Critical Types*

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Abstract

How can we know in advance whether simplifying assumptions about beliefs will make a difference in the conclusions of game-theoretic models? We define *critical types* to be types whose rationalizable correspondence is sensitive to assumptions about arbitrarily high-order beliefs. We show that a type is critical if and only if it exhibits common belief in some non-trivial event. We use this characterization to show that all types in commonly used type spaces are critical. On the other hand, we show that *regular* types (types that are not critical) are generic, although perhaps inconvenient to use in applications.

1 Introduction

Tractable game-theoretic models require simplifying assumptions, often made implicitly, sometimes without awareness or intention of all the implications. In models of incomplete information, this trade-off is a consequence of the standard use of Harsanyi type spaces. Harsanyi's model simplifies the description of assumptions about players' beliefs about payoffs (their *first-order beliefs*) but to some extent conceals the assumptions about higher-order beliefs.

For example, in a typical study of auction design, the modeler may use an independent, private-values type space to express the (intended) assumption that no bidder has private

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information about another bidder's willingness to pay. But this convenient model, widely used in practice, entails the additional assumption of a common prior as well as a less transparent assumption about higher-order beliefs: each bidder's beliefs about others' willingness to pay is common knowledge among all bidders and the auction designer himself.

When a game is analyzed using such a type space, how can we know in advance whether these simplifying assumptions will be important for the conclusions? What type spaces should be used to ensure that, no matter what game is played, solutions will be robust to minor mis-specifications of higher-order beliefs? And which type spaces deliver conclusions that are sensitive to these mis-specifications?

In this paper, we answer these questions by defining and then characterizing *regular* and *critical* types. A type is defined to be regular if, regardless of the game, the set of rationalizable actions is guaranteed to be robust to changes in (sufficiently) high-orders of belief. Conversely, a type is defined to be critical if there is no such guarantee: there always exist games for which small changes in beliefs at even arbitrarily high order lead to non-negligible changes in the set of rationalizable actions.

Our main result finds a precise characterization of critical types in terms of their higher-order beliefs. We show that all commonly used type spaces in applied analysis consist entirely of critical types. These include finite type spaces, common-prior type spaces, and type spaces that entail common knowledge of any non-trivial event. Indeed, our characterization shows that a type is critical if and only if it exhibits a form of common belief in some non-trivial event.

Thus, when using these simplifying type spaces, an analyst *cannot* guarantee in advance that the predictions do not depend on higher-order beliefs without a case-by-case analysis of the specific game and its solutions in all "nearby" type spaces. For example, in a mechanism design application, the analyst first specifies a type space and then searches for a game form that achieves a certain goal. The goal of *robust* mechanism design is to avoid type spaces whose optimal mechanisms rely on assumptions about higher-order beliefs. The only type spaces that ensure robustness *at the outset* are those composed of regular types. These exclude not just the independent private values model but all commonly used type spaces in mechanism design. For those critical type spaces, robustness would have to be checked once the optimal mechanism is found by analyzing the solutions of that mechanism in all similar type spaces, embodying different higher-order beliefs.[1]

[1]There are two ways robustness can potentially fail. There may be a failure of *upper* hemi-continuity implying that the set of rationalizable actions expands, or there may be a failure of *lower* hemi-continuity implying that the set collapses. The first kind of failure would cause problems for a planner who seeks to fully implement a social choice function as new, undesirable, solutions could appear. The second kind would

On the other hand, we show that regular type spaces exist and in fact are, in a certain sense, typical. Regular types comprise a generic subset within the universal space of types according to the natural topology induced by their hierarchies of belief. Caution is necessary in interpreting this result, however, because regular type spaces, while generic, are so difficult to describe and work with that they are probably intractable.

1.1 Related Literature

Our work builds on the literature studying the impact of higher-order beliefs in game theory. Rubinstein (1989) was the first to demonstrate the sensitivity of solution concepts to higher-order beliefs. In the terminology we use, Rubinstein showed that models of complete information are critical. We illustrate some of our results using his electronic-mail game example below. Morris (2002) analyzed a particular infinite "higher-order expectations game" and showed that for this game rationalizable behavior in finite type spaces and continuous type spaces with bounded densities is not robust to changes in higher-order beliefs.

We extend these results in two ways. First, we provide an exact characterization of critical types. The characterization relies on a notion of common p-belief introduced by Monderer and Samet (1989) as a measure of approximate common knowledge. Roughly, a type exhibits mutual p-belief in some event E if the type believes E holds with probability p, and believes that with probability p the other players believe E holds with probability p, etc., for some finite number of iterations. The type exhibits *common* p-belief if the statement holds for infinitely many iterations. In this sense, common p-belief is an assumption about beliefs of arbitrarily high order. We show that a slightly weaker version of this assumption characterizes critical types.[2] We use this characterization to prove that commonly used types, such as finite types and common prior types, are critical.

Because we define a regular type to be one that has robust behavior across all games, our criterion is demanding. On the other hand, Weinstein and Yildiz (2007) have analyzed robust types for a fixed game. More generally, it is important for applications to identify the specific games for which critical types may fail robustness. In a recent paper, Chen and Xiong (2009) make an important step in this direction. Say that type is n-critical, if there exists a game with at most n actions for which small changes in beliefs at even an arbitrarily

be problematic in standard implementation problems where the existence of at least one desirable solution is enough. Then, the collapse of the rationalizable set may remove the desired outcome. Because Dekel, Fudenberg, and Morris (2007) have shown that the first form of robustness is guaranteed (we cite this result as Lemma 1 below), critical types exhibit this second form of non-robustness.

[2]More precisely, we work with a slightly weaker version of the common repeated p-belief introduced in Monderer and Samet (1997).

high order lead to large changes in the set of rationalizable actions. Chen and Xiong (2009) show that all finite and almost all common prior types are 3-critical (an example of their construction is presented below). On the other hand, for any given n, there are critical types that are not n-critical.

Weinstein and Yildiz (2007) analyze finite games with a rich payoff structure and show that for any fixed game the set of types with a unique and robust rationalizable action contains an open and dense set. The rich payoff assumption says that, for each action, there is a state in which that action is dominant. Although natural in some situations, the assumption is restrictive and for example would not be appropriate in a mechanism design application. While it is clearly necessary for the uniqueness part of their result, we show that it can be dispensed with in the robustness part. Say that type is G-regular, if the rationalizable actions in game G are robust to changes in high orders of belief. We show that the set of G-regular types contains an open and dense set. Some additional differences between our result and theirs concerning common prior types are discussed in Section 5.2.

We define rationalizability using the interim correlated rationalizability from Dekel, Fudenberg, and Morris (2007). We discuss in Section 6 how to extend the results in this paper to interim *independent* rationalizability, which has been studied by Morris and Skiadis (2000) and Ely and Pęski (2006). See Battigalli, Tillio, Grillo, and Penta (2009) for further discussion of these solution concepts.

2 Examples

We present some examples illustrating our characterization of critical types, beginning with complete-information types.

2.1 Complete Information Types

Suppose there are two players, and the players are uncertain whether tomorrow there will be rain or sun. Let us represent their uncertainty by assuming there are two states of nature $\omega \in \Omega = \{-1,+1\}$, where $\omega = -1$ represents rain and $\omega = +1$ represents sun. A Harsanyi type space specifies a set of types for each player, and for each type a belief about Ω and about the type of the other player.

Complete information is modeled by the simplest type space in which each player has exactly one type and that type is certain of (i.e., attaches probability 1 to) a single state. Let's suppose that there is complete information, and the two players are certain that there will be sun tomorrow, i.e., $\omega = +1$.

Once we have fixed a type in a type space, we determine whether the type is a *regular* type by considering all possible games whose payoffs depend on the realization of Ω. The type is regular if, for every such game, its rationalizable behavior is robust to changes in higher-order beliefs. If not, then it is a critical type.

Rubinstein first demonstrated that complete-information types are critical with the use of the following coordination game. The two players must decide whether to meet for coffee tomorrow (action A) or to stay in the office and work (action B). If there will be sun, then both prefer to meet rather than work, but if there will be rain, then player 2 prefers to work. Player 1 prefers to take the same action as player 2, regardless of the weather. These preferences are captured by the following payoffs.

	A	B
A	3,3	0,0
B	0,0	2,2

$\omega = +1$

	A	B
A	0,3	0,0
B	2,0	2,2

$\omega = -1$

Figure 1: The E-mail game

For our complete-information types, there is common knowledge that there will be sun. Common knowledge ensures that going to the coffee shop, i.e., profile (A,A), is rationalizable and indeed constitutes a (Bayesian) Nash equilibrium. To show that these types are critical, Rubinstein constructed a sequence of types whose beliefs approach common knowledge but for whom action A is not rationalizable, or even approximately rationalizable.

Consider any type that has *mutual knowledge of order* k that there will be sun. Mutual knowledge of order k means knowing there will be sun, knowing that the opponent knows that there will be sun, etc. (k times). Consider a sequence of types t_i^k, as k approaches infinity, where t_i^k has mutual knowledge of order k that there will be sun. Intuitively, for very large k, the type t_i^k has something close to common knowledge. This statement can be made formal by describing types explicitly in terms of their Mertens and Zamir (1985) beliefs and higher-order beliefs. Then, "close to common knowledge" means that the sequence t_i^k converges to the complete-information types in the natural product topology on higher-order beliefs.

Rubinstein looked at sequences in which the type t_i^k, while having mutual knowledge of order k that there will be sun, nevertheless exhibits some uncertainty at higher orders. In particular, she attaches greater than $1/2$ probability to the opponent attaching greater than

$1/2$ probability to ... ($k+1$ times) to player 2 being almost certain there will be rain.[3]

Rubinstein showed, using an infection argument, that B is the unique rationalizable action for every type in the sequence. First, if player 2 is sufficiently sure of rain, she will play B. Now, if player 1 is type t_1^1 and attaches greater than $1/2$ probability to player 2 being sufficiently sure of rain, then player 1 maximizes his expected payoff by playing B as well. By induction, for $k \geq 0$, type t_i^{k+1} will attach probability greater than $1/2$ to the opponent playing B and will have a unique best-reply to play B as well.

In fact, action A is not even *approximately* rationalizable for any type. Formally, an action is ε-rationalizable if it survives an iterative procedure of elimination of actions that are not ε-best-replies to the surviving strategies. Action A is not ε-rationalizable for any type of either player for any $\varepsilon < 1/2$. This strong form of discontinuity is part of our formal definition of a critical type. A type is critical if its rationalizable behavior is "broken" by higher-order belief perturbations, and not only "bent."

2.2 Finite Types and General Common Knowledge Types

Rubinstein was able to show that complete-information types are critical by exploiting the fact that there was an event in the state space Ω, namely sun, that was commonly known. This allows the construction of a game whose payoffs differ on the complement of that event, and an infection argument based on small higher-order probabilities that the complement had occurred.

A similar idea explains why types in any finite type space are critical, even types that have no common knowledge of any (proper) subset of Ω. To see this, consider any finite type space T. Recall that a type's first-order belief is the probability assessment over elements of Ω, in this case, the probability of sun. There is a continuum of possible first-order beliefs, but a finite type space includes only finitely many distinct first-order beliefs. Each type's second-order belief must assign probability 1 to this set of first-order beliefs, each type's third-order belief must assign probability 1 to such second-order beliefs, etc. As a consequence, while there may not be common knowledge in any subset of Ω, all of the types in any given finite type space have common knowledge of a discrete set of possible first-order beliefs. This

[3] These types can formally be described using a type space, following Rubinstein. Each player i has a countable set of types $T_i = \{t_i^k\}_{k=0}^{\infty}$. There is a common prior $\rho \in \Delta(\{-1,+1\} \times T_1 \times T_2)$ from which the state and the type profile are drawn. The prior ρ is defined by $\rho(-1, t_1^0, t_2^0) = 1/2$ and for some β less than but close to 1,

$$\rho(+1, t_1^k, t_2^l) = \begin{cases} \frac{(1-\beta)\beta^{k+l}}{2} & \text{if } k = l \text{ or } k = l+1 \\ 0 & \text{otherwise} \end{cases}$$

property implies that the types are critical, as we now demonstrate.

For concreteness, let us suppose that all of the first-order beliefs, i.e., the belief that the state is equal to $\omega = +1$, lies outside the interval $(1/4, 3/4)$. Then, we can show that all of the types in T are critical by considering the game from Figure 2.

	L	$\varnothing$	R
U	$2, 2+\omega$	$0, 5/2$	$2, 2-\omega$
D	$1, 0$	$2, 5/2$	$1, 0$

Figure 2: A game illustrating why finite types are critical

In this game, there is a Bayesian Nash equilibrium in which all types of player 1 play U and player 2 plays L when his first-order probability of sun ($\omega = +1$) is at least $3/4$ and R when that probability is less than $1/4$. It follows that these actions are rationalizable for the respective types.

Now we can perturb the higher-order beliefs of any of the types t of player 1 in the finite type space and show that none of these actions are approximately rationalizable for the perturbed types. To begin with, we consider any type t^1 that has exactly the same first-order beliefs as player 1 and that knows[4] that player $2's$ first-order belief assigns equal probability to the two states. For such a type, the action $\varnothing$ is (interim) strongly dominant and hence the unique rationalizable action. More generally, consider a type t^n that has exactly the same first $2n-1$ orders of beliefs as type t, and knows that player 2 knows that ... ($2n-1$ times) that player 2 assigns equal probability to the two states. Clearly, the sequence of types t^n converges to the original type t.

