Sufficient algebraic conditions for stability of cones of polynomials

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Abstract

In this paper a sufficient condition for a cone of polynomials to be Hurwitz is established. Such condition is a matrix inequality, which gives a simple algebraic test for the stability of rays of polynomials. As an application to stable open-loop systems, a cone of gains $c$ such that the function $u = -kc^Tx$ is a stabilizing control feedback for all $k > 0$ is shown to exist. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The aim of this paper is to derive simple algebraic conditions for the stability of rays of polynomials. Since the Routh–Hurwitz criterion and related results become usually quite complicated when applied to theoretical matters, we have developed here an ad hoc approach to the problem of obtaining characterizations of Hurwitz polynomials in terms of the corresponding coefficients. More specifically, we obtain a sufficient condition for a conic set $p_0 + K$ to consist only of stable polynomials, where $p_0$ is an $n$-degree stable polynomial and $K$ is a convex cone of $n$, $(n-1)$- or $(n-2)$-degree polynomials. In the framework of [5] this is equivalent to infinite robustness of the polynomial $p_0$ with respect to perturbations in the directions contained in $K$. It is not necessary to study the cases where $K$ is a convex cone of polynomials with degree $< n-2$ as can be seen in [5]. The algebraic sufficient condition is the matrix inequality (2.2) below which is a very simple algebraic test. In [5], the authors present sufficient Rantzer-type conditions for a ray of polynomials to be Hurwitz. We will show that the ray of polynomials given by $p_0(t) + kp_1(t)$ ($k \geq 0$), where $p_0(t) = t^3 + 6t^2 + 11t + 6$ and $p_1(t) = 5t^2 + 11t + \frac{13}{2}$, consist of Hurwitz polynomials that does not satisfy the Rantzer-type conditions proposed in [5] but satisfies condition (2.2), proposed in this paper.

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As an application of the above-mentioned result, we address the problem of designing high-gain controllers. Define a feedback control $u$ given by

$$u = -kc^T x$$

(1.1)

where $c \in \mathbb{R}^n$, and $k > 0$. If $k \geq 1$ the control feedback (1.1) is known as high-gain feedback, since high control gains $kc^T$ are induced. In the last 30 years, the high-gain control design and analysis problems have been addressed using a variety of approaches (see for instance [11,13,15–17]). In practical applications, high-gain feedback is commonly used to reduce the effects of bounded disturbance and nonlinearities on system outputs and internal stability [10]. It is well known (see [13,17]) that when $k \to \infty$ a closed-loop eigenvalue, say $\lambda_1$, has the property that $\lambda_1/k \to -c_1$ and, the other eigenvalues converge to the roots of the polynomial $c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n$. Consequently, the closed-loop system is asymptotically stable at the origin when $k$ is sufficiently large. Nevertheless, the origin is not necessarily asymptotically stable for all $k > 0$, even when $c \in \mathbb{R}^n$ is chosen in such a way that the polynomial $c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n$ is Hurwitz. Therefore, one must find a value of $k$, say $k^*$, so that the closed-loop system is asymptotically stable at the origin for all $k > k^*$. However, there are applications in which the closed-loop stability is important not only for high values of the gain parameter but also for all intermediate values since the gain can be increased only gradually. In this paper, we obtain, by means of the matrix inequality (2.2) below, a conic set of gain vectors $c$ such that the control $u = -kc^T x$ is a stabilizing control for all values of $k > 0$. We shall illustrate the above ideas with the following example.

Consider the system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ((-216k, -5k, -6k)x)$$

(1.2)

whose open-loop polynomial $p_0(t) = t^3 + 6t^2 + 11t + 6 = (t+1)(t+2)(t+3)$ is Hurwitz and its closed-loop polynomial $p_c(t)$ is given by $p_c(t) = t^3 + (6+6k)t^2 + (11+5k)t + 6 + 216k$. One of the eigenvalues, say $\lambda_1$, has the property that $\lambda_1/k \to -6$ and the other two eigenvalues converge to the roots of the polynomial $p_c^*(t) = 6t^2 + 5t + 216$. Thus, the origin is locally asymptotically stable for large values of $k$. In fact, the origin is locally asymptotically stable for $k \in [0, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$. The closed-loop system is not asymptotically stable at the origin for $k \in [2-\sqrt{2}, 2+\sqrt{2}]$ since the corresponding Hurwitz condition $(6+6k)(11+5k) - (6+216k) > 0$ is not satisfied. In this example, it is easy to see that $k^* = 2 + \sqrt{2}$. However, for higher dimensions, to find $k^*$ is a difficult problem since it requires the analysis of the inequalities determined by the Hurwitz criterion. Nevertheless, we will prove that the control feedback $-kc^T x$, where $c$ is any solution to the inequalities:

