Digital Signal Processing

POSTECH

Department of Electrical Engineering

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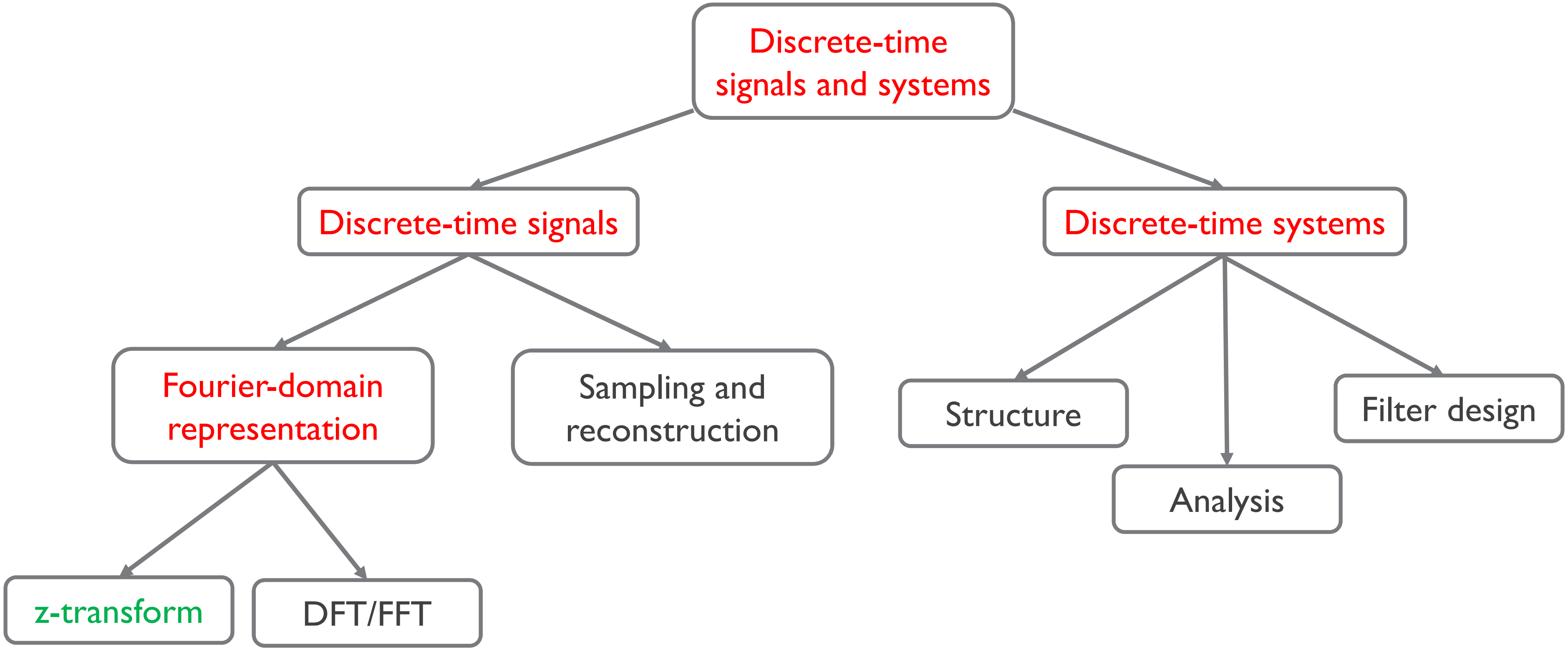
Course at glance



The z-Transform

Preliminaries

- Transforms
  - Fourier transform
  - Laplace transform

  Continuous time

  - Discrete-time Fourier transform (DTFT)
  - z-transform

  Discrete time

- z-transform is a generalization of DTFT

Limitation of Fourier transform

- Discrete-time Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Sufficient condition for the existence of DTFT

$$|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- DTFT may not exist for sequences that are not absolutely summable

  ➡ Require more generalized transform

z-transform definition

- Discrete-time Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- z-transform (can interpret as a function of z)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \Rightarrow x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