Then the proof is by induction on n. When $n = 1$, player 1 knows that player 2 assigns equal probability to the two states; hence, player 1 knows that player 2 will play $\varnothing$. The unique best-reply for player 1, regardless of the state, is to play D. Thus, D is the unique rationalizable action for the perturbed types of 1 with $n = 0$. When player 2 knows that player 1 knows that 2 assigns equal probability to the two states, player 2 knows that 1 will play D. Neither action L nor R is a best-reply to D, regardless of the state. Thus, only $\varnothing$ can be rationalizable for such player 2, and only D is rationalizable for player 1 type t^2. By induction, for arbitrary n, the types with beliefs t^n know that the opponent will play $\varnothing$ and the unique best-reply is D. We have shown that $\varnothing$ is the unique rationalizable action for all of the perturbed types for any n.

[4]Knowledge, i.e. belief with probability 1, is stronger than necessary. We need only belief with high probability. And by altering the payoffs in the game, we can get by with lower and lower probability belief.

We utilized only the property that in finite type spaces there is common knowledge that some interval of first-order beliefs was excluded. Thus, the previous construction would apply to show that any type in any type space with this property would be critical. For example, *none of the types in Rubinstein's approximate common knowledge sequence* have first-order beliefs in the interval $(1/4,3/4)$, and; hence, the types have common knowledge that this set is excluded. While these types do not have common knowledge of any proper event in Ω and do not belong to any finite type space, nevertheless they are also critical types, as can be shown using the same game and perturbations just described.

More generally, an extension of the previous idea applies to any type that exhibits common knowledge of some proper, closed subset of kth-order beliefs. A significant complication arises at this level, however. With finite types, by constructing the game with payoffs that depend in the appropriate way on Ω, we were able to induce types with special first-order beliefs to play differently and start the infection argument. But for $k > 1$, payoffs in a game do not depend directly on $k-1$st-order beliefs so it is not immediate how to initiate the induction. In this case, our construction involves two main steps. First, we construct a game in which *play* depends finely on kth-order beliefs. The construction of this game is built upon the result of Dekel, Fudenberg, and Morris (2006) showing that any pair of types with distinct higher-order beliefs have distinct rationalizable behavior in some game. Next, based on the structure of that game, we are able to construct a new coordination game on which to base the infection argument.[5]

2.3 Common p-belief and Common Prior Types

In fact, a further extension of this argument works to show that any type with common p-belief, for any $p > 0$, of any non-trivial closed subset of *belief hierarchies* (not just subsets of beliefs at some given order k) is critical. We apply this result to common prior type spaces. The result also extends to games with any number of players. In the n-player case critical types are characterized by a form of common p-belief that is weaker than the standard version.

[5]Although the construction of games that exhibit criticality may sound complicated, such games are not at all uncommon. Actions that depend on kth-order beliefs take the form of higher-order bets, where a first-order bet is a bet on the states, a second-order bet is a bet on the states and the first-order bets, and so on .... . Games involving first-order bets model trade in assets whose returns are linked to fundamentals. Games involving higher-order bets model derivative assets. Finally, as in the Rubinstein's example, the underlying coordination game can be interpreted as an investment game, where the payoff depends on the investment decisions of other players as well as the players' decisions concerning higher-order bets. We are grateful to an anonymous referee for this interpretation.

A sufficiently rich common prior type space will not have common knowledge or even common belief in any proper subset of finite-order beliefs. Therefore, to apply the characterization, we prove a new result showing that any common prior must attach probability 1 to a set of types that have a non-trivial common belief. The proof is based on an extension of (one half of) the Kajii and Morris (1997) critical path lemma that states that events with high ex-ante probability under a common prior must also be common belief with high probability.

Thus, in any common prior type space, critical types have probability 1. We contrast this result with the result on common prior types in Weinstein and Yildiz (2007).

2.4 Regular Types

We show that the non-trivial common p-belief is not only sufficient but also necessary for a type to be critical. The idea is to show that for each game G, there exists a non-trivial closed subset of hierarchies W and $p > 0$ such that if a type does not exhibit a common p-belief in W, then it is G-*regular*, i.e., it has a continuous rationalizable behavior in game G. The exact statement of this fact is contained in Lemma 5. Here, we present an intuition using the game from Figure 2.

Let $U = (1/4, 3/4)$ be an open interval of the first-order beliefs that the state is equal to $\omega = +1$ and let W be the closed set of player 2's hierarchies that do not have their first-order beliefs in U. Consider the types of player 2 with their first-order beliefs in set U. Denote such types as $E_2^{U,0}$. Each such type has a dominant action $\varnothing$. Moreover, such types have a continuous rationalizable behavior: any type obtained by a small perturbations of beliefs, i.e., any type with the first-order beliefs in a small neighborhood of W has $\varnothing$ as a dominant and, as the uniquely rationalizable action. Thus, all such types are G-regular.

Next, consider the types of player 1 that assign probability at least $1 - p$ to the types of player 2 in set $E_2^{U,0}$ for some $p < \frac{1}{2}$. Denote the set of such types as $E_1^{U,1}$. Such types believe with probabiltiy $1 - p$ that their opponent's unique rationalizable action is $\varnothing$. Because $p < \frac{1}{2}$, the only rationalizable action of such types is equal to D. Moreover, all types that have believes sufficiently close to the types in set $E_1^{U,1}$ assign a probability arbitrarily close to $1 - p$ to the opponent's unique rationalizable action being $\varnothing$, which implies that their unique rationalizable action is also equal to D. It follows that all types in set $E_1^{U,1}$ are G-regular.

Next, consider the types of player 2 that assign a probability at least $1 - p$ to the types of player 1 being in set $E_1^{U,1}$. Denote such types as $S_2^{U,2}$ and let $E_2^{U,2} = E_2^{U,0} \cup S_2^{U,2}$. All types in set $E_2^{U,2}$ have a unique rationalizable action equal to $\varnothing$: either because their first-order beliefs lie in the interval U, or because they assign probability at least $1 - p$ to the opponent's types with uniquely rationalizable action D. Moreover, because the unique rationalizable action remains the same after small perturbations of beliefs, all types in set $E_2^{U,2}$ have continuous rationalizable behavior, i.e, they are G-regular. More generally, define the set of types $E_i^{U,k}$ of player i that either belong to set $E_i^{U,k-2}$ or that assign probability at least $1-p$ to the opponent's types being in set $E_{-i}^{U,k-1}$. All such types are G-regular.

Finally, consider the union $S_1 = \bigcup_{k=0,1,\ldots} E_1^{U,2k+1}$ of sets of types of player 1. Such a union consists of player 1's types that assign the probability at least $1-p$ to the union $\bigcup_{k=0,1,\ldots} E_2^{U,2k}$, i.e., to the fact that either player 2's types do not belong to set W *or* that player 2's types assign probability at least $1-p$ to set S_1. Thus, S_1 is equal to the complement of the set of types that assign probability at least p to the fact that player 2's types belong to W *and* that player 2 assigns the probability at least p to the fact that player 1's types belong to S_1. In other words, S_1 is equal to the complement of the common p-belief in W. Because S_1 consists of G-regular types, this establishes our claim for game from Figure 2.

Our goal is to characterize the regular types, i.e., the types that are G-regular across all games G. Because the choice of W and p will depend on game G, a sufficient condition for a type to be regular is that has no common p-belief in any non-trivial closed set of hierarchies for any $p > 0$. But this completes our characterization of the cirtical (i.e., non-regular) types.

2.5 Genericity of Regular Types

While critical types are pervasive in applications, such types are, in a formal sense, very rare: their complement forms a residual subset of the universal type space relative to the natural product topology on higher-order beliefs. The regular types are the typical ones. Nevertheless, they are in a certain sense elusive: actually describing a regular type is a serious challenge in its own right. It is thus not surprising that they do not appear in applied analysis. Indeed, without the simplifying tools of either finite or common-prior type spaces to implicitly describe hierarchies, we are not well-equipped to describe them at all. In an online appendix we provide a constructive description of a regular hierarchy via a type space.

3 Model

If X is a measurable space, then ΔX refers to the set of probability measures on X. We assume that ΔX has a σ-algebra generated by sets $\left\{\mu \in \Delta X : \int f(x)\, d\mu(x) \leq a\right\}$ for some $a \geq 0$ and measurable function $f : X \to R$. When X is a topological space, we treat X as a measurable space equipped with the Borel σ-algebra and we assume that ΔX is equipped with weak$*$ topology. If $f : X \to Y$ is a measurable mapping between two measurable spaces, then we write $\Delta f : \Delta X \to \Delta Y$ for the induced mapping between the corresponding spaces of measures.[6]

We consider N-player games with incomplete information. For each player i, we use the standard notation $-i$ to denote all players $j \neq i$. We fix throughout a space of basic uncertainty (or states of nature) Ω. In a game with incomplete information, payoffs depend on action choices as well as the realization of Ω. We assume that Ω is a compact metrizable space with at least two elements.[7]

The players' uncertainty is modeled by a Harsanyi type space over Ω. A type space over Ω, denoted $T = (T_i, \mu_i)_{i \leq N}$, consists of a pair of measurable spaces T_i and two measurable belief mappings $\mu_i : T_i \to \Delta(\Omega \times T_{-i})$, where $T_{-i} = \times_{j \neq i} T_j$. The probability measure $\mu_i(t_i) \in \Delta(\Omega \times T_{-i})$ indicates the belief of type t_i about the basic uncertainty and the type of the opponent.

A *game* over Ω is a tuple $G = (A_i, g_i)_{i \leq N}$, where for each i, A_i is a finite set of actions and $g_i : A_i \times A_{-i} \times \Omega \to \mathbf{R}$ is a continuous payoff function. We extend the payoff functions to lotteries $g_i : \Delta A_i \times \Delta A_{-i} \times \Omega \to \mathbf{R}$ in the usual way.

3.1 Interim (Correlated) Rationalizability

We base our analysis on the concept of interim correlated rationalizability first introduced by Dekel, Fudenberg, and Morris (2007). Fix a type space T, and a game $G = (A_i, g_i)$. An *assessment* is a profile of measurable subsets $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \subseteq T_i \times A_i$. Alternatively, an assessment can be defined by the profile of correspondences $\alpha_i : T_i \rightrightarrows A_i$, with $\alpha_i(t_i) := \{a_i : (t_i, a_i) \in \alpha_i\}$. The image $\alpha_i(t_i)$ is interpreted as the set of actions that player i of type t_i could conceivably play. For each player i, each profile $t_{-i} \in T_{-i}$, let $\alpha_{-i}(t_{-i}) = \times_{j \neq i} \alpha_j(t_j)$.

A player *i's conjecture* is a measurable mapping $\sigma_{-i} : \Omega \times T_{-i} \to \Delta A_{-i}$.[8] The expected

[6]Formally, for any measure $\mu \in \Delta X$ and any measurable set $E \subseteq Y$, let $((\Delta f)\,\mu)(E) = \mu\left(f^{-1}(E)\right)$.

[7]This ensures that there is non-trivial incomplete information and there exists more than one (in fact infinitely many) hierarchies of belief.

[8]With such a conjecture, player i entertains the possibility that his opponents correlate their play with the play of the others, the types of the others, and the state of the world. This is an important feature of interim correlated rationalizability, and this feature plays a key role in Lemma 1 below, a starting point for our analysis. See Dekel, Fudenberg, and Morris (2007) and Ely and Pęski (2006) for elaboration of these points.

payoff to type t_i of player i from choosing action a_i against conjecture σ_{-i} is given by[9]

$$\pi_i\left(a_i,\sigma_{-i}|t_i\right) = \int_{\Omega\times T_{-i}} g_i\left(a_i,\sigma_{-i}\left(\omega,t_{-i}\right),\omega\right)\, d\mu_i\left(\omega,t_{-i}|t_i\right). \tag{3.1}$$

A conjecture is a *selection from assessment* α if for each profile of types $t_{-i}\in T_{-i}$, each state $\omega\in\Omega$,

$$\sigma_{-i}\left(\omega,t_{-i}\right)\left(\alpha_{-i}\left(t_{-i}\right)\right) = 1.$$

Let $\Sigma_i(\alpha)$ be the set of all conjectures for i that are selections from α.

For any $\varepsilon\geq 0$, an action a_i is an interim $\varepsilon-$best-response for t_i against σ_{-i} if $\pi_i(a_i,\sigma_{-i}|t_i)\geq \pi_i(a_i',\sigma_{-i},|t_i)-\varepsilon$ for all $a_i'\in A_{-i}$. Let $B_i(\sigma_{-i}|t_i;\varepsilon)$ denote the set of all interim $\varepsilon-$best-responses for t_i to σ_{-i}. If α is an assessment, then $B_i(\alpha|t_i;\varepsilon)$ is the set of all $\varepsilon-$best-responses to conjectures in $\Sigma_i(\alpha)$. Finally, $B_i(\alpha\mid\varepsilon)$ is the assessment given by the graph of the best-response correspondence $B_i(\alpha_{-i}|\cdot;\varepsilon)$:

$$(t_i,a_i)\in B_i(\alpha\mid\varepsilon) \text{ iff } a_i\in B_i(\alpha_{-i}|t_i;\varepsilon).$$

If we set $R_i^0(\cdot\mid G,\varepsilon)=T_i\times A_i$, and for natural numbers m, iteratively define assessments $R^m(\cdot\mid G,\varepsilon)$ by

$$R_i^m(\cdot\mid G,\varepsilon) = B_i\left(R^{m-1}(\cdot\mid G,\varepsilon)\mid\varepsilon\right),$$

then $R_i^m(t_i\mid G,\varepsilon)$ is the set of actions for type t_i that survive m rounds of elimination of never-best-replies.

An assessment α has the $\varepsilon-$*best-response property* if every action attributed to player i is an interim ε-best-reply to some selection from α_{-i}, i.e.,

$$\alpha_i\subseteq\left\{(t_i,a_i): a_i\in B_i\left(\alpha_{-i}|t_i;\varepsilon\right)\right\}$$

If the above is satisfied with equality, then we say that α has the ε fixed-point property.