1. $\frac{121}{60} c_1 < c_2$, and
2. $\frac{6}{11} c_2 < c_3 < 6c_2 - 11c_1$,

which correspond to the matrix inequality (2.2) below, asymptotically stabilizes system (1.2) for all $k > 0$. In this work we consider only single-input open-loop stable systems since the result is not true for open-loop unstable ones. Note that even for the one-dimensional case, the closed-loop system $\dot{x} = (a-kbc)x$ is not Hurwitz for all $k > 0$ when $a > 0$.

The rest of the paper is organized as follows: in Section 2, sufficient conditions on $c = (c_1, c_2, \ldots, c_n)^T$ assuring that the corresponding polynomial $c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n$ is Hurwitz are established (inequality (2.2)). Moreover, for $c$ satisfying those conditions it is proved that the corresponding closed-loop system is asymptotically stable for every value of the high-gain parameter $k$. In Section 3 it is shown that $H$, the set of solutions to (2.2), is not empty, which is equivalent to the existence of the desired controls. An algebraic
characterization of $H$ is given as well. Finally, in Section 4 an example illustrating the results of the paper is presented.

2. Main results

The aim of this section is twofold. First, we obtain algebraic conditions for the stability of rays of polynomials (matrix inequality (2.2) below). Second, we will use this to find a conic set of gains $c$ such that the control $u = -kc^Tx$ is a stabilizing control for all values of $k > 0$.

Given a real polynomial $p(t) = t^n + a_1t^{n-1} + \cdots + a_n$ define the matrix

$$D = \begin{pmatrix}
a_2 & a_1 & 0 & 0 & \cdots & 0 & 0 \\
a_4 & -a_3 & a_2 & -a_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{n-2} & -a_{n-3} \\
0 & 0 & 0 & 0 & \cdots & -a_n & a_{n-1}
\end{pmatrix} \tag{2.1}$$

and let $D_i$, $D_j$ denote the $i$th row and the $j$th column of the matrix $D$, respectively.

**Theorem 1.** Let $p(t) = t^n + a_1t^{n-1} + \cdots + a_n$ be a Hurwitz polynomial. Let $D$ be the corresponding matrix defined by (2.1). If the vector $c$ is a solution to the system of linear inequalities

$$Dc > 0 \tag{2.2}$$

then the polynomial $f(t) = \sum_{i=1}^n c_i t^{n-i}$ is Hurwitz. Here the symbol $>$ ($<$) means that every component of a given vector is positive (negative).

**Proof.** We present the proof for $n$ even (set $n = 2m$), the case when $n$ is odd happens to be analogous. Let $F(t) = p_0(-t)$. Then, $F(t)$ is a real polynomial of degree $n$ with all its roots in $C^+$. Consider the polynomial $F(t)f(t)$, which has degree $2n - 1$. Note that $p_0(i\omega)$ and $f(i\omega)$ can be written as

$$p_0(i\omega) = P(\omega^2) + i\omega Q(\omega^2) \quad \text{and} \quad f(i\omega) = p(\omega^2) + i\omega q(\omega^2),$$

where $P$, $Q$, $p$, and $q$ are real polynomials. We have

$$F(i\omega)f(i\omega) = [P(\omega^2) - i\omega Q(\omega^2)][p(\omega^2) + i\omega q(\omega^2)] = [Pp + \omega^2 Qq] + i\omega(Pq - Qp).$$

After some calculations we get

$$(Pq - Qp)(\omega^2) = -\sum_{i=1}^n (D_i c) \omega^{2(n-i)}.$$
On the other hand, we know that at least \( n \) roots of \( F(t)f(t) \) are in \( C^+ \), then \( r \geq n \) and \( l \leq n - 1 \). Hence, \( l - r < 0 \). Additionally, \( l - r \) is an odd number since \( l + r = 2n - 1 \). Thus, the equality \( \Delta_l^{\infty} \theta = -\pi/2 \) implies that \( \Delta_l^{\infty} \theta = -\pi/2 \). Consequently, \( l - r = -1 \), from where it follows that \( r = n \) and \( l = n - 1 \). Therefore, the \( n - 1 \) roots of \( f(t) \) are contained in \( C^- \), as we claimed.

\[ \Box \]

**Remark 1.** Note that the conclusion in Theorem 1 implies that \( c = \begin{pmatrix} c_1, c_2, \ldots, c_n \end{pmatrix}^T > 0 \) or \( c = \begin{pmatrix} c_1, c_2, \ldots, c_n \end{pmatrix}^T < 0 \) and since from the first inequality in (2.2) it follows that \( c_1 > 0 \) we have that \( c > 0 \) necessarily.

