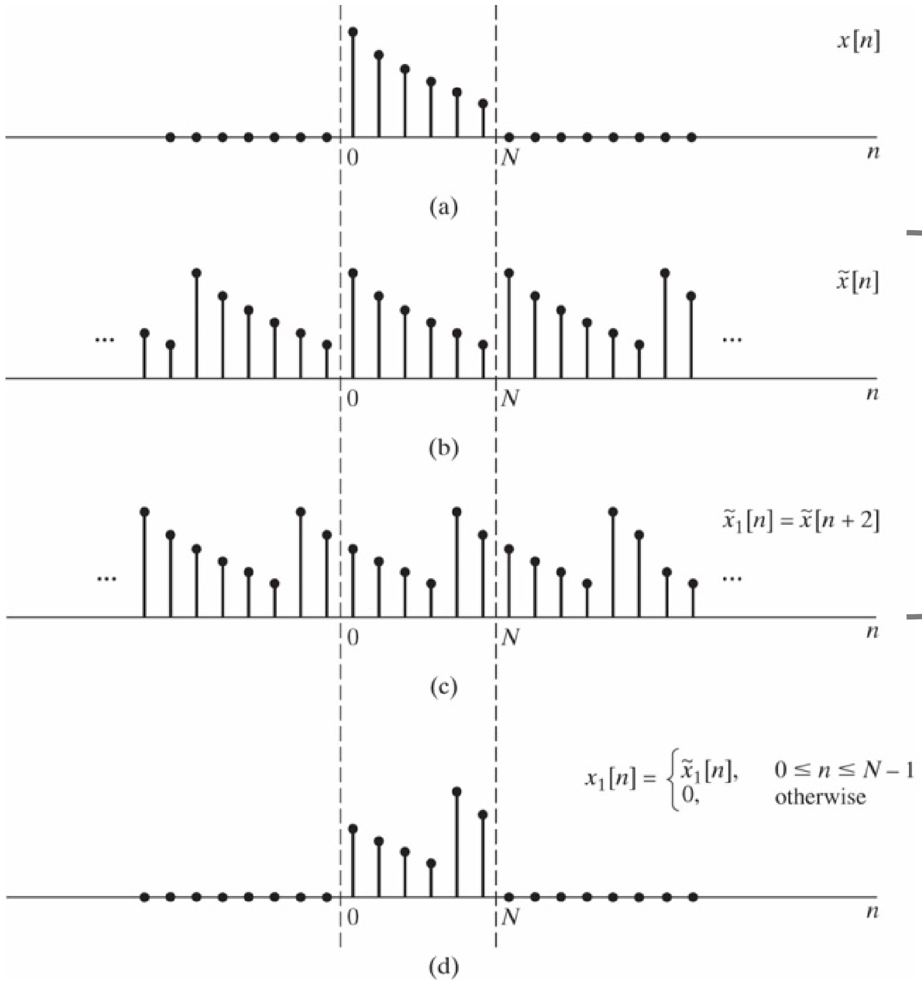
- $x_1[n]$ should be the length N sequence ➔ must be zero outside $0 \le n \le N-1$
  ➔ Cannot be a simple time shift of $x[n]$

- Correct result $x_1[n] = x[((n-m))_N], \quad 0 \le n \le N-1$

Circular shift example

$$N = 6$$

$$m = -2$$



DFS results in Section 8.2

Duality and symmetry properties

- Sections 8.6.3 and 8.6.4

- Useful when deriving complicated DFT and inverse DFT by hands
  - Not particular interest in this course

Circular convolution

- If $x_1[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k]$ and $x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_2[k]$ both with length *N*

$$X_3[k] = X_1[k]X_2[k] \overset{\mathcal{DFT}}{\longleftrightarrow} x_3[n] = \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N],\ 0 \le n \le N-1$$

$$= \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N],\ 0 \le n \le N-1$$

Circularly time reversed and shifted

➔ *N*-point circular convolution

- Define $x_3[n] = x_1[n] \circledN x_2[n]$
- Circular convolution is commutative as linear convolution
- Using duality: $x_1[n]x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} \frac{1}{N} X_1[k] \circledN X_2[k]$

Circular convolution with delayed impulse response

- Define $x_1[n] = \delta[n - n_0] = \begin{cases} 0, & 0 \le n < n_0 \\ 1, & n = n_0 \\ 0, & n_0 < n \le N-1 \end{cases}$ Make length *N* sequence