- The complex variable z in polar form $z = re^{j\omega}$

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

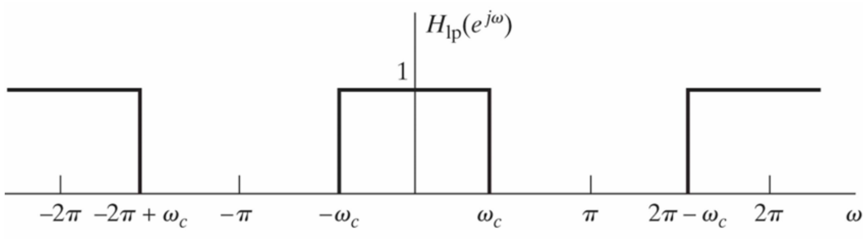
- If $|z| = r = 1, \quad X(z) = X(e^{j\omega})$

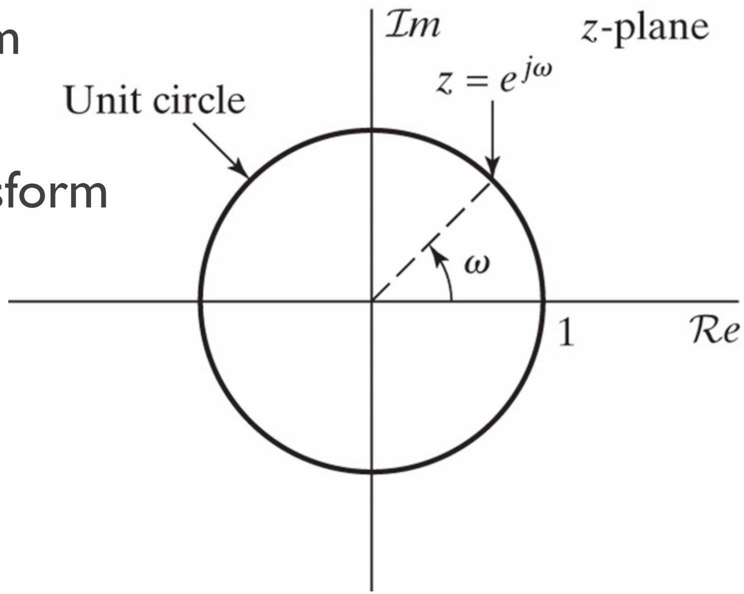
z-plane

- z-transform is a function of complex variable z
  - ➡ Interpret using the complex z-plan

- z-transform on unit circle = Fourier transform

- Linear frequency axis (x-axis) in Fourier transform
  - ➡ Unit circle in z-transform with period 2π





Region of convergence (ROC)

- DTFT does not converge for all sequences (depends on x[n])

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- z-transform does not converge for all sequences OR for all values of z

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

- ROC: **for any given sequence x[n],** the set of values of z for which z-transform converges

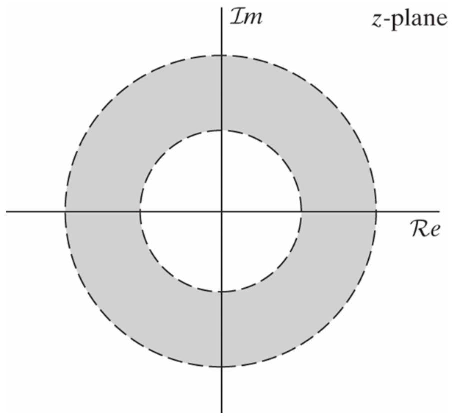
$$|X(re^{j\omega})| \leq \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| \leq \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

Shape of ROC

- Convergence depends only on $|z|$
  - If a point $z = z_1$ is part of ROC, then the circle with $|z| = |z_1|$ is part of ROC

- ROC consists of a ring centered at the origin
  - Outer boundary is a circle (may extend to infinity)
  - Inner boundary is a circle (may extend inward to include the origin)

- If ROC includes the unit circle $|z| = 1$
  - ➡ DTFT exists!