Proposition 1. *For every $\varepsilon\geq 0$, there exists a maximal (with respect to set inclusion) assessment $R(\cdot\mid G,\varepsilon)$ with the best-reply property. The assessment $R(\cdot\mid G,\varepsilon)$ has the fixed-point property and is equal to the assessment obtained by iterative elimination of never-best-replies:*

$$R(\cdot\mid G,\varepsilon) = \bigcap_m R^m(\cdot\mid G,\varepsilon).$$

[9] In order to avoid to many paranteses, we write $\mu_i\left(E|t_i\right)$ to denote the mass $\mu_i\left(t_i\right)(E)$ of measurable set $E\subseteq\Omega\times T_{-i}$ with respect to measure $\mu_i\left(t_i\right)$.

The proof of the proposition follows standard ideas. Dekel, Fudenberg, and Morris (2007) have shown it for the case of $\varepsilon = 0$ and the identical proof works for arbitrary $\varepsilon \geq 0$. The assessment $R(\cdot \mid G,\varepsilon)$ is the interim correlated ε-rationalizable correspondence, and we say that a_i is interim $\varepsilon-$rationalizable for type t_i if $a_i \in R_i(t_i|G,\varepsilon)$.

We refer to $\Sigma_i(R(t_i|G,\varepsilon))$ as the set of *ε-rationalizable conjectures* for i. An action is ε-rationalizable for some type if and only if the action is a best-response to an ε-rationalizable conjecture.

3.2 The Universal Type Space

A type space is an implicit description of a player's higher-order beliefs. Our characterization of critical types will be in terms of their hierarchies of beliefs, explicitly described. This ensures that our classification does not depend on any particular choice of type space.[10]

Throughout, we work with the Mertens and Zamir (1985) universal type space, which we denote $U(\Omega) = (U_i(\Omega), \mu_i)_{i=1}^{N}$. Here we briefly review the definition and emphasize only the properties that will be important for our results. For additional details, the reader can consult Mertens and Zamir (1985) or Brandenburger and Dekel (1993).

The set $U_i(\Omega)$ is taken to be the set of all *coherent* sequences of finite hierarchies of belief over the space Ω. If $U_i^k(\Omega)$ is the set of hierarchies up to order k, then $U_i(\Omega)$ is the set of coherent sequences from $\prod_{k=1}^{\infty} U_i^k(\Omega)$. Each element $u_i \in U_i(\Omega)$ is uniquely associated with a probability measure in $\Delta(\Omega \times U_{-i}(\Omega))$ by the belief mapping μ_i.

The *product topology* on $U_i(\Omega)$ is the Tychonoff topology inherited from the infinite product $\prod_{k=1}^{\infty} U_i^k(\Omega)$. Throughout the paper, we write $u_i^n \to u_i$ to denote convergence in the product topology. By standard results, this topology is compact metrizable. When the space of beliefs $\Delta(\Omega \times U_{-i}(\Omega))$ is endowed with the topology of weak* convergence, then the belief mapping is a homeomorphism. In particular, convergence of hierarchies $u_i^n \to u_i$ is equivalent to weak* convergence of the associated beliefs.

In light of the homeomorphism, to ease notation, we use the symbol u_i interchangeably to refer to either the hierarchy (i.e., the element of $U_i(\Omega)$) or the belief (i.e., the associated element $\mu_i(u_i) \in \Delta(\Omega \times U_{-i}(\Omega))$. Also, whenever no confusion results, we use the same symbol u_i to refer to marginal probabilities over $U_{-i}(\Omega)$. For example, if E is a measur-

[10] Friedenberg and Meier (2009) make the important observation that working with large type spaces as we do here may not be without loss of generality as usually supposed. The problem they identify is specific to the Bayesian Nash equilibrium solution concept and does not arise for rationalizability, the solution concept we study.

able subset of $U_{-i}(\Omega)$, then instead of writing $u_i(\Omega \times E)$, we simply write $u_i(E)$ for the probability of the set E.

Each type $t_i \in T_i$ from any type space T can be associated with its hierarchy of beliefs through the Mertens-Zamir type-morphism $\phi_i^T : T_i \to U_i(\Omega)$.[11] Dekel, Fudenberg, and Morris (2007) showed that (interim correlated) rationalizable behavior of any type is determined by this type's hierarchy.

Lemma 1 (Dekel, Fudenberg, and Morris (2007))**.** *For any type space $T = (T_i, \mu_i)$, any player i and type $t_i \in T_i$, any game G, any $\varepsilon > 0$*

$$R_i(t_i|G,\varepsilon) = R_i\left(\phi_i^T(t_i)\,|G,\varepsilon\right).$$

Moreover, the rationalizable correspondence $R_i(\cdot|G,\varepsilon) : U_i(\Omega) \rightrightarrows A_i$ on the universal type space is upper hemi-continuous.

The rationalizable correspondence on the right-hand side is defined over the universal type space. Lemma 1 allows us to consider R_i as a correspondence defined directly on $U_i(\Omega)$, i.e., $R_i(u_i|G,\varepsilon)$ is the set of ε-rationalizable actions for any type t_i whose hierarchy is u_i, independently of the type space to which t_i belongs. We note that for any finite m, the statement of the theorem also holds for the correspondences $R_i^m(\cdot \mid G,\varepsilon)$ of actions that survive m rounds of elimination of never-best-replies; see Lemmas 1 and 2 of Dekel, Fudenberg, and Morris (2007).

3.3 Continuity of Behavior

The product topology provides a concept of similarity of types according to their (exogenous) description, i.e., their beliefs. We are interested in types for whom similarity in beliefs corresponds to similarity in behavior. We will use the concept of *strategic convergence* introduced by Dekel, Fudenberg, and Morris (2006) to capture similarity in behavior. For any sequence of hierarchies u_i^n, any hierarchy u_i, and any game G, say that a sequence u_i^n *G-converges to* u_i, if for any action a_i, the following two statements are equivalent:

1. a_i is rationalizable for hierarchy u_i,

[11] The Mertens Zamir type morphism is uniquely defined as the profile of measurable mappings $\phi_i^T : T_i \to U_i(\Omega)$ such that for each player i, each type $t_i \in T_i$,

$$\mu_i\left(\phi_i^T(t_i)\right) = \Delta\left(\mathrm{id}_\Omega \times \phi_{-i}^T\right)\mu_i(t_i),$$

and $\mathrm{id}_\Omega \times \phi_{-i}^T : \Omega \times T_{-i} \to \Omega \times U_{-i}(\Omega)$ is defined in the natural way.

2. for each $\varepsilon > 0$, there exists N_ε such that for all $n \geq N_\varepsilon$, a_i is ε-rationalizable for u_i^n.

Say that sequence u_i^n *converges to* u_i *in the sense of strategic convergence*, written $u_i^n \to^{ST} u_i$, if it G-converges for each game G.[12]

Convergence in the strategic topology captures a notion of continuity of rationalizable behavior. The implication from 2 to 1 is a form of upper hemi-continuity. It requires that actions that are approximately rationalizable for approaching types are rationalizable in the limit. As we note in Lemma 1, Dekel, Fudenberg, and Morris (2006) show that this form of upper-hemicontinuity is guaranteed for sequences that converge in the product topology.

The implication from 1 to 2 is a form of lower hemi-continuity: actions that are rationalizable for some type should be approximately rationalizable for approaching types. Lower hemi-continuity of (approximate) rationalizable behavior is a stronger requirement than convergence in the product topology, and this was illustrated with the electronic mail game. There, a type with complete information could rationalize an action that was not approximately rationalizable for types approaching it in the product topology.[13]

4 Critical Types

4.1 Regular and Critical Hierarchies

We are now in a position to formalize our notion of a critical type as one for which similarity of higher-order beliefs is not enough to ensure similarity of behavior. With these notions of similarity defined, respectively, in terms of convergence in the product and strategic topologies, we have the following definition.

Definition 1. *We say that hierarchy* $u_i \in U_i(\Omega)$ *is*

1. G-regular *for a given game* G *if for any sequence* u_i^n*, the convergence in the product topology* $u_i^n \to u_i$*, implies that* u_i^n *G-converges to* u_i*,*

[12] Strategic convergence is defined only for sequences (and not more generally, nets). Specifying a set of convergent sequences is in general not sufficient to define a topology. In Ely and Pęski (2009), we formally define the coarsest topology on the universal type space with these convergent sequences. This *strategic topology* turns out to be metrizable and equivalent to the metric topology considered in Dekel, Fudenberg, and Morris (2006).

[13] It is important that the strategic topology is defined using ε-rationalizability. Rationalizability, like best-replies, allow for indifferences. These indifferences can always be broken by small perturbations in beliefs and higher-order beliefs. Thus, an alternative definition that substitutes exact rationality into 2 would yield no convergent sequences and (looking ahead) all types would be critical types. This is analogous to the statement that, for example, the Nash correspondence is not lower hemi-continuous, although the ε-Nash correspondence is. See Fudenberg and Levine (1986). Using ε-rationalizability gives the strategic topology the same convergence properties. See Dekel, Fudenberg, and Morris (2006) for further discussion.

2. regular *if it is G-regular for each game G, i.e., for any sequence u_i^n, the convergence in the product topology $u_i^n \to u_i$, implies strategic convergence, $u_i^n \to^{ST} u_i$,*

3. critical *(G-critical), if it is not regular (G-regular).*

Appealing to Lemma 1, we are going to say that a type in some type space is critical (or regular) if its hierarchy of beliefs is critical (or regular). Thus, for example, the complete-information types are G-critical when G is the electronic-mail game. The types in the Rubinstein sequence, while G-regular for the electronic-mail game,[14] are nevertheless critical because, for example, they are G-critical for the game in Figure 2.

Now the task is to identify the property of a type's belief hierarchy that determines whether the type is regular or critical. That property turns out to be a version of common p-belief due to Monderer and Samet (1989).

4.2 Common Belief

Monderer and Samet (1989) introduced the concept of common p-belief. Monderer and Samet (1997) discuss a related concept of common repeated p-belief. We will use a weaker form of the latter concept in our characterization (our definition coincides with Monderer and Samet (1997) in the case of $N = 2$). If we have measurable sets $W_j \subseteq U_j(\Omega)$ for each player j, then the set $W = \times_j W_j \subseteq \times_j U_j(\Omega)$ is called a *product event.* We say that a hierarchy u_i exhibits *(weak) common p-belief* in a product event W if $u_i \in W_i$, and u_i assigns probability at least p that the hierarchy of *at least one* player $j \neq i$ belongs to W_j and j assigns probability at least p to that the hierarchy of *at least one* player $j' \in j$ belongs to $W_{j'}$, and so on. Formally, for each player i, define

$$B_i^p(W) = W_i \cap \left\{ u_i \in U_i(\Omega) : u_i\left(\left\{u_{-i} : \exists_{j \neq i} u_j \in W_j\right\}\right) \geq p \right\}.$$

Thus, $B_i^p(W)$ is the set of hierarchies of player i that belong to W_i and that assign probability at least p that a hierarchy of at least one opponent $j \neq i$ belongs to W_j. It is sometimes convenient to write $V_j = U_j(\Omega) \setminus W_j$ and $V_{-i} = \times_{j \neq i} V_j$ and use the equivalent expression

$$B_i^p(W) = W_i \cap \left\{u_i \in U_i(\Omega) : u_i\left(V_{-i}\right) < 1 - p\right\}.$$

[14] This follows immediately because these types have a unique rationalizable action in the electronic mail game.

For any product event $W = \times_i W_i$, define the product event

$$B^p(W) = \prod_i B_i^p(W) \subseteq U(\Omega).$$

Note that $B^p(W) \subseteq W$. By the measurability of the belief-mapping on the universal type space (indeed it is a homeomorphism), the sets $B_i^p(W)$ and $B^p(W)$ are measurable. To refer to iterations of the belief operator, write $[B^p]^2(W) = B^p(B^p(W))$, and $[B^p]^k(W) = B^p([B^p]^{k-1}(W))$. Define *(weak) common p-belief* in W as[15]

$$C^p(W) = \bigcap_{k \geq 1} [B^p]^k (W).$$

Note that

$$\begin{aligned} C^p(W) &= \bigcap_{k \geq 0} \prod_i B_i^p [B^p]^k (W) \\ &= \prod_i \bigcap_{k \geq 0} B_i^p [B^p]^k (W) \end{aligned}$$

so that $C^p(W)$ is a product event with $C_i^p(W) = \bigcap_{k \geq 0} B_i^p [B^p]^k (W)$. In addition, we have the following version of the original fixed-point characterization due to Monderer and Samet (1989) (the proof can be found in Appendix A).

Lemma 2. *Let $W \subseteq U(\Omega)$ be a product event.*

$$C_i^p(W) = B_i^p (C^p(W)).$$

4.3 Characterization of Critical Types

The main result of the paper characterizes critical hierarchies. Say that product event $W = \times W_j$ is *closed*, if $W_j \subseteq U_j(\Omega)$ is closed in the product topology for each j. Say that W is *proper*, if $W_j \subset U_j(\Omega)$ (strict inclusion) is a proper subset for each j.

Theorem 1. *A hierarchy $u_i \in U_i(\Omega)$ is critical if and only if there exists $p > 0$ and a*

[15] The original Monderer and Samet (1989) concept would be defined analogously but with $B_i^p(W) = \{u_i : u_i(W_{-i}) \geq p\}$. In the common repeated p-belief from Monderer and Samet (1997), we would take $B_i^p(W) = W_i \cap \{u_i : u_i(W_{-i}) \geq p\}$. See Morris (1999) for discussion of various definitions of common belief.

proper and closed product event W such that

$$u_i \in C_i^p(W).$$

Thus, the critical types are all of those with any non-trivial common belief. The second result says that the regular types are generic in the standard sense of residual sets. Recall that the subset of topological space is residual if it is a countable intersection of dense and open sets. By the Baire Category Theorem, residual subsets of complete metrizable spaces are dense.