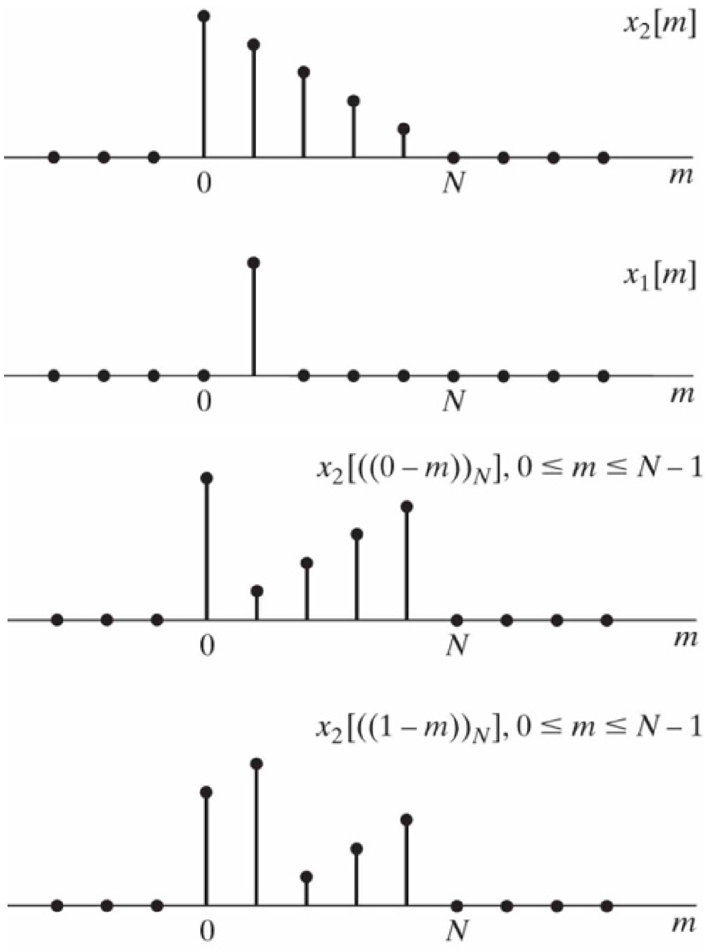
- $x_1[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k] = W_N^{kn_0}$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

- Form the product $X_3[k] = X_1[k] X_2[k] = W_N^{kn_0} X_2[k]$

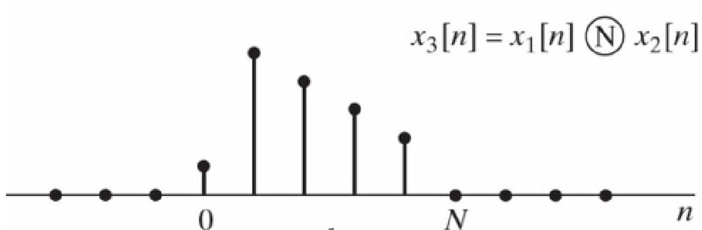
Circular shift of $x_2[n]$ with $n_0 \rightarrow x_3[n] = x_2[((n - n_0))_N]$

Circular convolution with delayed impulse response



$$N = 5,\ n_0 = 1$$

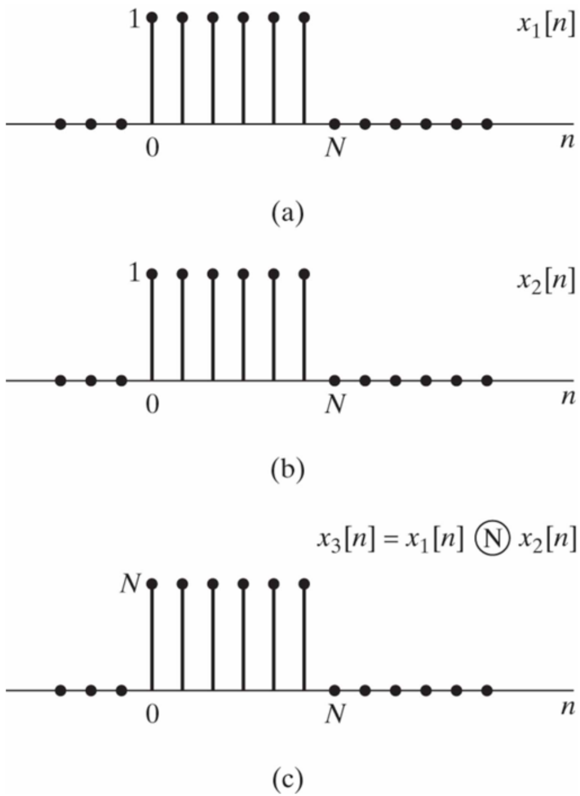
$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$