ROC example

Right-sided sequence

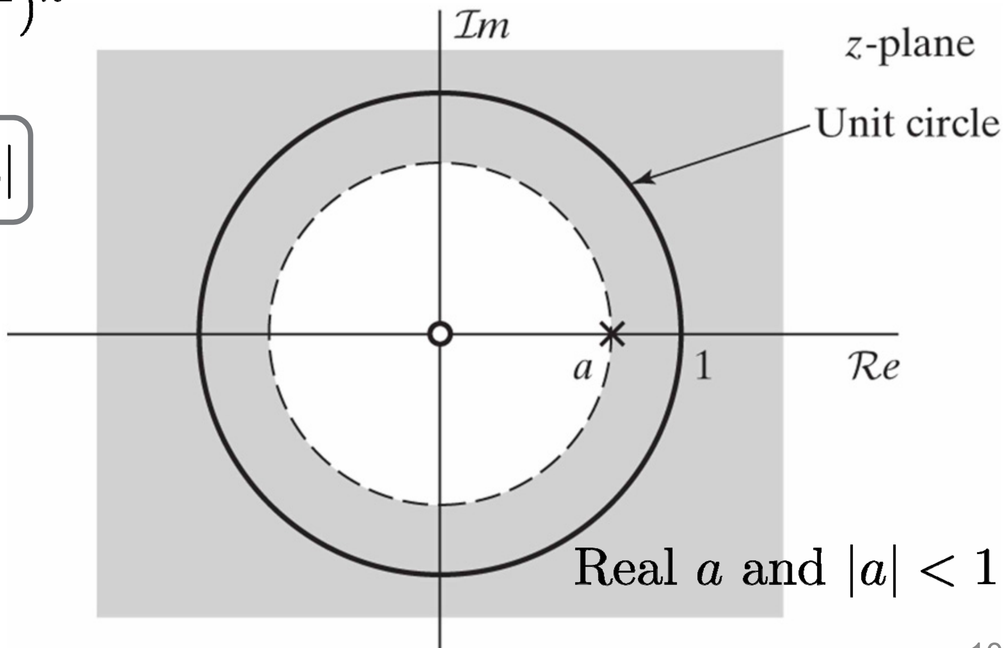
- Consider sequence $x[n] = a^n u[n]$
- z-transform of x[n]

$|az^{-1}| < 1$ to converge

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad \boxed{|z| > |a|}$$

ROC



Real a and $|a| < 1$

Zeros and poles

- Most important and useful z-transform expression – rational function of z

$$X(z) = \frac{P(z)}{Q(z)}$$

  - In previous example,

$$X(z) = \frac{z}{z-a}$$

- Zeros of $X(z)$: the values of z for $X(z) = 0$
  - ➡ Roots of polynomial $P(z)$

- Poles of $X(z)$: the values of z for $X(z) = \infty$
  - ➡ Roots of polynomial $Q(z)$

ROC example I

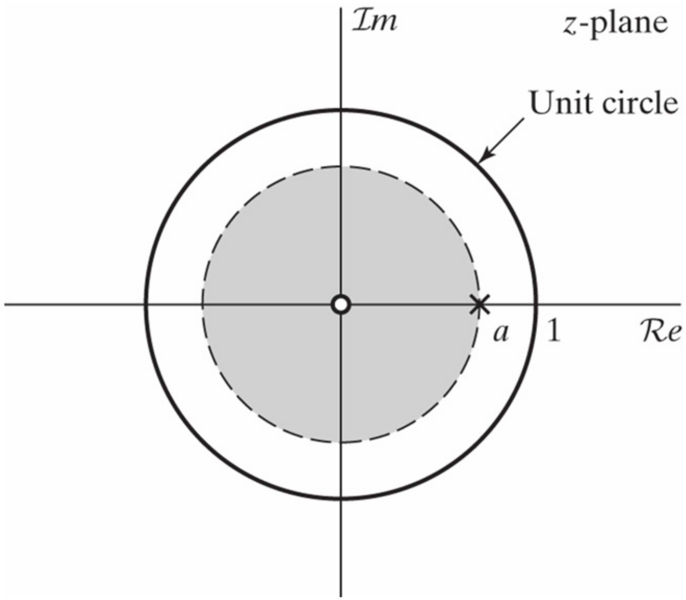
- Left-sided exponential sequence

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n \leq -1 \\ 0 & n > -1 \end{cases}$$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1 - a^{-1}z} = \frac{z}{z-a}$$