Theorem 2. *The set of regular hierarchies forms a residual subset (in the product topology) of $U_i(\Omega)$.*

We compare Theorem 2 with a result from Weinstein and Yildiz (2007). In that paper, the authors analyzed finite games with rich payoffs and showed that for any fixed game the set of types with a unique and robust rationalizable action is open and dense. In our terminology, it means that the set of G-regular types is an open and dense subset of the universal type space. The next result shows that the rich payoff assumption can be dispensed with for this generic robustness result. The observation is a simple consequence of the upper hemi-continuity of the rationalizable correspondence.[16]

Theorem 3. *For each game G, the set of G-regular hierarchies contains an open and dense subset of the space of hierarchies $U_i(\Omega)$.*

4.4 Preliminary Results

Our characterization of critical types is based on the following lemmas. The proofs of the results can be found in Appendix B.

Lemma 3. *Let V be any product event, and let $p > 0$. The complement of $C_i^p(V)$ is dense in the product topology. If V is closed, then the complement of $C_i^p(V)$ is open for each i.*

Thus, given any type u_i, there is a sequence of types that do not have common p-belief in V but whose higher-order beliefs converge to those of u_i. In fact, our proof of Lemma 3 shows that for every k, there is a hierarchy in the complement of $C_i^p(V)$ that is *identical* to u_i up to order k.

[16] The rich payoff structure is clearly necessary for the generic *uniqueness*. For example, consider a game G in which the payoffs do not depend on $\omega \in \Omega$. That game is effectively a complete-information game, and all types will have the same sets of rationalizable actions. If the complete-information game has multiple rationalizable actions, then generic uniqueness clearly fails.

Lemma 4. *For any proper and closed product event W, any $p > 0$ and any small enough $\varepsilon > 0$, there exists a proper product event $V \supseteq W$ and a game G, which for each player i has a subset of actions ζ_i such that*

1. *For each $u_i \in C_i^p(W)$, there is an element $a_i \in \zeta_i$ that is interim-rationalizable for u_i.*
2. *No element of ζ_i is interim-ε-rationalizable for any $u_i \notin C_i^{p-2\varepsilon(1-p)}(V)$.*

This lemma says that we can always find a game with a set of actions whose rationalizability hinges on whether there is common p-belief in certain events. There is a simple intuition behind the lemma. We construct a coordination game with a pair of actions (a_i, a_{-i}) such that player i plays a only if her hierarchy of beliefs belongs to V and she believes with probability at least p that the opponent plays a_{-i}. On the other hand, players $-i$ play a_{-i} only if they believe with probability at least p that player i plays a_i. The difficulty in the proof is finding the right set V and constructing the game so that types in V have distinctive rationalizable behavior.

We use Lemma 3 and Lemma 4 to show that a non-trivial common belief is a sufficient condition for a type to be critical. The fact that the condition is also necessary follows from the last result.

Lemma 5. *For any game G, there is a proper and closed product event W such that if $u_i \notin C_i^p(W)$ for all $p > 0$ then u_i is G-regular.*

The proof of Lemma 5 begins by showing that for every game G there is an open non-empty set of hierarchies V_j that are G-regular. We define the set W to be the complement of $V = \prod V_j$. Suppose $u_i \notin B_i^p(W)$. This means either that $u_i \notin W_i$ or u_i believes with at least $1 - p$ that each opponent j's hierarchy belongs to the complement of W_j. The first case is equivalent to $u_i \in V_i$ which implies that u_i is G-regular. Consider the second case. Any type u_i' close to u_i in the product topology will assign close to $1 - p$ probability to the G-regular types V_{-i}. We use this to show that any rationalizable conjecture σ_i for u_i can be approximated by an approximately rationalizable conjecture for u_i' that coincides with σ_i on the set V_{-i}, i.e., with probability at least $1 - p$. If a_i was a best-reply to σ_i then a_i will be a ε-best-reply for u_i', where ε is proportional to p. An inductive argument then extends to types $u_i \notin B_i(B^p(W))$, etc. Finally, quantifying over all $p > 0$ completes the proof that these types are G-regular.

4.5 Proof of Theorem 1

Suppose that $u_i \in C_i^p(W)$ for proper and closed product event W. Let $\varepsilon = \frac{p}{4(1-p)}$. Then, by Lemma 4, there exist $V_j \subset U_j(\Omega)$ for each j, $\varepsilon > 0$, and a game G with an action a_i, such that

$$a_i \in R_i(u_i|G, 0)$$

but

$$a_i \notin R_i(v_i|G, \varepsilon)$$

for any $v_i \notin C_i^{p-2\varepsilon(1-p)}(V)$. By Lemma 3, there is a sequence of hierarchies $u_i^n \to u_i$ such that $u_i^n \notin C_i^{p-2\varepsilon(1-p)}(V)$ for all n, and therefore $a_i \notin R_i\left(u_i^n|G, \varepsilon\right)$ for all n. Thus, u_i is critical because the sequence converges in the product topology to u_i but does not G-converge to u_i.

Now, suppose that $u_i \notin C_i^p(W)$ for each proper and closed product event W and for each $p > 0$. By Lemma 5, u_i is G-regular for any game G; hence, u_i is regular.

4.6 Proof of Theorem 2

For any $p > 0$ and any proper and closed product event W, the set $U_i(\Omega) \backslash C_i^p(W)$ is open and dense by Lemma 3. Notice also that if $p \leq p'$, W' is a product event and $W \subseteq W'$, then $C_i^p(W) \subseteq C_i^{p'}(W')$.

Find a sequence of proper and closed product events $W^1, W^2, \ldots \subset U(\Omega)$ such that for any proper and closed product event W, there is n, such that $W \subseteq W^n$. Such a sequence exists, since the space $U(\Omega)$ is separable and metrizable; hence, it has a countable basis.

Let $\mathcal{W}$ be the collection of all closed and proper product events. The set of regular hierarchies of player i is equal to

$$\begin{aligned} &\bigcap_{p>0, W\in\mathcal{W}} U_i(\Omega) \backslash C_i^p(W) \\ &= \bigcap_n \bigcap_m U_i(\Omega) \backslash C_i^{1/n}(W^m) \end{aligned}$$

and is therefore residual as an intersection of a countable family of open and dense sets.

4.7 Proof of Theorem 3

Fix game G. For each player i and action a_i, let

$$U_i(a_i) = \{u_i : a_i \in R(u_i|G, 0)\}$$

be the set of hierarchies of player i for which action a_i is rationalizable. By the upper hemi-continuity of the rationalizable correspondence, $U_i(a_i)$ is closed. Let $\operatorname{int} U_i(a_i)$ be the interior, and let $\operatorname{bd} U_i(a_i) = U_i(a_i) \setminus \operatorname{int} U_i(a_i)$ be the boundary of $U_i(a_i)$. Notice that each $\operatorname{bd} U_i(a_i)$ is nowhere dense and closed. Define set

$$Bd = \bigcup_{a_i} \operatorname{bd} U_i(a_i).$$

Set Bd is closed and nowhere dense as a finite union of nowhere dense and closed sets.

For each subset of actions $B_i \subseteq A_i$, let

$$V(B_i) = \left(\bigcap_{a_i \in B_i} \operatorname{int} U_i(a_i)\right) \setminus \left(\bigcup_{a_i \notin B_i} U_i(a_i)\right).$$

Each $V(B_i)$ is an open set. Moreover, each hierarchy $u_i \in V(B_i)$ is G-regular. If not, then there is an action $a_i \in B_i$ that is not rationalizable along some sequence $u_i^n \to u_i$. But this is impossible, because $V(B_i)$ is open, and there must be $u_i^n \in V(B_i)$ for all sufficiently large n.

Finally, notice that

$$U_i(\Omega) \setminus Bd = \bigcup_{B_i} V(B_i)$$

is open and dense.

5 Special Cases

In this section, we apply Theorem 1 to show that types in the most commonly used type spaces are critical.

5.1 Finite Type Spaces

Take any type space such that there is a player with finitely many types. The next result shows that all types of all players in such a type space are critical.

Theorem 4. *Take any type space* $T = (T_i, \mu_i)$ *such that* $\left|T_{i_0}\right| < \infty$ *for at least one player* i_0*. Then, for each player* i*, each type* $t_i \in T_i$ *is critical.*

Proof. Let $\phi_i^T : T_i \to U_i(\Omega)$ be the Mertens-Zamir homeomorphism. Define $W_{i_0} = \left\{\phi_{i_0}^T\left(t_{i_0}\right) : t_{i_0} \in T_{i_0}\right\}$. Then, $\left|W_{i_0}\right| < \infty$. For each $i \neq i_0$, let $W_i = U_i(\Omega)$. Then, $W = \times_i W_i$ is a proper and closed product event such that for each player i, each type t_i, $\phi_i^T(t_i) \in C_i^1(W)$. The result follows from Theorem 1. □

5.2 Common Prior Type Spaces

We will show that almost all the types from type spaces with a common prior are critical. Let $T = (T_i, \mu_i)$ be a type space. Say that $\psi \in \Delta(T)$ is a *common prior* on T if for any bounded measurable function $f : T \to \mathbf{R}$ and any player i, the law of total probability holds:

$$\mathbf{E}_\psi f = \int_{T_i} \mathbf{E}_{\mu_i(t_i)} f \, d\psi_i(t_i).$$

where $\psi_i(t_i)$ is the marginal of ψ on T_i. Note that this definition is weaker than the standard one because it imposes no restriction on beliefs about Ω. Thus, any type space that admits a common prior according to the standard definition also admits a common prior according to this one. Of course, not every type space admits a common prior in our sense. However, if T is a type space with common prior ψ, then there corresponds a common prior ψ^* on the universal type space $U(\Omega)$, obtained using the Mertens-Zamir mappings $\phi_i^T : T_i \to U_i(\Omega)$. The measure $\psi^* \in \Delta U(\Omega)$ is defined by setting

$$\psi^* = \Delta \prod_i \phi_i^T(\psi).$$

We show that every common prior attaches probability 1 to critical types.

Theorem 5. *Suppose that* ψ *is a common prior on a type space* $T = (T_i, \mu_i)$*. Then, for each player* i*, under* ψ_i*, almost every type is critical.*

The "almost every" quantifier in the Theorem cannot be avoided. To see why, recall that on general (uncountable) common prior type spaces, the conditional beliefs are determined only up to a set of types of probability 0. One can modify the interim beliefs on a zero-probability set of types by making them, for instance, regular, without violating the common prior assumption.

Weinstein and Yildiz (2007) argue that common prior types generically exhibit robust rationalizable behavior. There are two differences between our statement and theirs. First, Weinstein and Yildiz (2007) fix game G and focus on G-regular types. This first difference is less important than it appears.

For example, suppose that the space of basic uncertainty Ω is finite, and we consider finite action spaces. The set of games is isomorphic to the set of payoff functions over actions and Ω, and there is a countable dense set Γ of these. For each of the games G in Γ, there is an open and dense set of common-prior types with robust behavior in G. Thus, as a consequence of the results in Weinstein and Yildiz (2007), the set of types whose behavior is robust in *all* of the games in Γ is, in a topological sense, generic among common-prior types: a countable intersection of open and dense subsets.

Second, they show that the set of G-regular types contains an open and dense subset of the space of common prior types (more precisely, types from some finite common prior type spaces). Weinstein and Yildiz (2007) rely on a result due to Lipman (2003) that common prior types are dense *in the universal type space.* In a similar way, one shows that any type is close to a G-regular type, which in turn is close to a common prior type. If the latter distance is sufficiently small, the common prior type is also G-regular as well. It may happen that the G-critical type constructed in such a way has arbitrarily small common prior probability in the type space to which the type belongs.

On the other hand, our results imply that the common prior itself attaches probability zero to the set of regular types (i.e., types that are G-regular for each G). In applications, common priors are modeled using type *spaces,* not individual types, the negative result would seem to carry the more important message for applied work.[17]

The proof relies on the following lemma, which is an extension of (one half of) the critical path lemma due to Kajii and Morris (1997).