$$x_2[((n-1))_5]$$

Circular convolution of two rectangular pulses

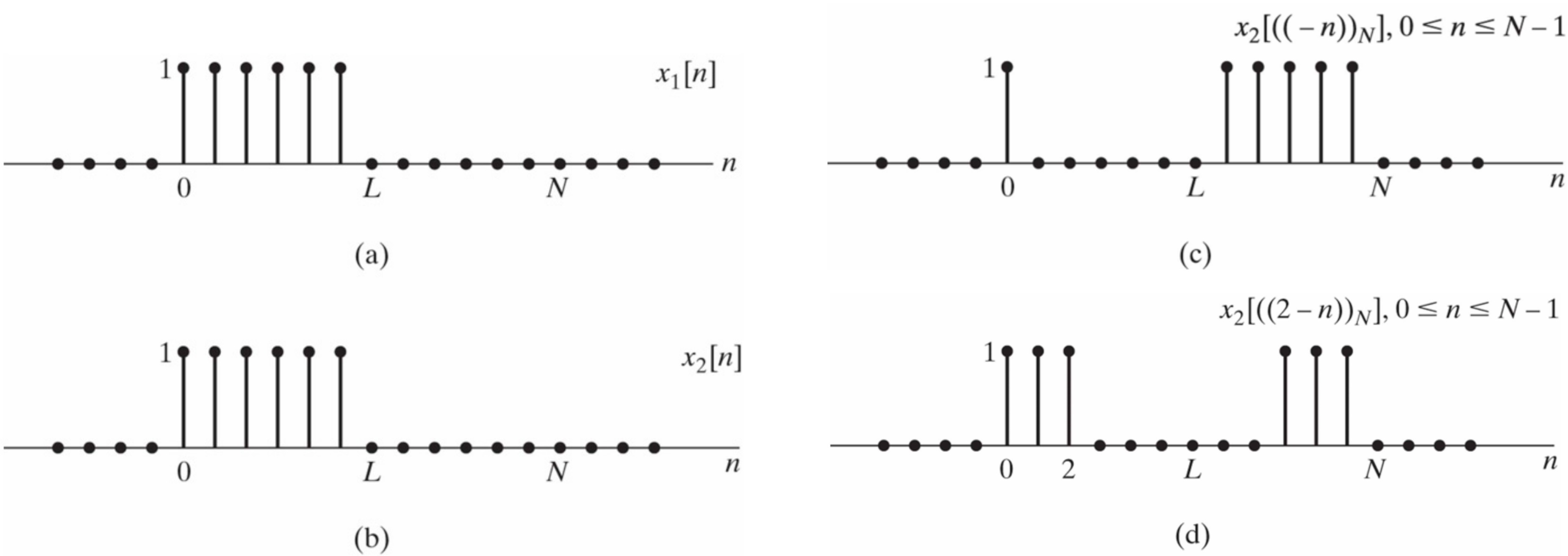
- N-point circular convolution of length N sequences



Circular shift same as original sequence

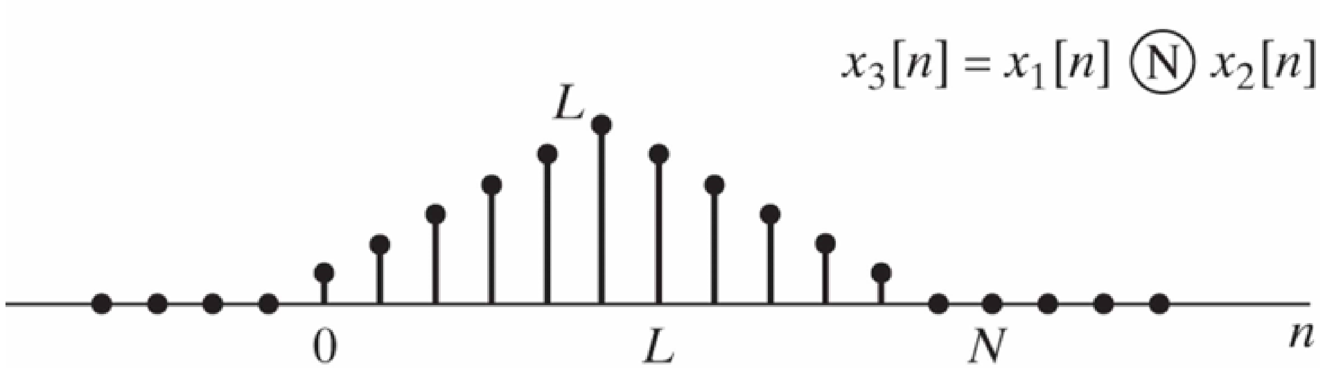
Circular convolution of two rectangular pulses

- N=2L-point circular convolution of length L sequences



Circular convolution of two rectangular pulses

- N=2L-point circular convolution of length L sequences



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➔ Same as linear convolution!

Computing Linear Convolution Using the DFT

Importance of linear convolution

- In many DSP applications, we want linear convolution
  → LTI systems represented with linear convolution
  - Filtering
  - Auto/cross-correlations

- DFT can be efficiently computed using Fast Fourier Transform (FFT)
  - Results in circular convolution, not linear convolution
  - Can we use DFT operations to get linear convolution?
    → Yes!

