Hadamard Factorization of Stable Polynomials

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Abstract. The stable (Hurwitz) polynomials are important in the study of differential equations systems and control theory (see [7] and [19]). A property of these polynomials is related to Hadamard product. Consider two polynomials \( p, q \in \mathbb{R}[x] \):

\[
p(x) = a_0x^n + a_{-1}x^{n-1} + \cdots + a_1x + a_0
\]

\[
q(x) = b_0x^m + b_{-1}x^{m-1} + \cdots + b_1x + b_0
\]

the Hadamard product \((p \ast q)\) is defined as

\[
(p \ast q)(x) = a_0b_0x^n + a_{-1}b_{-1}x^{n-1} + \cdots + a_1b_1x + a_0b_0
\]

where \( k = \min(m, n) \). Some results (see [16]) shows that if \( p, q \in \mathbb{R}[x] \) are stable polynomials then \((p \ast q)\) is stable, also, i.e. the Hadamard product is closed; however, the reciprocal is not always true, that is, not all stable polynomial has a factorization into two stable polynomials the same degree \( n \), if \( n \geq 4 \) (see [15]).

In this work we will give some conditions to Hadamard factorization existence for stable polynomials.

Keywords: Hurwitz polynomial, Hadamard product, Hadamard stable factorization.

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INTRODUCTION

It is known that the problem of finding conditions for verifying if a given polynomial has all of its roots with negative real part was proposed by Maxay [29]. The polynomials with such property are named stable (Hurwitz) polynomials. Interesting information about Hurwitz polynomials can be found in [13] and [22]. The Routh-Hurwitz criterion [20], the Hermite-Biehler theorem [17] and Stability test (see [7]) are perhaps the most famous criteria. Due to the presence of uncertainties when a physical phenomenon is modeled, we must often study the stability of families of polynomials. Excellent references about families of Hurwitz polynomials are [1], [6] and [7]. With respect to this subject it is important to mention Kharitonov’s theorem [21], which is the most famous result about families of stable polynomials. Kharitonov studied the stability of interval families. Other questions about interval polynomials can be consulted in [9] and [10]. Since the set of Hurwitz polynomials is not convex, the stability of segments of polynomials has also been investigated (see for instance [2], [3], [8] [18] and [24]). Excellent references about Hurwitz polynomials reported during 1987-1991 can be seen in [11]. Geometric and topological approaches have been used in some works. For example, in [19] it was shown that the set of Hurwitz polynomials of degree \( n \) with positive coefficients – \( \mathcal{H}^+_{n, e} \) – is contractible. With ideas of differential topology, in [4] it was proved that \( \mathcal{H}^+_{n, e} \) is a smooth trivial vector bundle over \( \mathcal{H}^+_{n-1, e} \). With this last approach the set of Schur polynomials was studied (see [5]).

In order to explain the relation between Hurwitz polynomials and the Hadamard factorization, consider two polynomials \( p(t) \) and \( q(t) \) with real coefficients

\[
p(t) = \beta_0t^n + \beta_{-1}t^{n-1} + \cdots + \beta_1t + \beta_0, \tag{1}
\]

\[
q(t) = \gamma_0t^m + \gamma_{-1}t^{m-1} + \cdots + \gamma_1t + \gamma_0. \tag{2}
\]

Then we define the polynomial \((p \ast q)(t)\) in the following way

\[
(p \ast q)(t) = \beta_0\gamma_0t^n + \beta_{-1}\gamma_{-1}t^{n-1} + \cdots + \beta_1\gamma_1t + \beta_0\gamma_0. \tag{3}
\]

It is said that \((p \ast q)(t)\) is the Hadamard product of \( p(t) \) and \( q(t) \).
In [16] it was proved that Hadamard products of Hurwitz polynomials are Hurwitz polynomials. However in [15] it was shown that there are Hurwitz polynomials of degree 4 that do not have a Hadamard factorization in two Hurwitz polynomials. Recently, in [25] necessary conditions for Hadamard factorizations of Hurwitz polynomials were obtained. With respect to the Hadamard product other questions can be read in [14]. The relation between stable multivariate polynomials and the Hadamard product can be seen in [26]. The result of Garloff and Wagner has also been used to do robustness analysis (see [12] and [23]). In this work, we give necessary and sufficient conditions for Hadamard factorizations of Hurwitz polynomials by means of a recursive approach. This lets us present necessary and sufficient conditions for Hadamard factorizations of Hurwitz polynomials of degree 4.

**MAIN RESULTS**

In this section, we present a result which is valid for polynomials with arbitrary degree. In order, first we present a result which is valid for Hurwitz polynomials of degree 3.

**Theorem 1.** Let \( f(t) = \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0 \) be Hurwitz. Then exists two Hurwitz polynomials \( p(t) \) and \( q(t) \) such as \( (p \ast q)(t) = f(t) \).

**Proof.** Since \( f \) is Hurwitz, then it holds \( \alpha_1 \alpha_2 - \alpha_0 \alpha_3 > 0 \) therefore \( \frac{\alpha_1 \alpha_2}{\alpha_0 \alpha_3} > 1 \). We can choose real numbers \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 > 0 \) such as

\[
\frac{\alpha_1 \alpha_2}{\alpha_0 \alpha_3} > 1,
\]

and we define \( p(t) = t^3 + \alpha_1 t^2 + \alpha_1 t + \alpha_0 \) and \( q(t) = \frac{\alpha_0}{\alpha_3} t^3 + \frac{\alpha_0}{\alpha_3} t^2 + \frac{\alpha_0}{\alpha_3} t + \frac{\alpha_0}{\alpha_3} \). Hence \( p \ast q \) are Hurwitz and satisfies that \( (p \ast q)(t) = f(t) \).