with ROC $|z| < |a|$



ROC example 2

- Sum of two exponential sequence

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

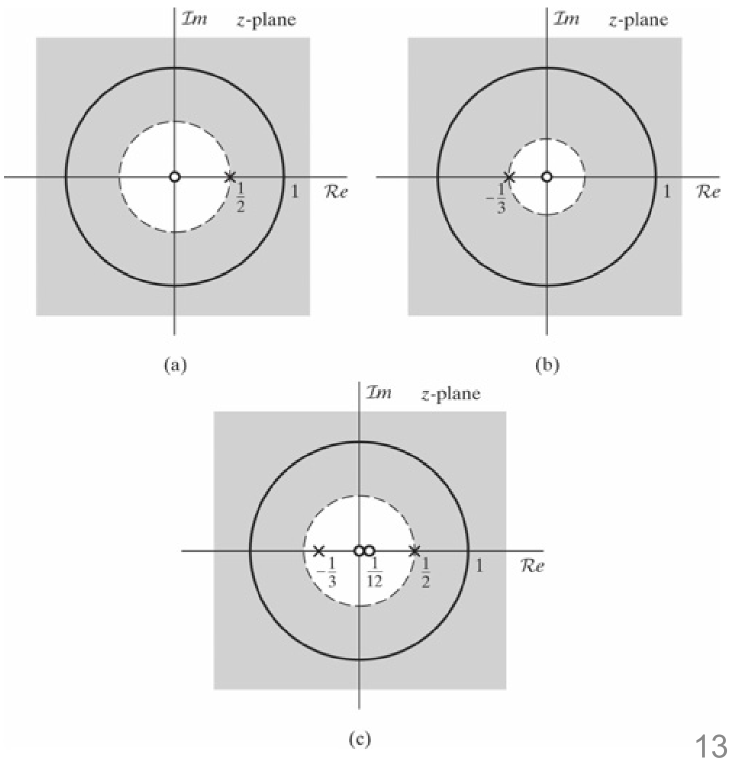
$$\text{ROC:} |z| > \frac{1}{2} \qquad \text{ROC:} |z| > \frac{1}{3}$$

Use linearity of z-transform

$$X(z) = \cdots = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}$$

$$\text{ROC:} |z| > \frac{1}{2} \text{ and } |z| > \frac{1}{3}$$



ROC example 3

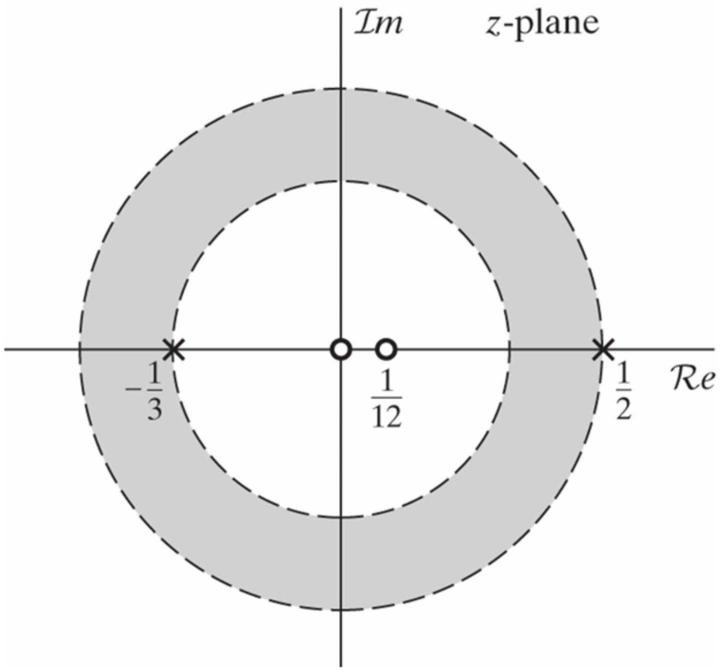
- Two-sided exponential sequence $x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$

$$\left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \frac{1}{3} < |z| < \frac{1}{2}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$



Finite-length sequence example 1

- Recall z-transform $X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}$
- No problem of convergence as long as $|x[n]z^{-n}|$ is finite
- May not have closed form expressions, may be unnecessary

$$x[n] = \delta[n] + \delta[n-5] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = 1 + z^{-5}$$