Length of linear convolution

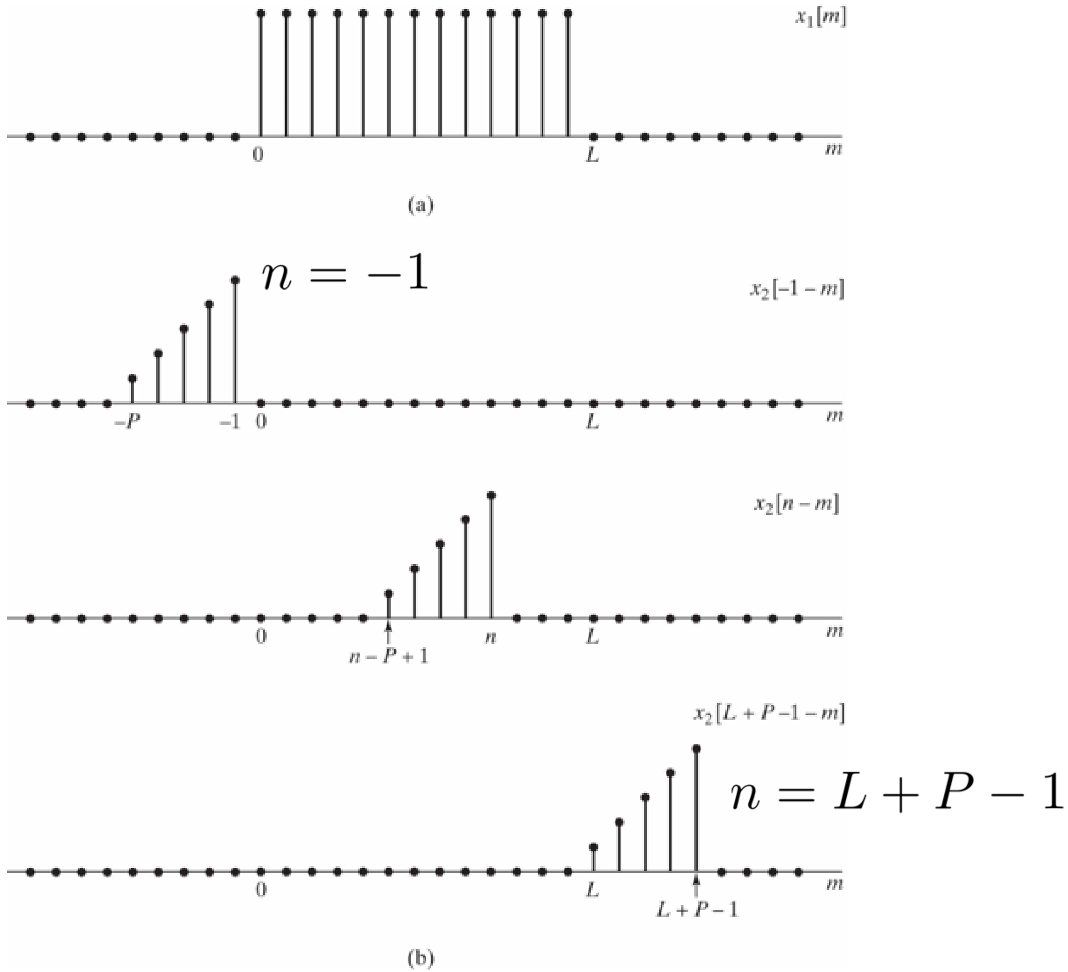
- Consider length *L* sequence $x_1[n]$ and length *P* sequence $x_2[n]$

- Linear convolution of the two sequences

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

is length *L*+*P*-1 sequence

Length of linear convolution



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Sampling vs. aliasing

- Sampling in time-domain → replicas in frequency-domain
- Sampling in frequency-domain → replicas in time-domain

Previous result on DFS

- Recall $X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}, \quad \tilde{X}[k] = X(e^{j(2\pi/N)k})$
- Synthesis equation

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \qquad \tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}\right] W_N^{-kn}$$

$$= \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right] = \sum_{m=-\infty}^{\infty} x[m]\tilde{p}[n-m]$$

$$= \sum_{r=-\infty}^{\infty} x[n - rN]$$ ➡ Periodic replicas of $x[n]$

Time domain aliasing

- Recall DFT $X[k] = \begin{cases} X(e^{j(2\pi k/N)}), & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$

  and inverse DFT $x_p[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

  $$x_p[n] = \tilde{x}[n](u[n] - u[n-N])$$

  where $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$

- If the length of *x*[*n*] is less than or equal to *N*, no time aliasing and $x_p[n] = x[n]$

- If the length of *x*[*n*] is greater than *N*, $x_p[n] \neq x[n]$ for some or all *n*

From DTFT to DFT

- Let $x_3[n] = x_1[n] * x_2[n]$ and $X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$

  Linear convolution

- If we define DFT $X_3[k] = X_3(e^{j(2\pi k/N)})$

$$
\begin{aligned}
&= X_1(e^{j(2\pi k/N)})X_2(e^{j(2\pi k/N)}) \\
&= X_1[k]X_2[k], \quad 0 \le k \le N-1
\end{aligned}
$$

- Inverse DFT of

$$
X_3[k] \overset{\mathcal{DFT}}{\longleftrightarrow} x_1[n] \textcircled{N} x_2[n] = x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n-rN], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}
$$

➔ Circular convolution = linear convolution followed by time aliasing!!!

When circular convolution = linear convolution?

- Consider length *L* sequence $x_1[n]$ and length *P* sequence $x_2[n]$

- Linear convolution of the two sequences

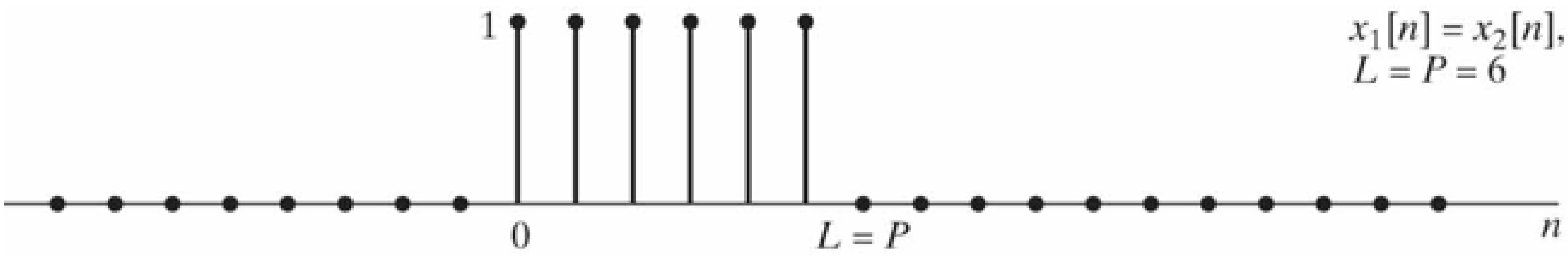
$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

is length *L*+*P*-1 sequence

- With *N*-point DFT where $N \geq L + P - 1$
➔ Circular convolution = linear convolution

