1. Online Supplement

Proof of Theorem 4

By using Proposition 2, and by denoting $\alpha_i = \rho_{i1}$, we have (12) is equivalent to

$$\min\limits_{\alpha, \lambda, s} \sum_{i=1}^{n} (\lambda_i + \mu_i \alpha_i)$$

$$\text{s.t. } \sum_{i=k}^{\min(n, j)} \max\limits_{p_i \in D_i} ((p_i - s_i)\pi_{ij} - \alpha_ip_i - \lambda_i) \leq 0 \quad \text{for } 1 \leq k \leq n, 1 \leq k \leq j \leq n + 1 \quad (A.1)$$

For $1 \leq i \leq n$ and $1 \leq i \leq j \leq n + 1$, we have

$$\max\limits_{p_i \in D_i} (p_i - s_i)\pi_{ij} - \alpha_ip_i - \lambda_i = \begin{cases} (\pi_{ij} - \alpha_i)(\mu_i + \bar{d}_i) - \pi_{ij}s_i - \lambda_i & \text{if } \pi_{ij} \geq \alpha_i \\ (\pi_{ij} - \alpha_i)(\mu_i - \bar{d}_i) - \pi_{ij}s_i - \lambda_i & \text{if } \pi_{ij} < \alpha_i. \end{cases}$$

Then (A.1) is equivalent to

$$\sum_{i=k}^{\min(n, j)} \pi_{ij}s_i + \lambda_i \geq \sum_{i=k}^{\min(n, j)} \max\left( (\pi_{ij} - \alpha_i)(\mu_i + \bar{d}_i), (\pi_{ij} - \alpha_i)(\mu_i - \bar{d}_i) \right) \quad \text{for } 1 \leq k \leq n, 1 \leq k \leq j \leq n + 1. \quad (A.2)$$

By introducing new variables $\xi$ to replace $\max\left( (\pi_{ij} - \alpha_i)(\mu_i + \bar{d}_i), (\pi_{ij} - \alpha_i)(\mu_i - \bar{d}_i) \right)$, (A.2) is equivalent to

$$\sum_{i=k}^{\min(n, j)} \xi_{ij} \leq \sum_{i=k}^{\min(n, j)} \lambda_i + s_i\pi_{ij} \quad \text{for } 1 \leq k \leq n, 1 \leq k \leq j \leq n + 1$$

$$\xi_{ij} \geq (\pi_{ij} - \alpha_i)(\mu_i + \bar{d}_i) \quad \text{for } 1 \leq i \leq n, 1 \leq i \leq j \leq n + 1$$

$$\xi_{ij} \geq (\pi_{ij} - \alpha_i)(\mu_i - \bar{d}_i) \quad \text{for } 1 \leq i \leq n, 1 \leq i \leq j \leq n + 1.$$

This completes the proof. \(\square\)
Proof of Theorem 5

In this proof, we provide a feasible solution to primal problem (31) and show its objective value is equal to \( u(\kappa^*) \) (part 1). Then we construct a feasible solution to the dual of problem (31) and show its objective value is also equal to \( u(\kappa^*) \) (part 2). Therefore, weak duality of linear programming implies Theorem 5.

Part 1: A Primal Solution.

We first claim that there exists a feasible solution \( \alpha^* \) to the following equations.

\[
\alpha_i^* = 0 \quad \text{for} \quad i \in \Upsilon_1 \\
\alpha_i^* = \pi_{i,n+1} \quad \text{for} \quad i \in \Upsilon_2 \\
\alpha_i^* \in (0, \pi_{i,n+1}) \quad \text{for} \quad i \in \Upsilon_3 \\
\sum_{i=1}^{n} \left( \mu_i + \bar{d}_i - \frac{\alpha_i^*}{\pi_{i,n+1}} (d_i + \bar{d}_i) \right) = T. \tag{A.3}
\]

Notice that \( u(\kappa) \) is a piecewise linear concave function. If \( \kappa^* \in (0, \gamma) \), it must be one of break points, i.e. \( \kappa^* = \pi_{i,n+1} d_i / (d_i + \bar{d}_i) \) for some \( i \). Consider a sufficiently small \( \epsilon > 0 \) such that \([\kappa^* - \epsilon, \kappa^* + \epsilon]\) contains exact one break point \( \kappa^* \). Since \( u(\kappa^*) \geq u(\kappa^* - \epsilon) \), we must have

\[
\sum_{i=1}^{n} \mu_i + \sum_{i \in \Upsilon_1 \cup \Upsilon_3} \bar{d}_i - \sum_{i \in \Upsilon_2} d_i \geq T.
\]