which is finite as long as $z \neq 0$

Finite-length sequence example 2

- Consider the sequence $x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

- z-transform becomes

$a < \infty$ and $z \neq 0$

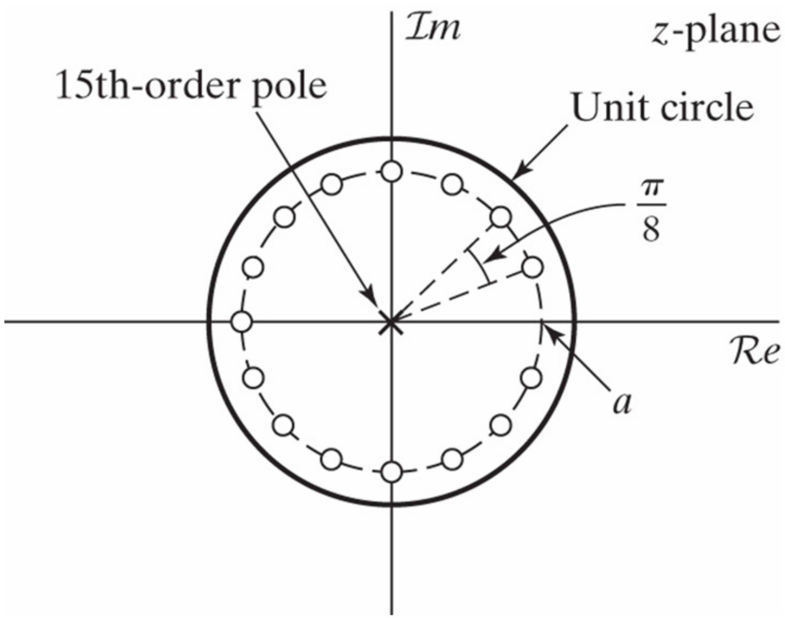
$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

- N roots of numerator polynomial

$$z_k = ae^{j2\pi k/N}, \quad k = 0, 1, \ldots, N-1$$

Pole-zero cancellation with k=0



$N = 16, \ 0 < a < 1$

Common z-transform pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Properties of ROC – read text 140p-142p

PROPERTY 1: The ROC will either be of the form $0 \le r_R < |z|$, or $|z| < r_L \le \infty$, or, in general the annulus, i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

ROC and z-transform may not exist

- Consider the sequence

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

- z-transform becomes

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

ROC: $|z| > \frac{1}{2}$ ROC: $|z| < \frac{1}{3}$

- No overlap in ROC ➔ no z-transform (nor Fourier transform) representation

ROC must be specified

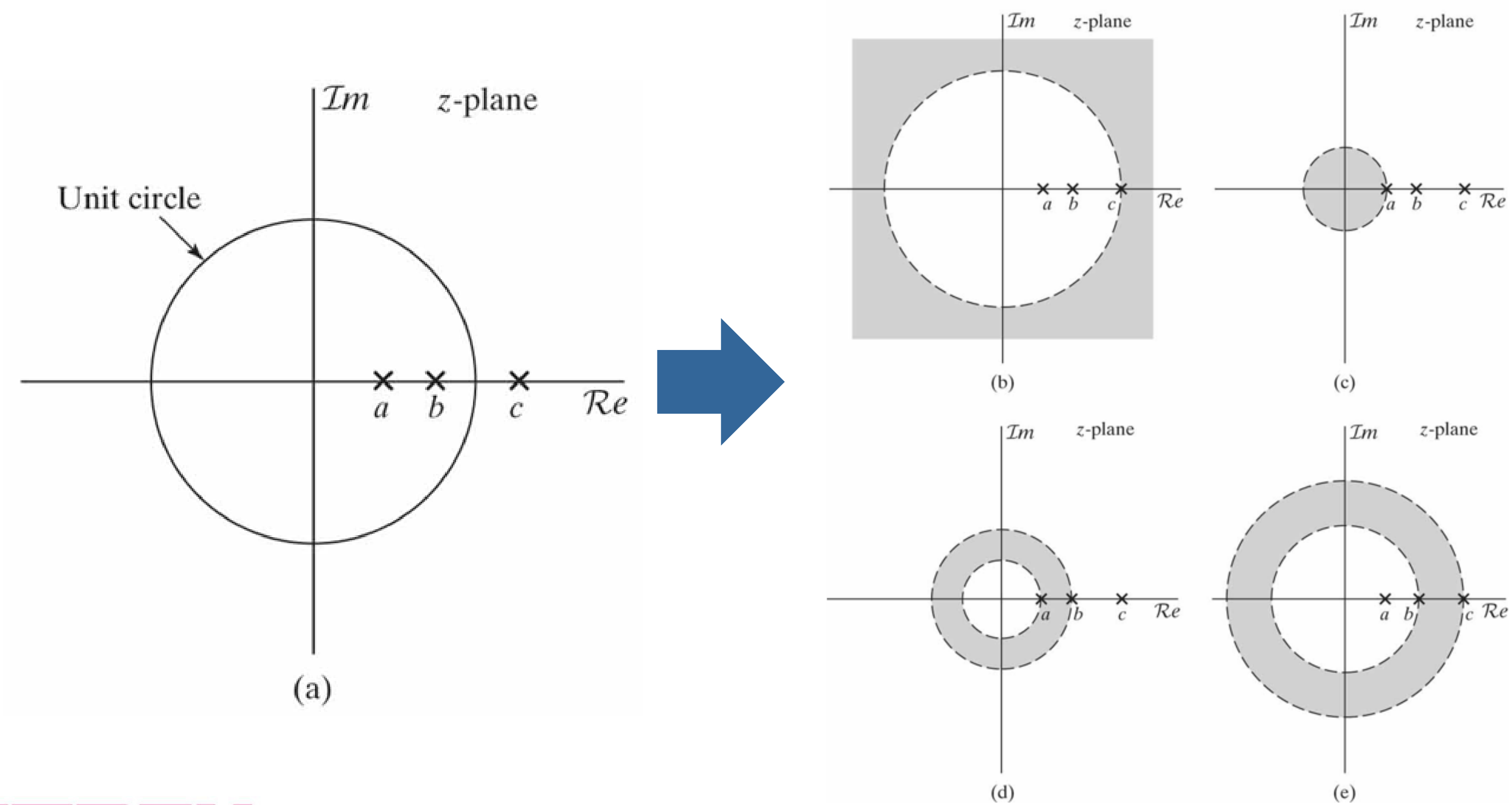
- Algebraic expression or pole-zero pattern does not completely specify the z-transform of the sequence
  - Example