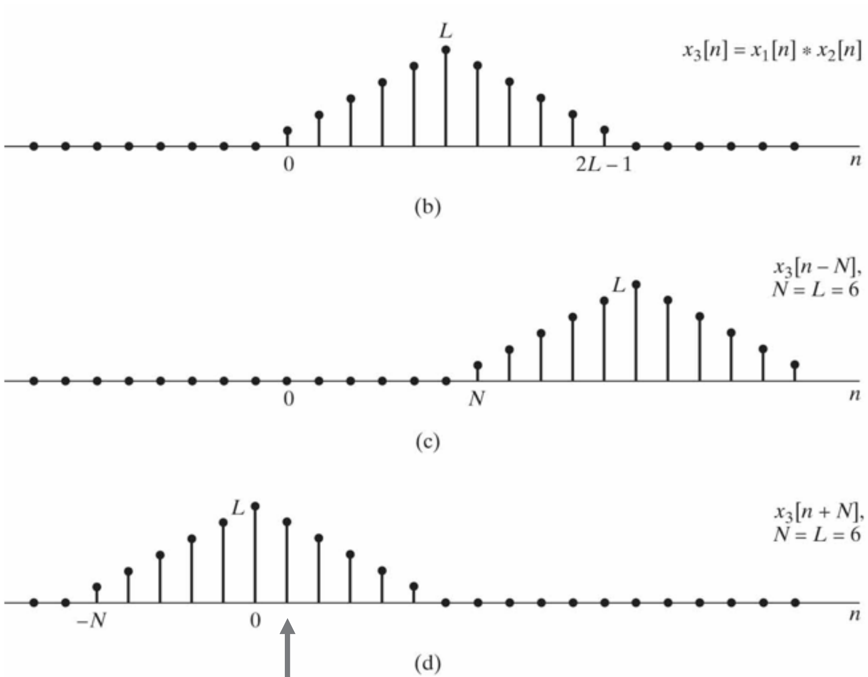
Previous examples

- Consider two sequences

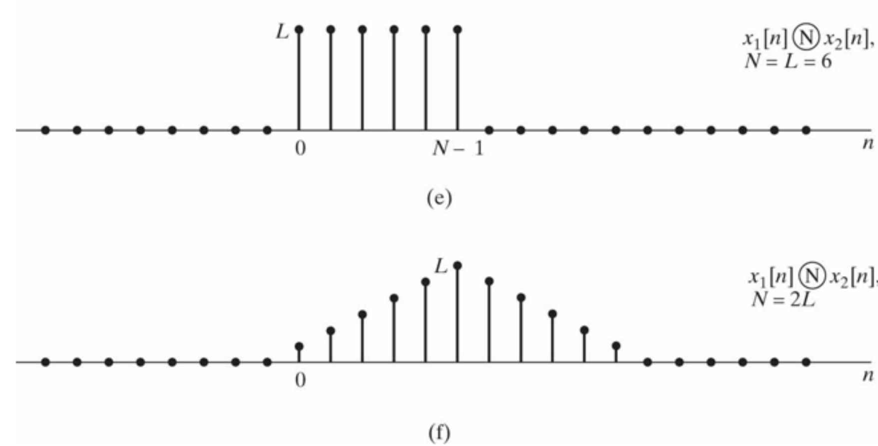


(a)