Lemma 6. *Let ψ^* be a common prior on the universal type space $U(\Omega)$. For any product event $V = \times_i V_i \subseteq U(\Omega)$, there is a product event $S = \times_i S_i \subseteq U(\Omega)$ such that $S_i \subseteq V_i$ for each player i,*

$$\psi^*(S) \geq \frac{3}{2}\psi^*(V) - \frac{1}{2}, \tag{5.1}$$

[17] Weinstein and Yildiz (2007) rely on a result due to Lipman (2003) that common prior types are dense *in the universal type space.* As a result, the sense in which regular types *within in a given type space* are generic according to Weinstein and Yildiz (2007) is that for any type there is a regular type with a nearby hierarchy but possibly, indeed typically, that regular type belongs to a different type space with a different common prior.

and for each player i, any type $u_i \in S_i$,

$$u_i\left(S\right) \geq \frac{1}{2N}. \tag{5.2}$$

Hence, $S \subseteq C^{1/2N}\left(S\right) \subseteq C^{1/2N}\left(V\right)$.[18]

Proof. Define inductively the sets: $V_i^0 = V_i$ and

$$V_i^{k+1} = \left\{u_i \in V_i^k : u_i\left(V_{-i}^k\right) \geq \frac{1}{2N}\right\}.$$

Put

$$S_i = \bigcap_{k\geq 0} V_i^k.$$

Since the sequence of sets V_i^k is decreasing, Equation 5.2 holds for any $u_i \in S_i$, for any player i. Recall that ψ_i^* denotes the marginal prior over types of player i. Notice that

$$V^{k+1} = V^k \setminus \bigcup_{i\leq N}\left[\left(V_i^k \backslash V_i^{k+1}\right) \times V_{-i}^k\right]. \tag{5.3}$$

Consider the indicator function $\mathbf{1}_X$ for the set $X = \left(V_i^k \backslash V_i^{k+1}\right) \times V_{-i}^k$. By the definition of a common prior,

$$\begin{aligned}\psi^*\left[\left(V_i^k \backslash V_i^{k+1}\right) \times V_{-i}^k\right] &= \mathbf{E}_{\psi^*}\mathbf{1}_X \\ &= \int_{U_i(\Omega)} \mathbf{E}_{u_i}\mathbf{1}_X \, d\psi_i^*(u_i) \\ &= \int_{V_i^k \backslash V_i^{k+1}} u_i(V_{-i}^k)\, d\psi_i^*(u_i)\end{aligned}$$

and since by definition of V_i^{k+1} we have $u_i(V_{-i}^k) \leq 1/2N$ for all $u_i \in V_i^k \setminus V_i^{k+1}$,

$$\psi^*\left[\left(V_i^k \backslash V_i^{k+1}\right) \times V_{-i}^k\right] \leq \frac{1}{2N}\psi_i^*\left(V_i^k \backslash V_i^{k+1}\right).$$

Hence, by Equation 5.3,

$$\psi^*\left(V^{k+1}\right) \geq \psi^*\left(V^k\right) - \frac{1}{2N}\sum_i \psi_i^*\left(V_i^k \setminus V_i^{k+1}\right).$$

[18] Notice that Equation 5.2 implies that the set S is $1/2N$-evident in the sense of Monderer and Samet (1989), and therefore $S \subset C^{1/2N}$ even in the strong sense of Monderer and Samet (1989).

By passing to the limit, we obtain

$$\psi^*(S) \geq \psi^*(V) - \frac{1}{2N}\sum_i \psi_i^*(V_i \setminus S_i). \tag{5.4}$$

On the other hand, for each player i,

$$\psi_i^*(V_i \setminus S_i) \leq 1 - \psi_i^*(S_i) \leq 1 - \psi^*(S).$$

so that

$$\frac{1}{2N}\sum_i \psi_i^*(V_i \setminus S_i) \leq \frac{1}{2}(1 - \psi^*(V)), \tag{5.5}$$

and Equation 5.1 follows from combining Equation 5.4 and Equation 5.5. □

Proof of Theorem 5. Take any $\varepsilon > 0$. Let

$$\psi^* = \Delta \prod_i \phi_i^T(\psi)$$

be the common prior on the universal type space associated with ψ, and ψ_i^* the marginal on $U_i(\Omega)$. Because the space $U_i(\Omega)$ is separable and infinite, there exists an infinite collection of open and disjoint subsets $U_i^n \subseteq U_i(\Omega)$. For at least one n_i, $\psi_i^*\left(U_i^{n_i}\right) \leq \frac{\varepsilon}{N}$ (otherwise ψ_i^* cannot be a probability measure). Then $V = \times_i \left(U_i(\Omega) \setminus U_i^n\right)$ is a proper closed product set such that $\psi^*(V) \geq 1 - \varepsilon$. Lemma 6 implies that

$$\psi^*\left(C^{1/2N}(V)\right) \geq 1 - \frac{3}{2}\varepsilon.$$

Hence, for any player i, with ψ^*-probability at least $1 - 3\varepsilon/2$, player i's hierarchy is critical. Since the latter is true for any $\epsilon > 0$, it means that ψ^*-almost all hierarchies are critical. In particular, ψ attaches probability 1 to types in T whose hierarchies are critical. □

6 Interim Independent Rationalizability

Ely and Pęski (2006) analyze the solution concept of interim independent rationalizability.[19] In that paper, we show that, with two players, the IIR actions depend on Δ-hierarchies of beliefs (hierarchies of beliefs about conditional beliefs) and that the rationalizable correspondence is upper hemi-continuous. The results have counterparts when the interim correlated

[19] An earlier version of this paper was formulated in terms of the IIR. All of the results mentioned in this section are available on the authors' websites.

rationalizability used in this paper is replaced by the IIR.

Assume that there are only two players. Define the product topology on the space $U(\Delta\Omega)$ of Δ-hierarchies, the strategic topology, and common p-belief in a similar way as in this paper. Additionally, the characterization of regularity in the IIR case requires a notion of a partial order on types. We say that $u_i \succeq u_i'$ if u_i has a weakly larger set of ε-rationalizable actions than u_i' in all games, for all $\varepsilon \geq 0$, i.e., $R_i(u_i'|G,\varepsilon) \subseteq R_i(u_i|G,\varepsilon)$. A set $V_i \subseteq U_i(\Delta\Omega)$ is called an *upper-contour set* if V includes all hierarchies that are larger than those in V under the relation $\succeq$. Formally $V = \cup_{v_i \in V}\{u_i : u_i \succeq v_i\}$.

The following counterpart to Theorem 1 can be shown. A hierarchy is critical *if and only if* it exhibits a common p-belief in an event $V = \times V_i$ hat is a product of upper-contour, closed, and proper subsets V_i of the space of Δ-hierarchies, $U_i(\Delta\Omega)$. Theorem 2 holds as well: The set of regular hierarchies is a residual subset of the space of all hierarchies.

The proofs of the results are analogous to the proofs described here. The upper-contour property of sets V_i is used in the proof of the counterpart of Lemma 4. The goal of the first part of the proof is to find a game with an action that is rationalizable if the type has a hierarchy in set V_i, but not rationalizable if the type's hierarchy is outside V_i. If V_i is not upper-contour, such a game may not exist. For example, if V_i is a *lower-contour set*, i.e., it is a complement of an upper contour set), then, by definition, any action that is rationalizable for all hierarchies in V_i must be rationalizable for all hierarchy in the space of Δ-hierarchies.

The partial order $\succeq$ is non-trivial. We illustrate it in the example from Ely and Pęski (2006), reproduced below.

	-1	$+1$
-1	0	1/4
$+1$	1/4	0

$\omega = -1$

	-1	$+1$
-1	1/4	0
$+1$	0	1/4

$\omega = +1$

Figure 3: A type space

The figure illustrates a type space over a space of basic uncertainty containing two elements, $\omega \in \{-1,+1\}$. There are two players, each with two types, also labeled $\{-1,+1\}$. The type space has a common prior, and the tables show the probabilities of various type-profile/state combinations. We can compare this type to a simpler type space in which each player has exactly one type, labeled $*$, and the common-prior attaches equal probability to the two states.

Let us first compare, for player 1 say, type $*$ with any of the types from Figure 3, say

$+1$. There is a close connection between their best-reply correspondences. For any game, take any action a played by type $*$ of player 2, and consider the set of best-replies for type $*$ of player 1. This is exactly the set of best-replies for type $+1$, to the strategy of player 2 that plays a irrespective of type. The same argument applies to ε-best-replies. Thus, any action that can be a best-reply for $*$ is also a best-reply for type $+1$. It follows that (the hierarchy represented by) $+1$ is weakly larger than (the hierarchy represented by) $*$. Indeed, the ordering is strict. As demonstrated by the example in Ely and Pęski (2006), there are games in which the set of rationalizable actions for $+1$ strictly includes the set of rationalizable actions for $*$. On the other hand, the types $+1$ and -1 have the same best-response correspondences, and therefore, their hierarchies are equivalent under the ordering.

This example illustrates properties of the order $\succeq$ that can be stated directly in terms of belief-hierarchies. First $u_i \sim u_i'$ if and only if $u_i = u_i'$. That is, two types have the same rationalizable actions in all games if and only if they have the same Δ-hierarchies. This was the main result in Ely and Pęski (2006). Second, $u_i \succeq u_i'$ only if u_i and u_i' represent the same Mertens-Zamir hierarchies of belief. This follows from a result in Dekel, Fudenberg, and Morris (2006), namely that for any two types with distinct Mertens-Zamir hierarchies there is a game in which they have mutually disjoint rationalizable sets.

Finally, although the definition of partial order is presented in terms of IIR actions, it is possible to find a primitive description, purely in terms of Δ-hierarchies. Suppose that Δ-hierarchy u_i is obtained from a type t_i and u_i' is obtained from type t_i' (the two types may lie in possibly two different type spaces). Then, $u_i \succeq u_i'$ if and only if there exists a (Mertens-Zamir) belief preserving mapping taking type t_i into type t_i'. (The argument uses the result from Ely and Pęski (2006) that shows that the space of Δ-hierarchies is isomorphic to the $\Delta\Omega$-based type space with types being equal to the descriptions of IIR behavior in all possible games.) In a sense, types that are lower in the order allow for fewer correlations between players. Using the characterization, we can show that type $*$ is minimal and types $+1$ and -1 are maximal in the partial order $\succeq$.

References

Battigalli, P., A. D. Tillio, E. Grillo, and A. Penta (2009): "Interactive Epistemology and Solution Concepts for Games with Asymmetric Information," .

Brandenburger, A., and E. Dekel (1993): "Hierarchies of Beliefs and Common Knowledge," *Journal of Economic Theory*, 59, 189–198.

CHEN, Y.-C., AND S. XIONG (2009): "The E-mail Game Phenomenon," http://www.owlnet.rice.edu/ xiong/n-regular.pdf.

DEKEL, E., D. FUDENBERG, AND S. MORRIS (2006): "Topologies on Types," *Theoretical Economics*, 1, 275–309.

——— (2007): "Interim Correlated Rationalizability," *Theoretical Economics*, 2, 15–40.

ELY, J. C., AND M. PĘSKI (2006): "Hierarchies of Belief and Interim Rationalizability," *Theoretical Economics*, 1(1), 19–65.

——— (2009): "A Characterization of the Strategic Topology," unpublished.

FRIEDENBERG, A., AND M. MEIER (2009): "The Context of the Game," http://www.public.asu.edu/ afrieden/cog-06-18-09.pdf.

FUDENBERG, D., AND D. LEVINE (1986): "Limit games and limit equilibria," *Journal of Economic Theory*, 38(2), 261–279.

KAJII, A., AND S. MORRIS (1997): "The Robustness of Equilibria to Incomplete Information," *Econometrica*, 65(6), 1283–1310.

LIPMAN, B. (2003): "Finite Order Implications of Common Priors," *Econometrica*, 71(4), 1255–1267.

MERTENS, J. F., AND S. ZAMIR (1985): "Formulation of Bayesian Analysis for Games with Incomplete Information," *International Journal of Game Theory*, 14, 1–29.

MONDERER, D., AND D. SAMET (1989): "Approximating Common Knowledge with Common Beliefs," *Games and Economic Behavior*, 1.

——— (1997): "Proximity of Information in Games with Incomplete Information," *Mathematics of Operations Research*, 21, 707–725.

MORRIS, S. (1999): "Approximate common knowledge revisited," *International Journal of Game Theory*, 28(3), 385–408.

——— (2002): "Typical Types," `http://www.econ.yale.edu/ sm326/typical.pdf`.

MORRIS, S., AND C. SKIADIS (2000): "Rationalizable Trade," *Games and Economic Behavior*, 31, 311–323.

Rubinstein, A. (1989): "The Electronic Mail Game: Strategic Behavior under "Almost Common-Knowledge."," *American Economic Review*, 79, 385–91.

Weinstein, J., and M. Yildiz (2007): "A Structure Theorem for Rationalizability with Application to Robust Predictions of Refinements," *Econometrica*, 75(2), 365–400.

A Proof of Lemma 2

Because by definition

$$B_i^p(C^p(W)) = C_i^p(W) \cap \left\{u_i : u_i\left(\left\{u_{-i} : u_j \in C_j^p(W) \text{ for some } j\right\}\right) \geq p\right\},$$

we prove the lemma by showing

$$C_i^p(W) \subseteq \left\{u_i : u_i\left(\left\{u_{-i} : u_j \in C_j^p(W) \text{ for some } j\right\}\right) \geq p\right\}.$$

To simplify the notation, define the product set $\beta_j^k \subseteq U_{-i}(\Omega)$ as follows,

$$\beta_j^k = \left\{u_{-i} : u_j \in B_j^p \left[B^p\right]^k (W)\right\}.$$

Note that $\beta_j^k \subseteq \beta_j^{k-1}$ for all k and

$$\bigcap_{k\geq 0} \beta_j^k = \left\{u_{-i} : u_j \in C_j^p(W)\right\}.$$

Let $u_i \in C_i^p(W)$. First, since for all k, $u_i \in B_i^p \left[B^p\right]^k (W)$, then by definition

$$u_i\left(\left\{u_{-i} : \exists j \text{ st. } u_j \in B_j^p \left[B^p\right]^{k-1} (W)\right\}\right) \geq p.$$

i.e.,

$$u_i\left(\bigcup_{j\neq i} \beta_j^k\right) \geq p,$$

for all k. Since for each j, β_j^k is a nested sequence of sets,

$$\begin{aligned} p &\leq u_i\left(\bigcap_{k\geq 0}\bigcup_{j\neq i}\beta_j^k\right) \\ &\leq u_i\left(\bigcup_{j\neq i}\bigcap_{k\geq 0}\beta_j^k\right) \\ &= u_i\left(\left\{u_{-i} : u_j \in C_j^p(W) \text{ for some } j\right\}\right) \end{aligned}$$

establishing the lemma.

B Proofs of Section 4

B.1 Technical Result

Here, we prove a useful topological observation.