Let us consider the following system:

\[ \dot{x} = Ax + bu, \tag{2.3} \]

where the pair \((A, b)\) is controllable, \( x, b \in \mathbb{R}^n \) and \( u \) is a control function. Without loss of generality, we suppose that the pair \((A, b)\) is given in the canonical form (see [2])

\[ A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.4} \]

It is well known that one of the eigenvalues of the closed-loop system \( \dot{x} = Ax - kbc^Tx \), say \( \lambda_1 \), has the property that \( \lambda_1/k \to -c_1 \) when \( k \to \infty \) and the remaining ones converge to the roots of the polynomial \( f(t) = c_1t^{n-1} + c_2t^{n-2} + \cdots + c_n \) [12, 17]. On the other hand, if \( c > 0 \) is a solution to (2.2), it follows from Theorem 1 that the polynomial \( f(t) \) is Hurwitz. Consequently, the closed-loop system is asymptotically stable at the origin for \( k \) sufficiently large. This shows that the control \( u(t) = -kc^Tx \) is a high-gain feedback. However, we have the following stronger result which is useful for the design of high-gain stabilizing feedbacks.

**Theorem 2.** Consider the linear system (2.3) written in the canonical form (2.4). Suppose \( A \) is Hurwitz, that is, the open-loop polynomial \( p_0(t) = t^n + a_1t^{n-1} + \cdots + a_n \) is Hurwitz. If \( c > 0 \) is a solution to (2.2), then, for all \( k > 0 \), the control \( u(t) = -kc^Tx \) is a stabilizing control feedback.

**Proof.** Suppose \( n \) is even (the case \( n \) odd is analogous). Let \( n = 2m \) and \( k > 0 \). It is enough to see that the closed-loop polynomial is Hurwitz. Let \( p_c(t) \) and \( p_0(t) \) denote the closed-loop and the open-loop polynomials, respectively. Consider the polynomial \( p_c(t)p_0(-t) \) and let \( \theta_1(\omega) \) be the argument of \( p_c(i\omega)p_0(-i\omega) \) and \( \Delta_0^\infty \theta_1 = \theta_1(\infty) - \theta_1(0) \) denote the net change in the argument of \( p_c(i\omega)p_0(-i\omega) \).

Following similar ideas as in the proof of Theorem 1 we get that \( |\Delta_0^\infty \theta_1| \leq \pi \). On the other hand, \( p_c(0)p_0(0) = a_{2m}(a_{2m} + kc_{2m}) \), which is a positive real number. Hence \( \theta_1(0) = \theta_1(\omega) \).

Now we will analyze \( \theta_1(\omega) \) when \( \omega \) is large. First, we have for large \( \omega \) that \( p_c(i\omega)p_0(-i\omega) \approx \omega^{4m} - ic_1\delta\omega^{3m-1} \). Therefore, \( p_c(i\omega)p_0(-i\omega) \) lies in the 4th quadrant when \( \omega \) is large and

\[ \frac{\text{Im}[p_c(i\omega)p_0(-i\omega)]}{\text{Re}[p_c(i\omega)p_0(-i\omega)]} \to 0 \]

when \( \omega \to \infty \). Hence, \( \theta_1(\infty) = 2s\pi \), where \( s \) is an integer. Then, since \( \Delta_0^\infty \theta_1 = \theta_1(\infty) - \theta_1(0) = 2s\pi \) and \( |\Delta_0^\infty \theta_1| \leq \pi \), we get that \( \Delta_0^\infty \theta_1(\omega) = 0 \).

Consequently, the polynomial \( p_c(t)p_0(-t) \) has as many roots in \( C^- \) as in \( C^+ \). Since such polynomial has degree \( 2n \), then there are \( n \) roots in \( C^+ \). In fact, the roots contained in \( C^+ \) correspond to the roots of \( p_0(-t) \).
because the open-loop polynomial \( p_0(t) \) is a Hurwitz polynomial. Finally, it follows that the \( n \) roots in \( \mathbb{C}^- \) correspond to the roots of \( p_c(t) \), which means that \( p_c(t) \) is Hurwitz. 