Previous examples



Correct 0



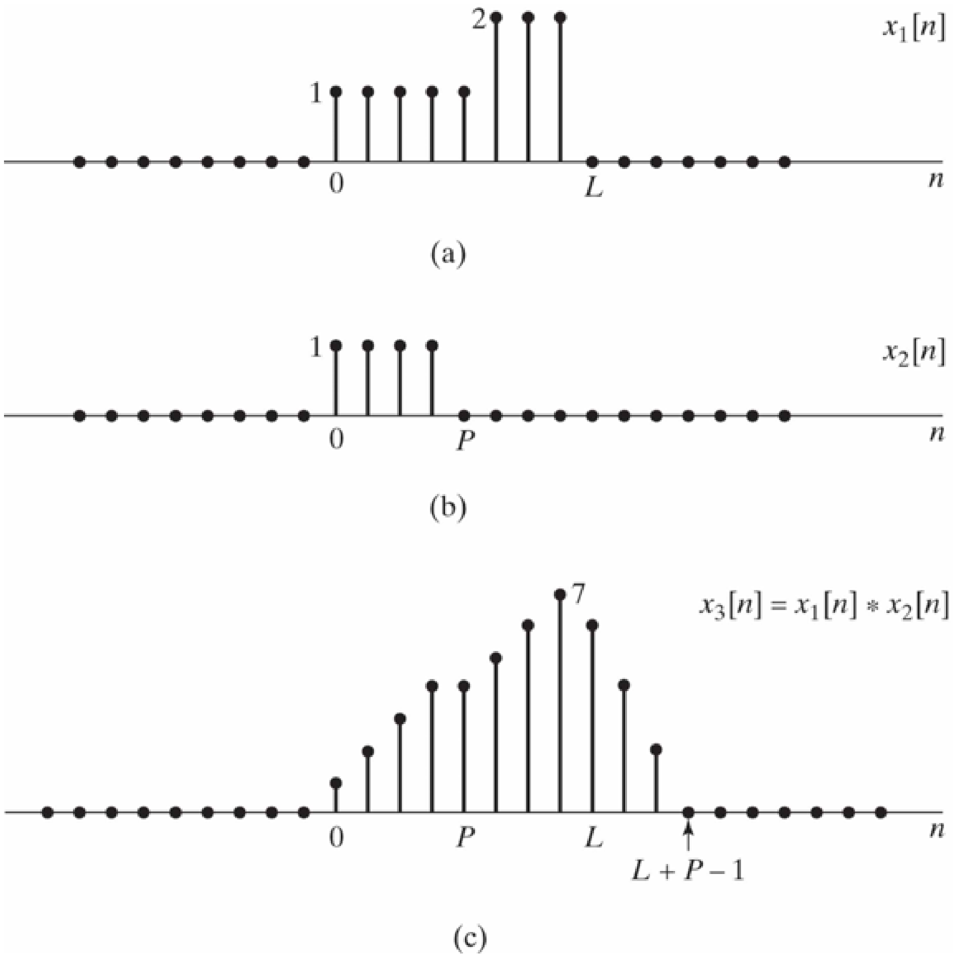
Partial time domain aliasing

- With *L*-point DFT (instead of *N*)

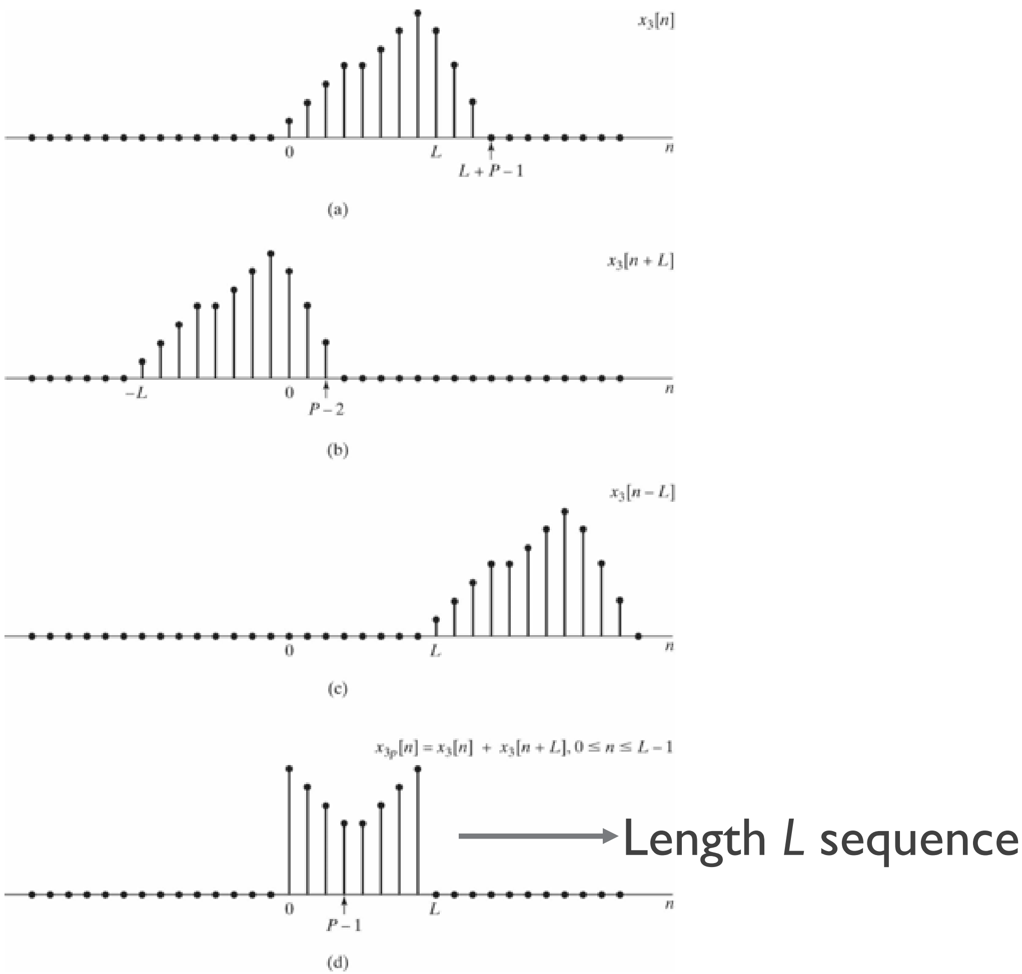
$$x_{3p}[n] = \begin{cases} x_1[n] \textcircled{L} x_2[n] = \sum_{r=-\infty}^{\infty} x_3[n - rL], & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

- How does it look like?