Similarly, since \( u(\kappa^*) \geq u(\kappa^* + \epsilon) \), we must have

\[
\sum_{i=1}^{n} \mu_i + \sum_{i \in \Upsilon_1} d_i - \sum_{i \in \Upsilon_2 \cup \Upsilon_3} \bar{d}_i \leq T.
\]

Then, we define a continuous function

\[
e(\alpha) = \sum_{i=1}^{n} \left[ (\mu_i + \bar{d}_i) - \frac{\alpha_i}{\pi_{i,n+1}} (d_i + \bar{d}_i) \right].
\]

We know that \( e(\alpha) \geq T \) where \( \alpha_i = 0 \) for \( i \in \Upsilon_1 \cup \Upsilon_3 \) and \( \alpha_i = \pi_{i,n+1} \) for \( i \in \Upsilon_2 \). On the other hand, we have \( e(\alpha) \leq T \) where \( \alpha_i = 0 \) for \( i \in \Upsilon_1 \) and \( \alpha_i = \pi_{i,n+1} \) for \( i \in \Upsilon_2 \cup \Upsilon_3 \). Thus, there must exist an \( \alpha^* \in [\alpha, \bar{\alpha}] \) such that \( e(\alpha^*) = T \) which implies the existence of a feasible \( \alpha^* \) for the system of equations (A.3).

We are now ready to construct the following solution to problem (31).

\[
\xi^*_{ij} = \begin{cases} 
(\pi_{ij} - \alpha_i^*) (\mu_i + d_i) & \text{for} \quad \alpha_i^* \leq \pi_{ij} \\
(\pi_{ij} - \alpha_i^*) (\mu_i - d_i) & \text{for} \quad \alpha_i^* > \pi_{ij}
\end{cases} \tag{A.4}
\]

\[
\lambda_i^* = \xi_{ii}^*, \quad \text{for} \quad i = 1, \cdots, n \tag{A.5}
\]

\[
s_i^* = \mu_i + \bar{d}_i - \frac{\alpha_i^*}{\pi_{i,n+1}} (d_i + \bar{d}_i), \quad \text{for} \quad i = 1, \cdots, n. \tag{A.6}
\]
This solution clearly satisfies constraints (32)-(36), and thus is a feasible solution to problem (31). Next, we show that its corresponding objective value $u(\kappa^*)$.

By construction, for $i \in \Upsilon_1$, we have $\alpha_i^* = \lambda_i^* = 0$ and thus $\lambda_i^* + \mu \alpha_i^* = 0$. For $i \in \Upsilon_2$, we have $\alpha_i^* = \pi_{i,n+1}$ and $\lambda_i^* = -\pi_{i,n+1}(\mu_i - d_i)$, which implies $\lambda_i^* + \mu \alpha_i^* = \pi_{i,n+1} d_i$. For $i \in \Upsilon_3$, we have

$$
\sum_{i \in \Upsilon_3} (\lambda_i^* + \mu \alpha_i^*)
= \sum_{i \in \Upsilon_3} (-\alpha_i^* (\mu_i - d_i) + \mu \alpha_i^*)
= \sum_{i \in \Upsilon_3} \pi_{i,n+1} (d_i + \bar{d}_i) \frac{\alpha_i^*}{\pi_{i,n+1}}
$$

where the last equality follows from the fact that $\kappa^* = \pi_{i,n+1} d_i/(d_i + \bar{d}_i)$ for $i \in \Upsilon_3$. However, by (A.3), $\alpha_i^* = 0$ for $i \in \Upsilon_1$, and

$$
\sum_{i=1}^n \left( \mu_i + \bar{d}_i - \frac{\alpha_i^*}{\pi_{i,n+1}} (d_i + \bar{d}_i) \right) = T.
$$