**Remark 2.** In the introduction it was pointed out that in general, a high-gain feedback is not necessarily a stabilizing control for every value of the parameter \( k \) and we illustrated this fact with an example. This enhances the importance of Theorems 1 and 2.

Theorem 1 can be rewritten in terms of polynomials as follows.

**Corollary.** Given a Hurwitz polynomial \( p_0(t) \) let \( G \) be the family of polynomials \( p_1(t) = c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n \) such that \( c^T = (c_1, c_2, \ldots, c_n)^T \succ 0 \) satisfies inequality (2.2). It holds that for each \( p_1 \in G \), the ray of polynomials \( p_0(t) + kp_1(t), \ k \geq 0 \) is Hurwitz.

**Remark 3.** Our results can be extended to the cases when \( \text{deg}(p_1(t)) = n \) and \( n - 2 \). Condition (2.2) must be satisfied for a matrix \( D \) below. Given a real polynomial \( p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n \) define the matrix \( D \in \mathbb{M}_{n \times (n+1)} \) by

\[
D = \begin{pmatrix}
    a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
    -a_3 & a_2 & -a_1 & 1 & \cdots & 0 & 0 \\
    a_5 & -a_4 & a_3 & -a_2 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & \cdots & -a_{n-2} & a_{n-3} \\
    0 & 0 & 0 & 0 & \cdots & a_n & -a_{n-1}
\end{pmatrix}
\]

and let \( D_i \) denote the \( i \)th row of matrix \( D \).

If \( c = (c_1, c_2, \ldots, c_{n+1})^T \succ 0 \) is a solution to \( Dc \succ 0 \), then, the polynomial \( p_1(t) = c_1 t^n + c_2 t^{n-1} + \cdots + c_{n+1} \) is Hurwitz and moreover \( p_0(t) + kp_1(t) \) is Hurwitz for all \( k \geq 0 \).

For the case when the degree of \( p_1 \) is \( n - 2 \), define the matrix \( D \in \mathbb{M}_{(n-1) \times (n-1)} \) by

\[
D = \begin{pmatrix}
    a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
    -a_3 & a_2 & -a_1 & 1 & \cdots & 0 & 0 \\
    a_5 & -a_4 & a_3 & -a_2 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & \cdots & a_{n-2} & -a_{n-3} \\
    0 & 0 & 0 & 0 & \cdots & -a_n & a_{n-1}
\end{pmatrix}
\]

Then, if \( c = (c_1, c_2, \ldots, c_{n-1})^T \succ 0 \) is a solution to \( Dc \succ 0 \) the polynomial \( p_1(t) = c_1 t^{n-2} + c_2 t^{n-3} + \cdots + c_{n-1} \) is Hurwitz and moreover \( p_0(t) + kp_1(t) \) is Hurwitz for all \( k \geq 0 \).

### 3. The set of solutions of the linear inequalities

In this section we explicitly characterize the solution set of the system of inequalities (2.2) and prove that it is not empty.
Given a Hurwitz real polynomial $p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n$, consider the matrix $D$ as defined in (2.1). Let $H$ be the set of solutions to (2.2), that is

$$H = \{c \in \mathbb{R}^n : c > 0 \text{ and } D_i c > 0, i = 1, \ldots, n\}.$$ 

It can be seen that for $n = 3$ and $4$, the set $H$ is not empty and the solutions can be explicitly given as follows:

- For $n = 3$, $H$ is the set of three-dimensional vectors ($c > 0$), whose coordinates satisfy
  1. $[a_2^2/(a_1 a_2 - a_3)]c_1 < c_2$, and
  2. $(a_3/a_2)c_2 < c_3 < a_1 c_2 - a_2 c_1$.

- For $n = 4$, $H$ is the set of four-dimensional vectors ($c > 0$), whose coordinates satisfy
  1. $[a_3(a_2 a_3 - a_1 a_4)/(a_3 a_1 a_2 - a_3) - a_1^2 a_4]c_1 < c_3$,
  2. $(a_4/a_3)c_3 < c_4 < [(a_1 a_4 - a_2 a_3)/a_4^2]c_1 + [(a_1 a_2 - a_3)/a_1^2]c_3$, and
  3. $(a_2/a_1)c_1 + (1/a_1)c_3 < c_2 < (a_4/a_3)c_1 + (a_2/a_3)c_3 - (a_1/a_3)c_4$.

In both cases, the nonemptiness of the set $H$ follows from the nonemptiness of the corresponding open intervals in the real line, which turns out to be a consequence of the stability conditions of the closed-loop system.