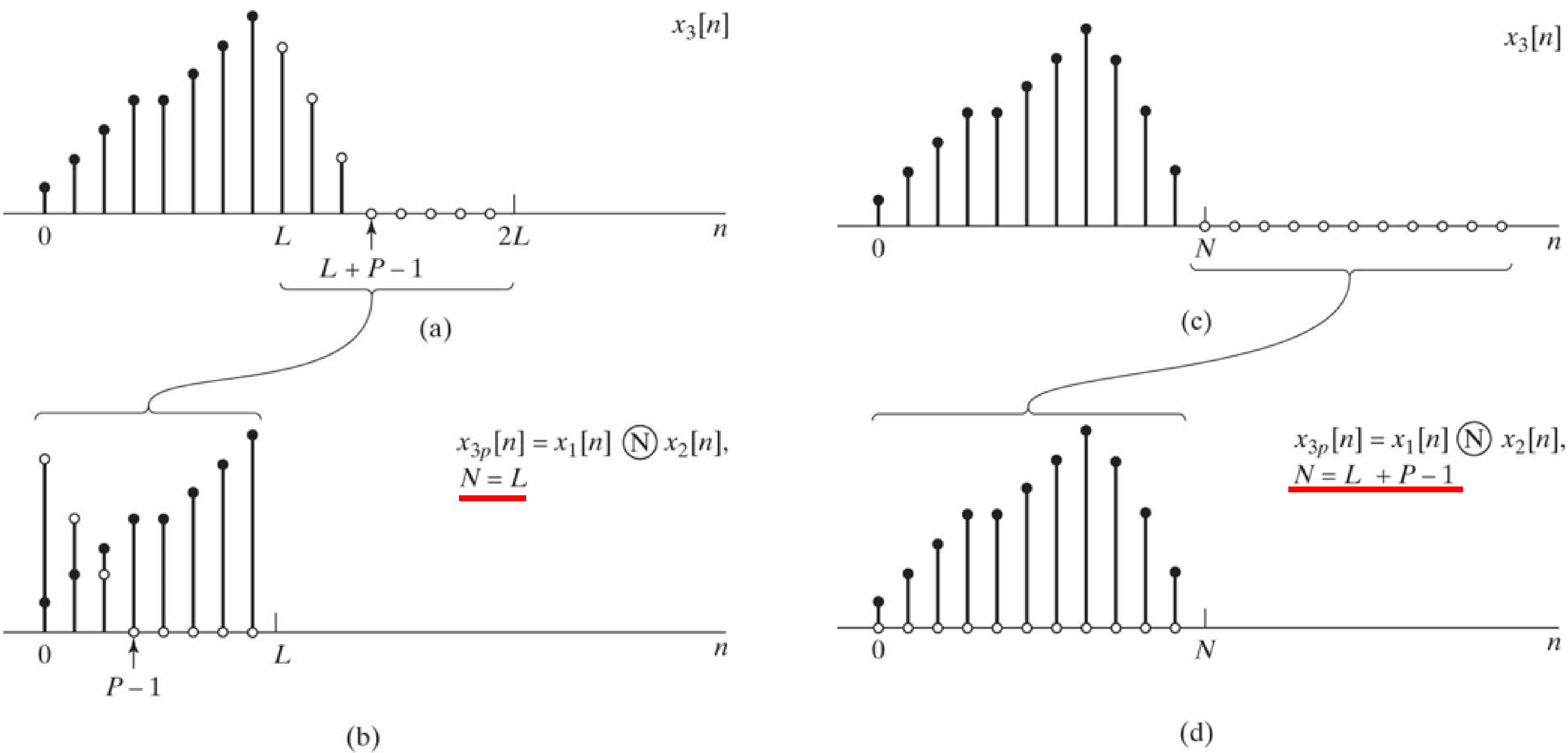
Partial time domain aliasing



Partial time domain aliasing



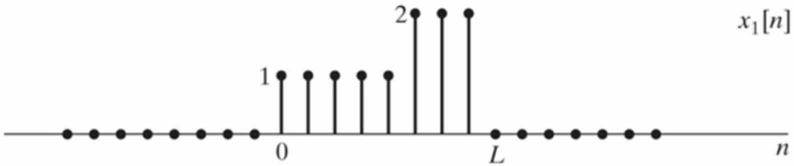
Partial time domain aliasing – systematic approach



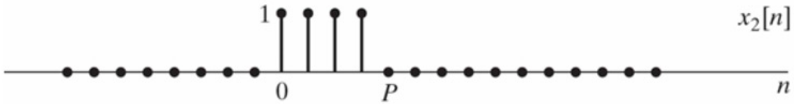
Efficient way to calculate circular convolution

- Consider the two sequences
- Want to have L-point circular convolution
- Possible to use circulant matrix

$$x_1[n] = \{1,1,1,1,1,2,2,2\}$$
$$x_2[n] = \{1,1,1,1,0,0,0,0\}$$

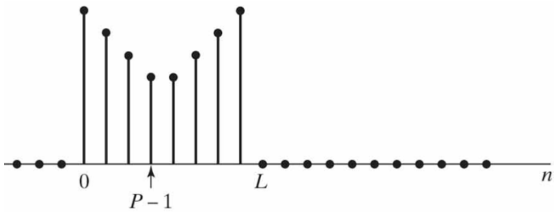


(a)