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} \quad |z| < |a| \quad \neq \quad X(z) = 1 - \frac{1}{1 - a^{-1}z} \quad |z| > |a|$$

  - ROC must be specified to represent the sequence

Possible ROC patterns

- Due to Properties 1,3,8, the number of possible ROC patterns is limited



Inverse z-Transform

Definition of inverse z-transform

- The inverse z-transform is defined as

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C represents a closed contour within the ROC of the z-transform

➡ Hard to evaluated!

- Take different approaches in practice
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

Inspection method

- Nothing but memorizing z-transform pairs or use lookup tables (e.g., Table 3.1)

- Frequently arising pairs

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

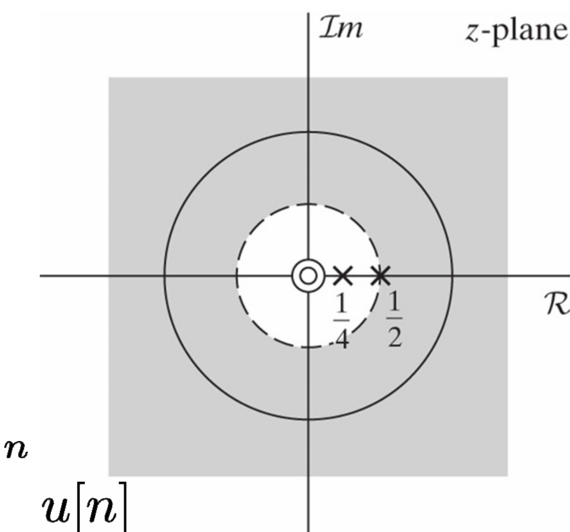
- Memorizing these forms will significantly reduce the time to solve problems!

Partial fraction expansion

- Hard to explain in words (check section 3.3.2) but concept is simple
- Consider

$$X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \quad |z| > \frac{1}{2}$$

Right-sided sequence



- Expand as

$$X(z) = \frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

where

$$A_1 = \left(1-\frac{1}{4}z^{-1}\right)X(z)\bigg|_{z=1/4} = -1$$

$$A_2 = \left(1-\frac{1}{2}z^{-1}\right)X(z)\bigg|_{z=1/2} = 2$$

- By inspection method, $x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$

Another example on partial fraction

- Find the inverse of

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{(1+z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}, \quad |z|>1$$

- Both numerator and denominator are the second-order
  ➔ There is a constant in X(z)
- By long division (or direct division), $X(z) = 2 + \dfrac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$
- Therefore,

$$X(z) = 2 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

  where $A_1 = -9, \ A_2 = 8$
- When $|z|>1$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Power series expansion