For higher dimensions the algebraic problem becomes more complicated. One of the methods that could be applied to address the higher-dimensional case is the elimination procedure of Kuhn–Fourier (see [7,14]). This method is a generalization of the elimination method for systems of linear equations. For the questions of existence of solutions this procedure allows one to find explicitly the whole set of solutions, by means of an iteration scheme. Although this method can be applied to solve specific problems, it is quite involved to be useful in theoretical developments.

In our case, since the matrix $A$ is stable and the matrix $D$ is defined in terms of the elements of $A$, we will be able to get the whole set of solutions of the linear inequalities by using matrices of monotone kind.

**Definition.** A $m \times s$ real matrix $R$ is a matrix of monotone kind if $Rz \succeq 0$ implies $z \succeq 0$ (see [8]).

It is known that a square real matrix $R$ is of monotone kind if and only if the inverse $R^{-1}$ exists and $R^{-1} \succeq 0$, where $\succeq 0$ means that all its entries are nonnegative for vectors or matrices (see [3]).

**Proposition 1.** The matrix $D$ is of monotone kind.

**Proof.** $\det(D)$ is a minor of the Hurwitz matrix of the stable polynomial $p_0(t)$, hence $D$ is nonsingular. We will prove that the elements of $D^{-1}$ are nonnegative analyzing the cofactors of $D$, which we denote by $\tilde{D}_{ij}$.

$\tilde{D}_{11} = \det(D)$ which is nonnegative and it is immediate that $\tilde{D}_{s1} = 0 \forall 2 \leq s \leq n$.

Now for $2 \leq m, p \leq n$ we have that $\tilde{D}_{mp}$ is a minor of order $n - 2$ of the Hurwitz matrix of $p_0(t)$.

Finally for $2 \leq r \leq n$, $\tilde{D}_{1r}$ can be written as $\tilde{D}_{1r} = b_1 M_1 + \cdots + b_{n-1} M_{n-1}$ where some $b_i$’s are zero and others are coefficients of $p_0(t)$ and the $M_i$’s are minors of order $n - 2$ of the Hurwitz matrix of $p_0(t)$.

Since the minors of a Hurwitz matrix are positive [1,6] we get that the cofactors of $D$ are nonnegative, consequently $D$ is of monotone kind. $\square$

Now, denote by $U = \{z \in \mathbb{R}^n | z_i > 0, \forall i = 1, 2, \ldots, n\}$ the positive orthant. The set of solutions to (2.2) is characterized in the next theorem.

**Theorem 3.** The set of solutions to the system of linear inequalities (2.2) can be written as $H = D^{-1} U$. 

Proof. \( H \subseteq D^{-1}U \). Let \( v \in H \), then \( v \succ 0 \) and \( Dv \succ 0 \). Consequently, \( v = D^{-1}Dv \) with \( Dv \succ 0 \). That is \( v \in D^{-1}U \).

\( H \supseteq D^{-1}U \). Let \( v \in D^{-1}U \), then \( v = D^{-1}u \) with \( u \succ 0 \). Hence, \( Dv = u \succ 0 \). Since the matrix \( D \) is a matrix of monotone kind, we finally get that \( v \succ 0 \). Therefore, \( v \in H \). \( \square \)

Remark 4. In contrast to Theorem 3, where the set of solutions to the system of linear inequalities (2.2) has been completely described \( (H = D^{-1}U) \), from the Kuhn–Fourier theorem [7] it follows the existence of a solution to (2.2) but it is not clear how to characterize the whole set \( H \).

4. An example

Consider the following system:

\[
\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (-6.5k, -11k, -5k)x. \tag{4.1}
\]

The matrix \( D \) and its inverse \( D^{-1} \) are given by

\[
D = \begin{pmatrix} 1 & 0 & 0 \\ -11 & 6 & -1 \\ 0 & -6 & 11 \end{pmatrix}, \quad D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2.0167 & 0.1833 & 0.0167 \\ 1.1 & 0.1 & 0.1 \end{pmatrix}.
\]

Then, the set of solutions to (2.2) is the following conic set:

\[
H = \{ c \in \mathbb{R}^3 | c_1 = z_1, c_2 = 2.0167z_1 + 0.1833z_2 + 0.0167z_3, \\
\phantom{H = \{ c \in \mathbb{R}^3 | } c_3 = 1.1z_1 + 0.1z_2 + 0.1z_3 \text{ for } z_1 > 0, z_2 > 0, z_3 > 0 \}.
\]