(b)

$$x_3[n] = x_1[n] \textcircled{L} x_2[n] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 5 \\ 4 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$$



Interesting feature of DFT

- Consider $x[n] = e^{j2\pi \frac{k_0}{N} n} \{u[n] - u[n-N]\}$

$$H_N[k_0] = H(e^{j\omega}) \big|_{\omega = 2\pi \frac{k_0}{N}}$$

- *N*-point DFT gives $x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_N[k] = N\delta[k-k_0]$

- Consider *h*[*n*] of length $M \leq N$ and its *N*-point DFT $H_N[k]$

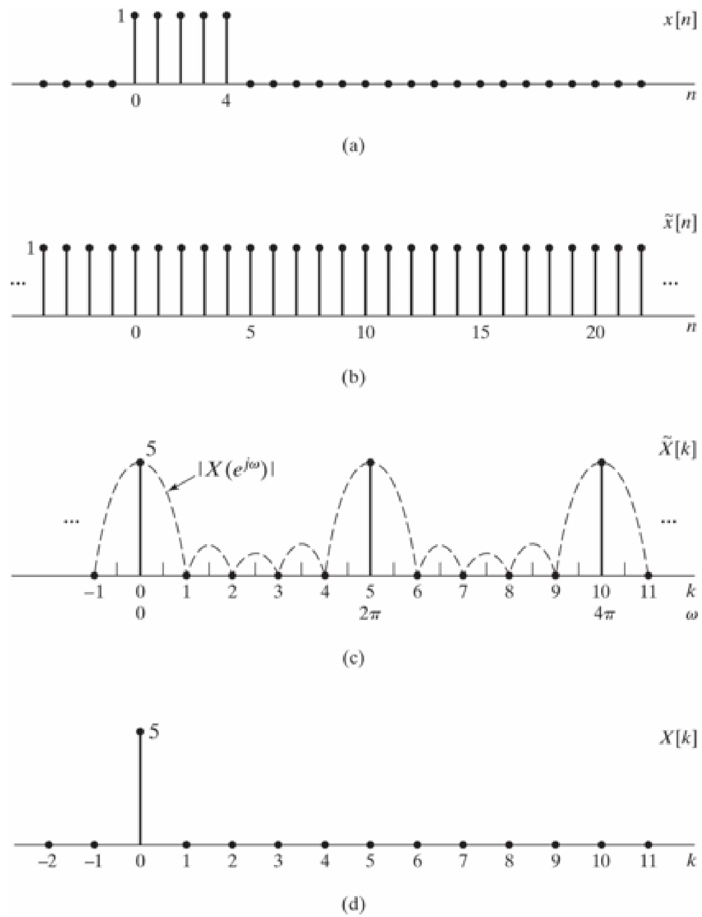
- Note $Y_N[k] = X_N[k]H_N[k] = NH_N[k_0]\delta[k-k_0] \overset{\mathcal{DFT}}{\longleftrightarrow} y_p[n] = H_N[k_0]x[n]$
  - In other words

$$y_p[n] = \left(e^{j2\pi \frac{k_0}{N} n} \{u[n] - u[n-N]\}\right) Ⓝ h[n] = H_N[k_0] \left(e^{j2\pi \frac{k_0}{N} n} \{u[n] - u[n-N]\}\right)$$

which mimics $y[n] = e^{j\omega_0 n} * h[n] = H(e^{j\omega_0})e^{j\omega_0 n}$

Linear convolution

Example (8.7 in the textbook)



Interesting feature of DFT

- Because *x*[*n*] is length *N* and *h*[*n*] is length *M*<*N*, the linear convolution

$$y[n] = h[n] * x[n]$$

is length *N*+*M*-*1* sequence
➔ Time-domain aliasing occurs in $y_p[n] = h[n] \circledN x[n]$

- But these aliased *M*-1 points amazingly yield "good" points to have

$$y_p[n] = \left(e^{j2\pi\frac{k_0}{N}n}\{u[n] - u[n-N]\}\right) \circledN h[n] = H_N[k_0]\left(e^{j2\pi\frac{k_0}{N}n}\{u[n] - u[n-N]\}\right)$$

Interesting feature of DFT

- Because DFT operation is linear (all DFT operations below are *N*-point DFT)

$$x[n] = \sum_{\ell=0}^{N-1} \alpha_\ell e^{j2\pi \frac{\ell}{N} n} \{u[n] - u[n-N]\} \overset{\mathcal{DFT}}{\longleftrightarrow} X_N[k]$$

$$h[n] \overset{\mathcal{DFT}}{\longleftrightarrow} H_N[k]$$

Length $M \leq N$

$$Y_N[k] = H_N[k] X_N[k] \overset{\mathcal{DFT}}{\longleftrightarrow} y_N[n] = \sum_{\ell=0}^{N-1} \alpha_\ell H_N[\ell] e^{j2\pi \frac{\ell}{N} n} \{u[n] - u[n-N]\}$$