**Example 1.** Consider the Hurwitz polynomial \( f(t) = t^3 + 3t^2 + 3t + 1 \). We choose \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 > 0 \) such that

\[
3 > \frac{\alpha_1 \alpha_2}{\alpha_0 \alpha_3} > 1,
\]

for example \( \alpha_0 = \alpha_1 = \alpha_2 = 2 \) and \( \alpha_3 = 1 \). Then, there are two Hurwitz polynomials \( p(t) = t^3 + 2t^2 + 2t + 2 \) and \( q(t) = t^3 + \frac{3}{2}t^2 + \frac{3}{4}t + \frac{1}{2} \) satisfying the relation \( (p \ast q)(t) = f(t) \).

**Remark 1.** In [15] it was indicated that the result in theorem 1 is valid, but we have decided to include our proof because it is illustrative of our approach.

**Theorem 2.** Consider the Hurwitz polynomial \( f(t) = \alpha_4 t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0 \), which has positive coefficients. Then we have that \( f(t) \) has a Hadamard factorization in two stable polynomials if and only if

\[
\frac{\alpha_0 \alpha_2}{\alpha_1} < \left( \sqrt{\frac{\alpha_0 \alpha_4}{\alpha_2}} - \sqrt{\alpha_3} \right)^2.
\]

**Example 2.** Consider the Hurwitz polynomial \( f(t) = t^4 + t^3 + 9t^2 + t + 1 \). Let be \( \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 1, \alpha_4 = 9 \), these coefficients satisfy (6). We choose \( c_1 \) such that \( 1 < c_1 < 4 \); for example \( c_1 = 3 \). Then we take \( r(t) = t^3 + 3t^2 + t + 1 \) which is also Hurwitz. Applying theorem 1 to \( r(t) \), there exists two stable polynomial \( p(t) \) and \( q(t) \). So, we choose \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) such that satisfy (4); for example \( \alpha_1 = \alpha_2 = 2, \alpha_0 = 2 \) and \( \alpha_3 = 1 \). Now, from \( p(t) \) and \( q(t) \) are stable polynomials, using the stability test, we find two 4-degree stable polynomials:

\[
g(t) = dt^4 + t^3 + (2 + 2da)t^2 + 2t + 2,
\]

\[
h(t) = \frac{1}{da} t^4 + t^3 + \left( \frac{3}{2} + \frac{1}{2da} \right)t^2 + \frac{1}{2} t + \frac{1}{2}.
\]

Then, making the Hadamard product \( (g \ast h)(t) \) and matching with \( f(t) \), we finally solve quadratic equation resulting and we find the value of the parameter \( da \):

\[
(2 + 2da) \left( \frac{3}{2} + \frac{1}{2da} \right) = 9 \quad \Rightarrow \quad da = \frac{5 \pm \sqrt{13}}{6}.
\]
Replacing one of the roots, we get

\[ g(t) = \frac{5 - \sqrt{13}}{6} t^2 + t^3 + \left( 2 \cdot \frac{5 - \sqrt{13}}{6} \right) t^2 + 2t + 2, \]  

(10)

\[ h(t) = \frac{6}{5 - \sqrt{13}} t^4 + t^3 + \left( \frac{3}{2} \cdot \frac{3}{5 - \sqrt{13}} \right) t^2 + \frac{1}{2} t + \frac{1}{2}, \]  

(11)

and hence \((g \ast h)(t) = f(t)\).

Next, we state the general result for an \(n\)-degree stable polynomial.

**Theorem 3.** Let \( f(t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_n t^n \) be a stable polynomial with real positive coefficients. The existence of two \((n-1)\)-degree stable polynomials \( R_G(t), R_H(t) \) and one solution of the quadratic equations system:

\[
\frac{cd_i}{d_i^{\sqrt{d_i-1}}} x^2 + \left( \frac{c_{i+1}}{d_i^{\sqrt{d_i-1}}} - \alpha_i \right) x + \frac{d_i^{\sqrt{d_i-1}} \alpha_{i-1} \alpha_i}{d_i^{\sqrt{d_i-1}} \alpha_{i-1}^{\sqrt{d_i-1}}} = 0,
\]

for at least \(x \geq 0\) with \(i = 2, 4, \ldots, n - 2\) (or \(i = 1, 3, \ldots, n-2\), if \(n\) is even [odd], is equivalent to the statement that \(f(t)\) has a Hadamard stable factorization in two \(n\)-degree stable polynomials.

**Remark 2.** The last theorem leads us to solve the system of quadratic equalities (12). We are interested in a criterion that is only based in the coefficients of the polynomials. Note, in (12) the values \(d_i\) are the polynomial coefficients of \(R_G(t)\). We have tried to study such problem using results about resultants (see [27]) but we have not been able to translate this result to obtain conditions in terms of the coefficients of the given polynomial, except for the case when the polynomial has degree equal to 4.

**CONCLUSIONS**

These results open new possibilities for robust stability analysis of families with non-linear dependencies of parameters, allowing in some cases, separate parameters via Hadamard stable factorization to obtain two stable polynomials, with fewer parameters and consequently obtaining a simpler problem to solve.

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**REFERENCES**