Lemma 7. *Suppose that E is separable and metrizable, A is a finite set, and let $\{V_a\}_{a\in A}$ be an open covering of E. Let $\mu \in \Delta E$ be a measure over E. Consider a (measurable) mapping $\sigma : E \to \Delta A$ such that σ is adapted to the covering $\{V_a\}_{a\in A}$, i.e.,*

$$\sigma(e)(a) > 0 \implies e \in V_a.$$

There is a sequence of continuous mappings $\sigma^m : E \to \Delta A$, each adapted to $\{V_a\}_{a\in A}$ such that

$$\sigma^m \to \sigma, \ \mu\text{-almost surely.}$$

Proof. By standard topological arguments, for each V_a, there exists a sequence of continuous functions $\alpha_a^m : E \to [0,1]$ such that $\alpha_a^m(e) > 0$ if and only if $e \in V_a$ and α_a^m converges pointwise to the indicator function for V_a. Also, because E is separable and metrizable and ΔA is compact metrizable, there is a sequence of continuous mappings $\tau^m : E \to \Delta A$, such that $\tau^m \to \sigma$, μ-almost surely. Note that for any $e \in E$, $\sum_{a\in A_i} \alpha_a^m(e) > 0$. Construct the sequence of mappings $\sigma^m : E \to \Delta A$ as follows. For any $e \in E$, for any $a \in A$, let

$$\sigma^m(e)(a) := \frac{\alpha_a^m(e)\left[\tau^m(e)(a) + \frac{1}{m}\right]}{\sum\limits_{a'\in A}\alpha_{a'}^m(e)\left[\tau^m(e)(a') + \frac{1}{m}\right]}.$$

By construction, σ^m is continuous. Moreover, for each m, $\sigma^m(e)(a) > 0$ if and only if $e \in V_a$. For μ-almost all $e \in V_a$,

$$\lim_{m\to\infty} \alpha_a^m(e)\,\tau^m(e)(a) \to \sigma(e)(a).$$

Thus, $\sigma^m \to \sigma$, μ-almost surely. □

B.2 Proof of Lemma 3

For any $p > 0$ and any proper and closed product event V, the set $C_i^p(V)$ is closed as the intersection of closed sets. Hence, $U_i(\Omega)\setminus C_i^p(V)$ is open.

To show that set $U_i(\Omega)/C_i^p(V)$ is dense, let u_i be an arbitrary hierarchy for i. For each integer k, we will construct a hierarchy $z_i \notin C_i^p(V)$ that agrees with u_i up to order $k-1$. The sequence of such hierarchies $(z)^k$ converges to u_i in the product topology.

Let $T = (T_j,\mu_j)$ be any type space such that there is a type t_i^* for player i that has hierarchy u_i and for each player j, there is a type y_j that has a hierarchy $\hat{u}_j$ that is not in V_j. We begin by constructing an alternate type space T' that represents the same hierarchies as T but has a convenient structure. The idea is to "factorize" T into infinitely many replicas.

Let T_j^k for each k be mutually disjoint "copies" of the space T_j. Let $\eta_j^k : T_j \to T^k$ the natural bijection between T_j^k and T_j. We construct a type space T' in which the set of types for player j is

$$T_j' = \bigcup_k T_j^k.$$

The belief mapping μ' is derived from μ as follows. For each k and for each j,

$$\mu_j'(\eta_j^k(t_j)) = \Delta(\mathrm{id}_\Omega \times \eta_{-j}^{k+1})\left[\mu_j(t_j)\right]$$

It is immediate that the hierarchy of beliefs for any type $\eta^k(t_j)$ is identical to that of t_j.

Fix any $\bar{k} \geq 1$. We next construct a new type space from T' by redefining the belief mapping. Define the belief mapping $\hat{\mu}$ such that for each $k \neq \bar{k}$, any player j, any type $t_j^k \in T_j^k$,

$$\hat{\mu}_j\left(t_j^k\right) = \mu_j'\left(t_j^k\right),$$

and for any player j, any type $t_j^{\bar{k}} \in T_j^{\bar{k}}$,

$$\hat{\mu}_j\left(t_j^{\bar{k}}\right)\left(O \times \left\{\eta_{-j}^{k+1}(y_{-j})\right\}\right) = \mu_j'\left(t_j^{\bar{k}}\right)\left(O \times T_{-j}'\right)$$

for all measurable subsets $O \subset \Omega$. Note in particular that the belief $\hat{\mu}_j\left(t_j^{\bar{k}}\right)$ has the same marginal on Ω as $\mu'_j\left(t_j^{\bar{k}}\right)$ but assigns probability 1 to the type profile $\eta_{-j}^{\bar{k}+1}(y_{-j})$ for the opponents.

Consider the type space with type sets T' and belief mappings $\hat{\mu}$. In this type space, every type $\eta_j^{\bar{k}}\left(t_j\right) \in T_j^{\bar{k}}$ has the same first-order beliefs as type t_j, but is certain that opponent j' has the hierarchy of $\hat{u}_{j'}$ for each opponent $j' \neq j$. Every type $\eta^{\bar{k}-1}(t_j)$ for j in $T_j^{\bar{k}-1}$ has the same first- and second-order beliefs as t_j but is certain that each opponent j' is certain that each of her opponents j'' has the hierarchy of $\hat{u}_{j''}$. Continuing inductively, the type $\eta^1(t_i^*)$ has a hierarchy of beliefs z_i that coincides with that of t_i^*,(i.e., the hierarchy u_i) up to order $\bar{k}$, but is certain that the opponents are certain that ... that the opponents' hierarchies are $\hat{u}_j$. That is,

$$z_i \in B_i^1\left(B^1\right)^{\bar{k}-1}(\{\hat{u}\}),$$

and so in particular, $z_i \notin C_i^p(V)$ for any $p > 0$ since $u_j \notin V_j$. □

B.3 Proof of Lemma 4

We divide the proof into four main sections. First, we present some results on the continuity of the correspondences of actions surviving finitely many rounds of elimination of never-best-replies. Next, we construct a series of games culminating with the game described in Lemma 12 that distinguishes hierarchies and has properties that will be used in the main construction. Then we construct the game G and conclude the proof.

B.3.1 Continuity of the Correspondences $R_i^m(\cdot \mid G,\varepsilon)$.

Recall that Lemma 1 and the subsequent remark allow us to view $R_j^m(\cdot|G,\varepsilon)$ as an upper hemi-continuous correspondence whose domain is $U_i(\Omega)$. For any m, player j, game G, action a_j and hierarchy u_j, define

$$h_j^m\left(a_j, G, u_j\right) = \inf\left\{\varepsilon : a_j \in R^m(u_j|G,\varepsilon)\right\}.$$

Lemma 8. $h_j^m\left(a_j, G, u_j\right)$ *is continuous in* u_j.

Proof. Suppose that $u_j^n \to u_j$ and let $h = \liminf_{n\to\infty} h_j^m(a_j, G, u_j^n)$. Then for each $\varepsilon > 0$, there exists a subsequence $u_j^n \to u_j$ such that $h_j^m(a_j, G, u_j^n) \leq h + \varepsilon$ for every n. In other words, $a_j \in R^m(u_j^n|G, h+\varepsilon)$. By the upper hemi-continuity of the correspondence $R^m(\cdot|G, h+\varepsilon)$, we have $a_j \in R^m(u_j|G, h+\varepsilon)$, i.e. $h_j^m(a_j, G, u_j) \leq h+\varepsilon$. Since this holds for each $\varepsilon > 0$, we have shown that $h_j^m\left(a_j, G, \cdot\right)$ is lower semi-continuous. We prove upper semi-continuity by induction on m. When $m = 0$, by definition $h_j^0(a_j, G, \cdot) \equiv 0$. Now assume that $h_j^{m-1}(a_j, G, u_j)$ is upper semi-continuous. Fix a player j, a game G, an action a_j^* and a hierarchy u_j^*. Let $h^* = h_j^m(a_j^*, G, u_j^*)$. We are going to show that for any $\varepsilon > 0$, there is a neighborhood $V \ni u_j^*$, such that $h_j^m(a_j^*, G, u_j) < h^* + \varepsilon$ for any $u_j \in V$. Let σ_{-j} be an $(h^* + \varepsilon/3)$-rationalizable conjecture of player j that makes a_j^* an $(h^* + \varepsilon/3)$-interim best response for type u_j^*. For any action profile $a_{-j} \in A_{-j}$, define

$$V_{a_{-j}} = \left\{u_{-j} : h_{j'}^m\left(a_{j'}, G, u_{j'}\right) < h^* + \varepsilon/2 \text{ for each } j' \neq j\right\}.$$

By the induction hypothesis, the collection $\{V_{a_{-j}}\}_{a_{-j} \in A_{-j}}$ is an open covering of $U_{-j}(\Omega)$. Also, if $\sigma_{-j}(u_{-j}, \omega)(a_{-j}) > 0$ for some type profile u_{-j} and state ω, then $u_{-j} \in V_{a_{-j}}$. By Lemma 7, there is a sequence of continuous strategies σ_{-j}^n converging u_j^*-almost surely to σ_{-j} such that for any $u_{-j} \in U_{-j}(\Omega)$ and action profile $a_{-j} \in A_{-j}$, if $\sigma_{-j}^n(u_{-j}, \omega)(a_{-j}) > 0$, then $u_{-j} \in V_{a_{-j}}$. Take any sequence of hierarchies $u_j^k \to u_j^*$. Then by the Mertens-Zamir homeomorphism, the corresponding sequence of beliefs $u_j^k \in \Delta(\Omega \times U_{-j}(\Omega))$ converges in the weak*-topology to the belief u_j^*. Hence, for any action $a_j \in A_j$, and for any n,

$$\lim_{k\to\infty}\left[\pi_i(a_j, \sigma_{-j}^n \mid u_j^k) - \pi_i(a_j^*, \sigma_{-j}^n \mid u_j^k)\right] = \left[\pi_i(a_j, \sigma_{-j}^n \mid u_j^*) - \pi_i(a_j^*, \sigma_{-j}^n \mid u_j^*)\right].$$

Taking limits in n, by the dominated convergence theorem

$$\begin{aligned}
\lim_{n\to\infty}\lim_{k\to\infty}\left[\pi_i(a_j, \sigma_{-j}^n \mid u_j^k) - \pi_i(a_j^*, \sigma_{-j}^n \mid u_j^k)\right] &= \lim_{n\to\infty}\int g_j\left(a_j, \sigma_{-j}^n, \omega\right) - g_j\left(a_j^*, \sigma_{-j}^n, \omega\right)\, du_j^* \\
&= \int g_j\left(a_j, \sigma_{-j}, \omega\right) - g_j\left(a_j^*, \sigma_{-j}, \omega\right)\, du_j^* \\
&= \pi_i(a_j, \sigma_{-j} \mid u_j^*) - \pi_i(a_j^*, \sigma_{-j} \mid u_j^*) \\
&\leq h^* + \varepsilon/2
\end{aligned}$$

This implies that for large enough k, there is a large enough n such that the action a_j^* is an $(h^* + \varepsilon/2)$-interim best-reply to the $(h^* + \varepsilon/2)$-rationalizable conjecture σ_{-j}^n, i.e. a_j^* is $(h^* + \varepsilon/2)$-interim rationalizable and

$$h_j^m\left(a_j^*, G\right) < h^* + \varepsilon.$$

□

B.3.2 Constructing a Game that Distinguishes Hierarchies

In this section, we construct a series of games that are used to separate rationalizable behavior of hierarchies of beliefs. First, recall a result from Dekel, Fudenberg, and Morris (2006).

Lemma 9 (Dekel, Fudenberg, and Morris (2006), Lemma 4)**.** *Suppose that $v_i \neq u_i^*$. There is a game $G = \left(A_j, g_j\right)$, with an action $a_i \in A_i$, an $\varepsilon > 0$ and an integer m such that*

$$a_i \in R_i^m\left(v_i|G, 0\right) \text{ and } a_i \notin R_i^m\left(u_i^*|G, \varepsilon\right).$$

Lemma 10. *Suppose that $v_i \neq u_i^*$. There is a neighborhood $U_i \ni v_i$, a game $G = \left(A_j, g_j\right)$ with an action $a_i \in A_i$, an $\varepsilon > 0$ and an integer m such that*

$$a_i \in R_i^m\left(u_i|G, 0\right) \text{ for all } u_i \in U_i \text{ and } a_i \notin R_i^m\left(u_i^*|G, \varepsilon\right).$$

Proof. Let $\bar{G}$ and a_i be the game and action from Lemma 9. Because of Lemma 8, there is a neighborhood $U_i \ni v_i$ open in the product topology and $0 < \varepsilon'' < \varepsilon'$ such that

$$a_i \in R_i^m\left(u_i|\bar{G}, \varepsilon''\right) \text{ for all } u_i \in U_i \text{ and } a_i \notin R_i^m\left(u_i^*|\bar{G}, \varepsilon'\right).$$

Now consider the game G that is identical to $\bar{G}$ except that ε'' is added to player i's payoff whenever he plays $\bar{a}_i$, independent of the state and action profile of the remaining players. In the game G,

$$a_i \in R_i^m\left(u_i|G, 0\right) \text{ for all } u_i \in U_i \text{ and } a_i \notin R_i^m\left(u_i^*|G, \varepsilon' - \varepsilon''\right).$$

This proves the lemma when we take $\varepsilon = \varepsilon' - \varepsilon''$. □

Lemma 11. *Suppose that $v_i \neq u_i^*$. There is a neighborhood $U_i \ni v_i$, an $\varepsilon > 0$, and a game $G = \left(A_j, g_j\right)$ whose action sets have a product structure, i.e., $A_j = X_j \times Y_j$ and that satisfies the following properties.*

1. *The first coordinate of i's action, x_i does not affect the payoffs of i's opponents: For any $a_{-i} \in A_{-i}$, $y_i \in Y_i$, $x_i, x_i' \in X_i$ and $\omega \in \Omega$*

$$g_{-i}\left(a_{-i}, (x_i, y_i), \omega\right) = g_{-i}\left(a_{-i}, (x_i', y_i), \omega\right),$$