- Expand $X(z)$ as a sum of polynomials z
- Example 1:

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

From inspection method, $x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$

- Example 2: $X(z) = \log(1 + az^{-1}),\ \ |z| > |a|$

Use Taylor series expansion for $\log(1+x)$ with $|x| < 1$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} \stackrel{z}{\longleftrightarrow} x[n] = \begin{cases} (-1)^{n+1}\frac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

POSTECH

Common z-transform pairs

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13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

z-Transform Properties

z-transform properties - preliminaries

- Some definition

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \quad \mathrm{ROC} = R_x$$

A set of values of z in ROC

- Consider two sequences

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z), \quad \mathrm{ROC} = R_{x_1}$$

$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z), \quad \mathrm{ROC} = R_{x_2}$$

Useful z-transform properties

- Linearity
- Time shifting
- Multiplication by an exponential sequence
- Differentiation of X(z)
- Conjugation of a complex sequence
- Time reversal
- Convolution of sequences

- Major difference from DTFT

  ➡ Need to carefully consider ROC

Linearity property

- Property:

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

- Proof is easy!

$$\sum_{n=-\infty}^{\infty} (ax_1[n] + bx_2[n])z^{-n} = a \underbrace{\sum_{n=-\infty}^{\infty} x_1[n]z^{-n}}_{|z| \in R_{x_1}} + b \underbrace{\sum_{n=-\infty}^{\infty} x_2[n]z^{-n}}_{|z| \in R_{x_2}}$$

- There can be pole-zero cancellations that make ROC larger

Pole-zero cancellation example

- Consider the sequence $x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

$$= a^n u[n] - a^n u[n-N]$$

Finite-length sequence
= ROC entire z-plane except z=0

Both sequences have ROC of $|z| > |a|$

- The pole at z=a is canceled by a zero at z=a

➡ ROC extends to the entire z-plane except z=0

Finite-length sequence example 2

- Consider the sequence $x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

- z-transform becomes

$a < \infty$ and $z \neq 0$

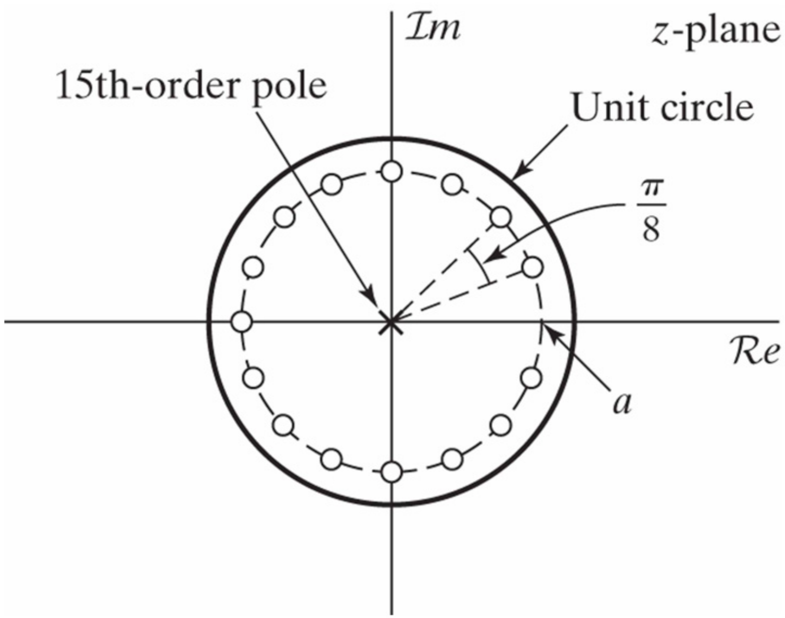
$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

- N roots of numerator polynomial

$$z_k = ae^{j2\pi k/N}, \quad k = 0, 1, \ldots, N-1$$

Pole-zero cancellation with k=0



$N = 16, \; 0 < a < 1$

Table of z-transform properties

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

How to use z-transform properties?

- Show

$$X(z) = \log(1 + az^{-1}), \ |z| > |a| \overset{\mathcal{Z}}{\longleftrightarrow} x[n] = \begin{cases} (-1)^{n+1} \frac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$