Local Energy, Resolvents, and Wave Decay

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Preliminaries

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- The flat Minkowski metric which is given in standard coordinates (t, x_1, x_2, x_3) by

$$m = \mathrm{diag}(1, -1, -1, -1)$$

(Note: $\Box_m = \Box$)

Tataru 2013

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Assumptions

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$$g = m + \mathcal{O}(|x|^{-1})$$
where m is the Minkowski metric

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Local Energy Decay

HEURISTICS

- Energy of solutions to the wave equation within compact spatial regions decays quickly enough to be time integrable
- Trapping: an enemy to local energy decay
- Sufficiently “nice” trapping $\longrightarrow$ weak local energy decay may hold

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Results

$$|u(t,x)| \lesssim \langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-2} (\text{energy of initial data})$$

My Research

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ASSUMPTIONS

1. The metric g is stationary and asymptotically flat in the sense
$$g = m + \mathcal{O}(|x|^{-k})$$
where m is the Minkowski metric
2. The "weak local energy decay" estimate holds

RESULTS

$$|u(t,x)| \lesssim \langle t + |x| \rangle^{-?} \langle t - |x| \rangle^{-?} (\text{energy of initial data})$$

The Approach - Part I

Fixing a Coordinate System

$$\Box_g = -\Delta_x + \partial_t^2 + \partial_t P_x^1 + P_x^2$$

where $P_x^1, P_x^2 \in \mathcal{O}(r^{-k-2})$

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Defining the Resolvent

Here the resolvent is NOT an operator minus a spectral parameter.

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The resolvent R_τ is defined via the Fourier Transform:

$$P_\tau := -\Delta_x - \tau^2 - i\tau P_x^1 + P_x^2$$
$$R_\tau := P_\tau^{-1}$$

R_τ is well-defined for $\text{Im}\tau < 0$.

The Approach - Part II

Using Weak Local Energy Decay

Our definition of the resolvent (and integration by parts) yields:

$$\hat{u}(\tau) = R_\tau(i\tau u_0 - P^1 u_0 + u_1)$$

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The weak local energy decay assumption is key in obtaining L^2 bounds on the resolvent and extending R_τ to $\tau \in \mathbb{R}$.

High vs. Low Frequency

High Frequency

- The “easier” case
- Trapping is a high frequency issue and weak local energy decay handles trapping, so heuristically it makes sense that the WLED assumption is in some sense sufficient for the high frequency analysis

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Low Frequency

- Need careful analysis of R_0
- The expansion of R_0 appears to depend on the metric behavior at spatial infinity, i.e. the "flatness rate"

Conclusion

CONNECTION BETWEEN LOCAL ENERGY DECAY AND THE RESOLVENT

Metcalfe, Sterbenz, and Tataru (arXiv)

- Non-trapping, stationary, asymptotically flat
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  - Complex eigenvalues outside the continuous spectrum $\mathbb{R}$ and in the lower half-space
  - Zero eigenvalues/resonances
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QUESTION

Is it possible to classify the asymptotically flat geometries for which the restrictions given by MST apply?

Thank you for your attention!