2. *The rationalizable correspondence has a product structure: For each j, there are corre-*

spondences $\mathcal{X}_j : U_j(\Omega) \rightrightarrows X_j$ *and* $\mathcal{Y}_j : U_j(\Omega) \rightrightarrows Y_j$, *such that for all* u_j,

$$R_j\left(u_j|G,0\right) = \mathcal{X}_j\left(u_j\right) \times \mathcal{Y}_j\left(u_j\right).$$

3. *There is an element* $\bar{x}_i$ *of* X_i *that distinguishes* U_i *from* u_i^*.

$$\{\bar{x}_i\} \times \mathcal{Y}_i\left(u_i\right) \subseteq R_i\left(u_i|G,0\right) \text{ for all } u_i \in U_i,$$
$$\{\bar{x}_i\} \times \mathcal{Y}_i\left(u_i^*\right) \cap R_i\left(u_i^*|G,\varepsilon\right) = \emptyset.$$

Proof. Take the neighborhood $U_i \ni v_i$, the game $\bar{G} = \left(\bar{A}_j, \bar{g}_j\right)$, the action $a_i \in \bar{A}_i$, and $\varepsilon > 0$ and m from Lemma 10. Define the game $G = \left(A_j, g_j\right)$ as follows. For any player j, let $A_j = \left(\bar{A}_j\right)^m$, and

$$g_j\left(\left(a_j^1, ..., a_j^m\right), \left(a_{-j}^1, ..., a_{-j}^m\right), \omega\right) = \sum_{k=1}^{m-1} \bar{g}_j\left(a_j^k, a_{-j}^{k+1}, \omega\right).$$

Notice that for any $\varepsilon' \geq 0$, for all j,

$$R_j\left(\cdot|G,\varepsilon'\right) = R_j^{m-1}\left(\cdot|\bar{G},\varepsilon'\right) \times ... \times R_j^0\left(\cdot|\bar{G},\varepsilon'\right).$$

(This is shown by induction on on m.) Set $X_j = \bar{A}_j$, $Y_j = \left(\bar{A}_j\right)^{m-1}$, and

$$\mathcal{X}_j\left(\cdot\right) = R_j^{m-1}\left(\cdot|\bar{G},0\right),$$
$$\mathcal{Y}_j\left(\cdot\right) = R_j^{m-2}\left(\cdot|\bar{G},0\right) \times ... \times R_j^0\left(\cdot|\bar{G},0\right),$$

and $\bar{x}_i = a_i$. The thesis of the Lemma follows. □

Lemma 12. *Fix player* i. *Let* $W_i \subset U_i\left(\Omega\right)$ *(strict inclusion) be a closed proper subset. For any type* $u_i^* \notin W_i$, *there is* $\varepsilon > 0$, *a game* $G = \left(A_j, g_j\right)$ *whose action sets have a product structure, i.e.,* $A_j = X_j \times Y_j$ *and that satisfies the following properties:*

1. *The* X_i *coordinate of* i*'s action does not affect the payoffs of* i*'s opponents: For any* $a_{-i} \in A_{-i}$, $y_i \in Y_i$, $x_i, x_i' \in X_i$ *and* $\omega \in \Omega$

$$g_{-i}\left(a_{-i}, (x_i, y_i), \omega\right) = g_{-i}\left(a_{-i}, (x_i', y_i), \omega\right).$$

2. *The rationalizable correspondence has a product structure: for each j there are correspondences $\mathcal{X}_j : U_j(\Omega) \rightrightarrows X_j$ and $\mathcal{Y}_j : U_j(\Omega) \rightrightarrows Y_j$, such that for all u_j,*

$$R_j\left(u_j|G,0\right) = \mathcal{X}_j\left(u_j\right) \times \mathcal{Y}_j\left(u_j\right).$$

3. *There is a non-empty subset $\bar{X} \subseteq X_i$ such that*

$$\bar{X} \times \mathcal{Y}_i\left(u_i\right) \subseteq R_i\left(u_i|G,0\right) \text{ for all } u_i \in W_i,$$
$$\bar{X} \times \mathcal{Y}_i\left(u_i^*\right) \cap R_i\left(u_i^*|G,\varepsilon\right) = \emptyset.$$

The Lemma provides a game with three important features. First, the action set has a product structure and the first dimension of i's action is irrelevant for $-i$'s payoffs. Second, the rationalizable correspondence has a product structure. Finally, there is a distinguishing subset of actions for player i that are rationalizable only for a proper subset of types that includes W_i.

Proof. For any $v_i \in W_i, v_i \neq u_i^*$ and we can apply Lemma 11 to find $\varepsilon^{(v_i)} > 0$, neighborhoods $U_i^{(v_i)} \ni v_i$, and games $G^{(v_i)}$. By the compactness of W_i, we can find a finite sequence of hierarchies $v_i^1, ..., v_i^K \in W_i$ such that $W_i \subseteq \bigcup_k U_i^{(v_i^k)}$. Let $\varepsilon = \min \varepsilon^{(v^k)}$. To shorten the notation, let $G^k = (A_j^k, g_j^k) := G^{(v_i^k)}$. We define the game $G = (A_j, g_j)$ to be the product game $G = G^1 \times ...G^K$. Specifically we set $A_j = X_j \times Y_j$ where

$$X_j := \prod_{k=1...,K} X_j^k \qquad Y_j := \prod_{k=1...,K} Y_j^k$$

for all j. The payoff to profiles $a = (a_j^k), j = 1, \ldots, N; k = 1, \ldots, K$ is given by

$$g_j\left(a,\omega\right) = \sum_{k=1}^{K} g_j^k\left(a^k,\omega\right).$$

Part 1 of the lemma then follows from part 1 of Lemma 11. The product structure of the game G yields a product structure for the rationalizable correspondence. In particular,

$$R\left(\cdot|G,\varepsilon\right) = R\left(\cdot|G^1,\varepsilon\right) \times ... \times R\left(\cdot|G^K,\varepsilon\right).$$

and this proves part 2 of the lemma. Finally, we define

$$\bar{X} = \left\{ x_i \in X_i : x_i^k = \bar{x}_i^k \text{ for at least one } k = 1, ..., K \right\},$$

and part 3 of the lemma follows from part 3 of Lemma 11. □

B.3.3 Construction of the Game G

Next we describe the construction of the game G that will satisfy the thesis of the lemma. Let $W = \prod_{i=1}^{N} W_i$ be a product event such that W_i are closed and proper subsets for each i. For each j, fix a hierarchy $u_j^* \notin W_j$ and $\varepsilon_j > 0$ and apply Lemma 12 to find a game denoted $G^j = (A_i^j, g_i^j)_{i=1}^N$ with the properties provided there. All objects from the game G^j will be designated with j superscripts. For example, $A_i^j = X_i^j \times Y_i^j$ is the set of actions of player i in game G^j, and

$$\bar{X}^j \times \mathcal{Y}_j^j (u_j) \subseteq R_j \left(u_j | G^j, 0 \right) \text{ for all } u_j \in W_j,$$
$$\bar{X}^j \times \mathcal{Y}_j^j \left(u_j^* \right) \cap R_j \left(u_j^* | G^j, \varepsilon \right) = \emptyset.$$

Let $\varepsilon = \min_j \varepsilon_j > 0$.

We construct a single game $G = (A_i, g_i)_{i=1}^N$ out of the player-indexed games G^j. The action sets are given by

$$A_i := X_i \times Y_i \times Z_i$$

where

$$X_i := \prod_{j=1}^{N} X_i^j$$
$$Y_i := \prod_{j=1}^{N} Y_i^j$$
$$Z_i = \{0, 1\}.$$

Thus, $X_i \times Y_i$ is the product of i's action sets in the games G^j, and Z_i is a binary variable.

For each j, let $\zeta_j \subseteq A_j$ denote the subset of j's actions in G such that player j plays both $z_j = 1$ and $x_j^j \in \bar{X}^j$. In an abuse of notation, write ζ_{-i} for the set of action profiles a_{-i} such that $a_j \in \zeta_j$ for at least one $j \neq i$.

Next we define the payoffs. If the action profile is $a = \left((a^j)_{j=1}^N, z \right)$ (where each a^j is a profile in G^j and z is a profile from $Z = \{0,1\}^N$), then

$$g_i(a,\omega) = \sum_{j=1}^{N} g_i^j(a^j,\omega) + \begin{cases} 1 & \text{if } z_i = 1 \text{ and } a_{-i} \in \zeta_{-i}, \\ -\frac{p}{1-p} & \text{if } z_i = 1 \text{ and } a_{-i} \notin \zeta_{-i}, \\ 0 & \text{if } z_i = 0. \end{cases}$$

Thus, G is a product of the games G^j together with a binary coordination game. The payoffs from the latter ensure that z_i is part of a best-reply if and only if i assigns probability at least p to the event that at least one opponent plays both $z_j = 1$ and $x_j^j \in \bar{X}^j$.

For each j, we define a mapping $\mathcal{L}_j\left(.|u_j\right) : R_i\left(u_j|G,0\right) \to A_j$. If $u_j \in C_j^p(W) \subseteq W_j$, then it follows from the additive part of the payoff function g_j and parts 2 and 3 of Lemma 12 that if $a_j \in R_j(u_j|G,0)$ is any rationalizable action in G for u_j, then there is another rationalizable action $a_j^* \in R_j(u_j|G,0)$ that is identical except possibly on the x_j^j and z_j dimensions where $(x_j^j)^{**} \in \bar{X}^j$. Using this observation, we pick an action $\mathcal{L}_j(a_j|u_i)$ to be an action that is identical to a_j^* except also that $z_j^* = 1$. On the other hand, if $u_j \notin C_j^p(W)$, then we set $\mathcal{L}_j(a_j|u_i) = a_j$. We extend the mapping to profiles $\mathcal{L}_{-j}(a_{-j}|u_{-j})$. Note that $\mathcal{L}_j(a_j|u_i) \in \zeta_j$ if $u_j \in C_j^p(W)$ and $\mathcal{L}_{-j}(a_{-j}|u_{-j}) \in \zeta_{-j}$ if $u_i \in C_i^p(W)$ for at least one $i \neq j$.

B.3.4 Concluding the Proof of Lemma 4

First we show that $\zeta_i \cap R_i(u_i|G,0) \neq \emptyset$ for all $u_i \in C_i^p(W)$. Consider the following assessment α.

$$\alpha_i(u_i) = R_i(u_i|G,0) \cup \bigcup_{a_i \in R_i(u_i|G,0)} \mathcal{L}_i(a_i|u_i).$$

We will show that α has the best-reply property and $\alpha_i\left(u_i\right) \subseteq R_i(u_i|G,0)$. This will conclude the proof of the first part of the lemma.

Since α includes the rationalizable correspondence, which by definition has the best-reply property, it remains only to show that for any type $u_i \in C_i^p(W)$, and for any action $a_i \in R_i(u_i|G,0)$, the action $\mathcal{L}_i(a_i|u_i)$ is a best-reply for u_i to a conjecture that is a selection from α. By construction, $\mathcal{L}_i(a_i|u_i)$ is identical to an action a_i^* on all except possibly the z_i dimension and $a_i^* \in R_i(u_i|G,0)$. There is a conjecture σ_{-i} that is a selection from the rationalizable correspondence, and hence also from α, against which a_i^* is a best-reply for u_i. We will modify σ_{-i} as follows; set

$$\hat{\sigma}_{-i}\left(u_{-i},\omega\right) := \left(\Delta\mathcal{L}_{-i}(.|u_{-i})\right)\left[\sigma_{-i}\left(u_{-i},\omega\right)\right].$$

Thus, $\hat{\sigma}_{-i}$ is a selection from α that differs from σ_{-i} only in terms of the dimensions z_j and x_j^j dimensions of conjectured actions. In particular, if $u_j \in C_j^p(W)$ for at least one j, then $\hat{\sigma}_{-i}(u_{-i},\omega)$ attaches probability 1 to profiles in ζ_{-i}. This means that the payoffs to all dimensions of i's action apart from z_i are unaffected, and $z_i = 1$ is better than $z_i = 0$ if u_i attaches probability at least p to at least one j having $u_j \in C_j^p(W)$. By Lemma 2, this is true for u_i since $u_i \in C_i^p(W)$. Thus, $\mathcal{L}_i(a_i|u_i)$ is a best-reply to $\hat{\sigma}_{-i}$ concluding this part of the proof.

Let $q = p - 2\varepsilon(1-p)$. Define

$$V_i = \left\{u_i : \bar{X}^i \cap R_i(u_i|G^i, \varepsilon) \neq \emptyset\right\}.$$

Since $u_i^* \notin V_i$ for each i, the sets V_i are proper subsets and moreover $W_i \subseteq V_i$.

Now we prove that $\zeta_i \cap R_i(u_i|G,\varepsilon) = \emptyset$ for all $u_i \notin C_i^q(V)$. From the product structure of G, we have $\zeta_i \cap R_i(u_i|G,\varepsilon) = \emptyset$ for all $u_i \in U_i(\Omega) \setminus V_i$. We will show by induction on k that $\zeta_i \cap R_i(u_i|G,\varepsilon) = \emptyset$ for all $u_i \in U_i(\Omega) \setminus B_i^q \left[B^q\right]^k(V)$. Assume the induction hypothesis for $k-1$ and let $u_i \in U_i(\Omega) \setminus B_i^q \left[B^q\right]^k(V)$. By definition, u_i either belongs to $U_i(\Omega) \setminus B_i^q \left[B^q\right]^{k-1}(V)$ or satisfies

$$u_i\left(\left\{u_{-i} : \forall j\, u_j \in U_j(\Omega) \setminus B_j^q \left[B^q\right]^{k-1}(V)\right\}\right) \geq 1-q.$$

In the former case, the induction hypothesis delivers the conclusion directly, so consider the latter case. Then the induction hypothesis implies that u_i assigns at least $1-q$ probability to the set of profiles u_{-i} such that $\zeta_j \cap R_j(u_j|G,\varepsilon) = \emptyset$ for all j. Thus, with any conjecture that is a selection from the rationalizable correspondence, u_i assigns probability less than q to action profiles in ζ_{-i}. No action in ζ_i can be a best-reply to such a conjecture, proving the claim for k.