Observe that the vector \( c = (5, 11, 6.5)^T \) is an element of \( H \), since

\[
Dc = \begin{pmatrix} 5 \\ 9 \\ 11/2 \end{pmatrix} \succ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

We have that \( p_0(t) = t^3 + 6t^2 + 11t + 6 \) is the open-loop polynomial and \( p_1(t) = 5t^2 + 11t + \frac{13}{2} \) is the closed-loop polynomial of the system. Consequently, \( p_0(t) + kp_1(t), k \geq 0 \), is a ray of Hurwitz polynomials. On the other hand, we will see that it is not possible to verify that this ray consists of Hurwitz polynomials using the Rantzer-type conditions of [5] as we claimed in the Introduction.

The conditions in [5] are the following: Suppose that \( p_0 \) is a Hurwitz polynomial and \( p_1 \) is a semistable polynomial, then, the ray of polynomials \( p_0(t) + kp_1(t) \) consists of Hurwitz polynomials if one of the following four conditions holds:

(i) The difference \( d = p_1 - p_0 \) satisfies

\[
\frac{\partial \arg(d(i\omega))}{\partial \omega} < 0, \quad \omega \in \{ \omega > 0 | d(i\omega) \neq 0 \}.
\]

1 A polynomial is semistable if the real part of its roots is not strictly positive.
(ii) Each of the polynomials \( p_0, p_1 \) has at least one root in the open left half-plane and
\[
\frac{\partial \arg(d(i\omega))}{\partial \omega} < \left| \frac{\sin(2\arg(d(i\omega)))}{2\omega} \right|, \quad \omega \in \{ w > 0 \mid d(iw) \neq 0 \}.
\]

(iii) Each of the polynomials \( p_0, p_1 \) has at least one root in the open left half-plane and
\[
\frac{\partial \arg(d(i\omega))}{\partial \omega} \leq 0, \quad \omega \in \{ w > 0 \mid d(iw) \neq 0 \}.
\]

(iv) Each of the polynomials \( p_0, p_1 \) has at least two roots in the open left half-plane and
\[
\frac{\partial \arg(d(i\omega))}{\partial \omega} \leq \frac{\left| \sin(2\arg(d(i\omega))) \right|}{2\omega}, \quad \omega \in \{ w > 0 \mid d(iw) \neq 0 \}.
\]

For this example \( p_0(t) = t^3 + 6t^2 + 11t + 6, \quad p_1(t) = 5t^2 + 11t + \frac{13}{2}, \) are Hurwitz polynomials, and \( d(t), d(iw) \) and \( \arg(d(iw)) \) are given by
\[
d(t) = (p_1 - p_0)(t) = -t^3 - t^2 + \frac{1}{2}, \quad d(iw) = \frac{1}{2} + \omega^2 + i\omega^3, \quad \arg(d(iw)) = \arctan \left( \frac{\omega^3}{\frac{1}{2} + \omega^2} \right).
\]

It is not difficult to verify that (i)–(iv) are not satisfied:

1. \( \frac{\partial \arg(d(i\omega))}{\partial \omega} = \frac{3}{2}\omega^2 + \omega^4}{(\frac{1}{2} + \omega^2)^2 + \omega^6} > 0 \quad \text{for all} \quad \omega \in \{ w > 0 \mid d(iw) \neq 0 \} = (0, \infty) \). Then, (i) is not satisfied.

2. \( \sin(2\arg(d(i\omega))) = \frac{2\omega^3(\frac{1}{2} + \omega^2)}{(\frac{1}{2} + \omega^2)^2 + \omega^6}, \) hence \( \frac{\partial \arg(d(i\omega))}{\partial \omega} < \frac{\sin(2\arg(d(i\omega)))}{2\omega} \)

is satisfied if and only if
\[
\frac{\frac{3}{2}\omega^2 + \omega^4}{(\frac{1}{2} + \omega^2)^2 + \omega^6} < \frac{2\omega^3(\frac{1}{2} + \omega^2)}{2\omega(\frac{1}{2} + \omega^2)^2 + \omega^6}.
\]

that is \( \frac{3}{2} < \frac{1}{2}, \) which is a contradiction. Consequently (ii) is not satisfied.

(3) From the above inequalities it is immediate that (iii) and (iv) are not satisfied either.

Consequently, although by our results the ray \( p_0(t) + kp_1(t), \ k \geq 0 \) consist of Hurwitz polynomials, it is not possible to verify this fact using the Rantzer-type conditions obtained in [5].

References