B.4 Proof of Lemma 5

Fix a game G. First, we show that there exists $\varepsilon^* > 0$ and for each player i set of player i's actions $Z_i^* \subseteq A_i$ such that set of hierarchies

$$V_i = \{u_i : R_i\left(u_i|G, \varepsilon^*\right) = Z_i^*\}$$

is open and for each $0 \le \varepsilon \le \varepsilon^*$,

$$V_i = \{u_i : R_i(u_i|G,\varepsilon) \subseteq Z_i^*\}. \tag{B.1}$$

Indeed, for each player i and every $\hat{\varepsilon} \ge 0$, define a collection of subsets of actions

$$\mathcal{A}_i(\hat{\varepsilon}) = \{Z_i \subseteq A_i : \text{ There exists } u_i \in U_i(\Omega) \text{ such that } R_i(u_i|G,\hat{\varepsilon}) = Z_i\}.$$

The collection $\mathcal{A}_i(\hat{\varepsilon})$ is a non-empty collection of non-empty sets, and it is ordered with respect to ε in the following way: if $\varepsilon' < \varepsilon''$ then for any $Z_i'' \in \mathcal{A}_i(\varepsilon'')$, there is $Z_i' \in \mathcal{A}_i(\varepsilon')$ such that $Z_i' \subseteq Z_i''$. Since A_i is finite, the number of all subsets of A_i is also finite and therefore there is $\varepsilon^* > 0$, such that for all $0 \le \hat{\varepsilon} \le \varepsilon^*$, we have $\mathcal{A}_i(\hat{\varepsilon}) = \mathcal{A}_i(0)$. Let Z_i^* be a minimal element of $\mathcal{A}_i(0)$, i.e., $Z_i^* \in \mathcal{A}_i(0)$, and there is no $Z_i \subsetneq Z_i^*$ belonging to $\mathcal{A}_i(0)$. Then, set V_i defined as above is open because, by the choice of minimal Z_i^*

$$V_i = \{u_i : R_i(u_i|G,\varepsilon^*) \subseteq Z_i^*\}$$

and that set on the right-hand side is open by the upper hemi-continuity of the correspondence $R_i(\cdot|G,\varepsilon^*) : U_i(\Omega) \rightrightarrows A_i$. Equality (B.1) follows from the choice of ε^* and Z_i^*.

Next, consider the closed, proper subsets

$$W_i = U_i(\Omega) \setminus V_i,$$

and define the proper product event $W = \times_i W_i$. Let K be an upper bound on payoffs

$$\max_{j,a,\omega} \left| g_j(a,\omega) \right| \le K \tag{B.2}$$

(such a bound exists due to the compactness of Ω and the continuity of payoffs). Fix any p so that $0 < p \le \varepsilon^*/6K$. We will show if $u_i \notin C_i^p(W)$, then u_i is G-regular.

It is convenient to represent the complement of the common p-belief set $C_i^p(W)$ as an countable union of certain open sets. Write $E_i^0 = V_i$ and inductively define,

$$E_i^k := U_i(\Omega) \setminus B_i^p\left([B^p]^k(W)\right), \tag{B.3}$$

and $E^k_{-i} = \times_{j\neq i} E^k_j$. Notice that

$$\begin{aligned}
&B^p_i\left([B^p]^k(W)\right)\\
&= B^p_i\left([B^p]^{k-1}(W)\right) \cap \left\{u_i : u_i\left(\times_{j\neq i}\left(U_j(\Omega)\setminus B^p_j\left([B^p]^{k-1}(W)\right)\right)\right) < 1-p\right\}\\
&= B^p_i\left([B^p]^{k-1}(W)\right) \cap \left\{u_i : u_i\left(E^{k-1}_{-i}\right) < 1-p\right\},
\end{aligned}$$

and

$$\begin{aligned}
E^k_i &= \left[U_i(\Omega)\setminus B^p_i\left([B^p]^{k-1}(W)\right)\right] \cup \left[U_i(\Omega)\setminus \left\{u_i : u_i\left(E^{k-1}_{-i}\right) < 1-p\right\}\right]\\
&= E^{k-1}_i \cup \left\{u_i : u_i\left(E^{k-1}_{-i}\right) \geq 1-p\right\}
\end{aligned}$$

In other words, set E^0_i consists of hierarchies that do not belong to closed set W_i; set E^1_i consists of hierarchies that do not belong to W_i or that assign at least $1-p$ probability that for all players $j \neq i$, player j's hierarchy does not belong to W_j; set E^2_j consists of hierarchies that belong to E^1_i or that assign at least $1-p$ probability that for all players $j \neq i$, player j's hierarchy belongs to set E^1_j, etc. Observe that sets E^k_i are open because W is closed and $B^p_i(D)$ is a closed set for any closed product set D. Finally, notice that

$$\begin{aligned}
U_i(\Omega)\setminus C^p_i(W) &= U_i(\Omega)\setminus \bigcap_{k\geq 0} B^p_i\left([B^p]^k(W)\right)\\
&= \bigcup_{k\geq 0} U_i(\Omega)\setminus B^p_i\left([B^p]^k(W)\right)\\
&= \bigcup_{k\geq 0} E^k_i.
\end{aligned}$$

Thus, if $u_i \notin C^p_i(W)$, then $u_i \in E^k_i$ for some k.

We will prove the following claim by induction on k.

Claim 1. *Suppose $u_i \in E^k_i$ and let u^n_i be a sequence converging to u_i in the product topology. If a^*_i is rationalizable for u_i, then there exists n^* such that a^*_i is $6Kp$-rationalizable for u^n_i for all $n > n^*$.*

The claim will imply that u_i is G-regular. To see why, take any action a_i that is rationalizable for u_i. Then the claim implies that a_i is $6Kp$-rationalizable for types in the tail of any sequence converging to u_i. Since $p < \varepsilon^*/6K$ was arbitrary, this establishes that u_i is G-regular.

The remainder of the proof establishes the claim. When $k = 0$, then the claim is a consequence of the fact that $E_i^0 = V_i$ is open and for hierarchies in V_i and any $\varepsilon \leq \varepsilon^*$ the set of ε-rationalizable actions is constant (equal to Z_i^*). Now assume that the statement holds for all $u_i \in E_i^{k-1}$ for $k > 0$. Take any $u_i \in E_i^k$. Since the induction hypothesis already covers the case of $u_i \in E_i^{k-1}$, it remains to show that the statement holds when

$$u_i\left(\Omega \times E_{-i}^k\right) \geq 1 - p. \tag{B.4}$$

Let u_i^n be a sequence of hierarchies converging to u_i in the product topology. Because of the continuity of the belief mapping on the universal type space, the beliefs associated with these hierarchies also converge,

$$u_i^n \to u_i \text{ (in the sense of weak}^* \text{ topology).} \tag{B.5}$$

Recall that a conjecture $\sigma_{-i} : \Omega \times U_{-i}(\Omega) \to \Delta A_i$ is ε-*rationalizable* if for any type profile $u_{-i} \in U_{-i}(\Omega)$, any state $\omega \in \Omega$, the conjectured actions are ε-rationalizable, i.e., $\sigma_{-i}(u_{-i})(R_{-i}(u_{-i}|G, \varepsilon)) = 1$. An action is ε- rationalizable if and only if it is a best-reply to a ε-rationalizable conjecture.

Let a_i^* be any 0-rationalizable action for u_i and let σ_{-i} be a 0-rationalizable conjecture against which a_i^* is a best-reply for u_i. We first show that there is a sequence of $6Kp$-rationalizable conjectures $\sigma_{-i}^m : \Omega \times U_{-i}(\Omega) \to \Delta A_i$, such that σ_{-i}^m is continuous on $\Omega \times E_{-i}^k$ and

$$\sigma_{-i}^m \to \sigma_{-i} \qquad u_i\text{-almost surely.} \tag{B.6}$$

To show this, we use the inductive hypothesis which provides a collection of sets of hierarchies

$$V_j\left(a_j\right) \subseteq E_j^{k-1},$$

such that for each a_i set $V_i(a_i)$ is open in the product topology and

$$\left\{u_j \in E_j^{k-1} : a_j \in R_j\left(u_j|G, 0\right)\right\} \subseteq V_j\left(a_j\right) \subseteq \left\{u_j \in E_j^{k-1} : a_j \in R_j\left(u_j|G, 6Kp\right)\right\}.$$

In particular, $\{\Omega \times \prod_j V_j(a_j)\}_{a_{-i} \in A_{-i}}$ is an open cover of $\Omega \times E_{-i}^{k-1}$. By Lemma 7, there is a sequence of continuous functions $\tau^m : \Omega \times E_{-i}^{k-1} \to \Delta A_i$, such that for any $u_{-i} \in E_{-i}^{k-1}$, each $\omega \in \Omega$

$$\tau^m(\omega, u_{-i})(a_{-i}) > 0 \text{ iff } u_j \in V_j\left(a_j\right) \text{ for each } j \neq i\text{, and}$$

and $\tau^m \to \sigma_{-i}$ almost surely with respect to the conditional measure $u_i(\cdot|\Omega \times E_{-i}^{k-1})$. Define $\sigma_{-i}^m : \Omega \times U_{-i}(\Omega) \to \Delta A_i$ as

$$\sigma_{-i}^m(\omega, u_{-i}) = \begin{cases} \tau^m(\omega, u_{-i}) & \text{if } u_{-i} \in E_{-i}^{k-1} \\ \sigma_{-i}(\omega, u_{-i}) & \text{otherwise.} \end{cases}$$

Thus, σ_{-i}^m is $6Kp$-rationalizable and continuous on $\Omega \times E_{-i}^k$.

Let $\alpha^m : U_{-i}(\Omega) \to [0,1]$ be a sequence of continuous functions, such that $\alpha^m(u_{-i}) = 0$ for any $u_{-i} \notin E_{-i}^{k-1}$ and $\lim_{m\to\infty} \alpha^m(u_{-i}) = 1$ for any $u_{-i} \in E_{-i}^{k-1}$. Since E_{-i}^{k-1} is open, such a sequence exists.

For any a_i, consider the difference in payoffs,

$$\left|\pi_i(a_i, \sigma_{-i}^m | u_i^n) - \pi_i(a_i, \sigma_{-i} | u_i)\right|.$$

By the triangle inequality, it is bounded by

$$\left|\pi_i(a_i, \sigma_{-i}^m | u_i^n) - \pi_i(a_i, \sigma_{-i}^m | u_i)\right| + \left|\pi_i(a_i, \sigma_{-i}^m | u_i) - \pi_i(a_i, \sigma_{-i} | u_i)\right| \tag{B.7}$$

By Equation B.6 and the dominated convergence theorem, the second term converges to zero in m. We can further bound the first term as follows, using Equation B.2

$$\begin{aligned} \left|\pi_i(a_i, \sigma_{-i}^m | u_i^n) - \pi_i(a_i, \sigma_{-i}^m | u_i)\right| \le & \left|\int \alpha^m g_i(a_i, \sigma_{-i}^m, \omega) du_i^n - \int \alpha^m g_i(a_i, \sigma_{-i}^m, \omega) du_i\right| \\ & + K\left(\left|\int 1 - \alpha^m du_i^n\right| + \left|\int 1 - \alpha^m du_i\right|\right). \end{aligned} \tag{B.8}$$

For any n, the last two terms converge to $1 - u_i^n(\Omega \times E_{-i}^{k-1})$, and $1 - u_i(\Omega \times E_{-i}^{k-1})$ respectively. By Equation B.4, the latter is less than p. Thus, we can find m^* large enough so that for all n and for all a_i,

$$\begin{aligned} \left|\pi_i(a_i, \sigma_{-i}^{m^*} | u_i^n) - \pi_i(a_i, \sigma_{-i} | u_i)\right| \le & \left|\int \alpha^{m^*} g_i(a_i, \sigma_{-i}^{m^*}, \omega) du_i^n - \int \alpha^{m^*} g_i(a_i, \sigma_{-i}^{m^*}, \omega) du_i\right| \\ & + K\left(1 - u_i^n(\Omega \times E_{-i}^{k-1}) + 2p\right). \end{aligned} \tag{B.9}$$

Finally, because of the convergence in Equation B.5 and since $g_i(a_i, \sigma_{-i}^{m^*}, \omega)$ is continuous, the first term on the right converges in n to zero and $u_i^n(\Omega \times E_{-i}^{k-1})$ converges in n to

$1 - u_i(\Omega \times E_{-i}^{k-1}) \leq p$. Thus, we can take n^* large enough so that

$$\left| \pi_i(a_i, \sigma_{-i}^{m^*} | u_i^n) - \pi_i(a_i, \sigma_{-i} | u_i) \right| \leq 3Kp$$

for all $n > n^*$ for all actions a_i.

Since a_i^* is a best-response for u_i to the conjecture σ_{-i}, it follows that for all $n > n^*$, a_i^* is a $6Kp$-best-response for u_i^n to the $6Kp$-rationalizable conjecture $\sigma_i^{m^*}$, and hence $a_i^* \in R_i(u_i^n | G, 6Kp)$. This completes the proof of the claim and the lemma.