It then follows that

$$
\kappa^* \sum_{i \in \Upsilon_3} (d_i + \bar{d}_i) \frac{\alpha_i^*}{\pi_{i,n+1}} = \kappa^* (-T + \sum_{i=1}^n (\mu_i + \bar{d}_i) - \sum_{i \in \Upsilon_2} (d_i + \bar{d}_i)).
$$

In sum,

$$
\sum_{i=1}^n (\lambda_i^* + \mu \alpha_i^*)
= \sum_{i \in \Upsilon_2} (\lambda_i^* + \mu \alpha_i^*) + \sum_{i \in \Upsilon_3} (\lambda_i^* + \mu \alpha_i^*)
= \sum_{i \in \Upsilon_2} \pi_{i,n+1} d_i + \kappa^* \sum_{i \in \Upsilon_3} (d_i + \bar{d}_i) \frac{\alpha_i^*}{\pi_{i,n+1}}
= \sum_{i \in \Upsilon_2} \pi_{i,n+1} d_i + \kappa^* \left( -T + \sum_{i=1}^n (\mu_i + \bar{d}_i) - \sum_{i \in \Upsilon_2} (d_i + \bar{d}_i) \right)
= \left( \sum_{i=1}^n \mu_i - T \right) \kappa^* + \sum_{i \in \Upsilon_1 \cup \Upsilon_3} \bar{d}_i \kappa^* + \sum_{i \in \Upsilon_2} \min(\bar{d}_i, d_i, (\pi_{i,n+1} - \kappa^*))
= \left( \sum_{i=1}^n \mu_i - T \right) \kappa^* + \sum_{i=1}^n \min(\bar{d}_i, \kappa^*, d_i, (\pi_{i,n+1} - \kappa^*))
= u(\kappa^*)
$$

where the second last equality holds by the definition of $\Upsilon_i$ for $i = 1, 2, 3$.

Part 2: A Dual Solution.

Let $\delta_{kj}$ be the dual variable associated with constraint (32) for $1 \leq k \leq n$ and $k \leq j \leq n + 1$, $\vartheta_{ij}$ be the dual variable associated with constraint (33) for $1 \leq i \leq n$ and $i \leq j \leq n + 1$, $\iota_{ij}$ be the dual
For any $i$ as that of Theorem 3, we have

**Proof of Theorem 6**

For any $i$ associated with constraint (34). Then the dual problem of (31) is.

\[
\max_{\kappa, \delta, \psi, \mu \geq 0} \sum_{i=1}^{n} \sum_{j=1}^{n+1} \left[ \pi_{ij}(\mu_i - \bar{d}_j) \psi_{ij} + \pi_{ij}(\mu_i + \bar{d}_j) \gamma_{ij} \right] - \kappa T
\]  

(A.7)

s.t. \[
\sum_{k=1}^{i} \sum_{j=1}^{n+1} \delta_{kj} = 1 \quad \text{for } 1 \leq i \leq n
\]  

(A.8)

\[
\sum_{k=1}^{i} \pi_{ij} \delta_{kj} \leq \kappa \quad \text{for } 1 \leq i \leq n
\]  

(A.9)

\[
\sum_{j=1}^{n+1} \delta_{kj} = \psi_{ij} + \gamma_{ij} \quad \text{for } 1 \leq i \leq n, 1 \leq i \leq j \leq n+1
\]  

(A.10)

\[
\sum_{j=1}^{n+1} \left[ \psi_{ij}(\mu_i - \bar{d}_j) + \gamma_{ij}(\mu_i + \bar{d}_j) \right] = \mu_i \quad \text{for } 1 \leq i \leq n.
\]  

(A.11)

We construct a dual solution as follows. Let

\[
\delta_{i,n+1}^{*} = \frac{\kappa^{*}}{\pi_{i,n+1}}
\]

\[
\delta_{i,n+1}^{*} = \frac{\kappa^{*}}{\pi_{i,n+1}} - \frac{\kappa^{*}}{\pi_{i-1,n+1}} \quad \text{for } i = 2, \ldots, n
\]

\[
\delta_{i}^{*} = 1 - \frac{\kappa^{*}}{\pi_{i,n+1}} \quad \text{for } i = 1, \ldots, n
\]

\[
\delta_{ij}^{*} = 0 \quad \text{for } i < j \leq n.
\]

For any $i$ with $\bar{d}_k \kappa^* \leq \bar{d}_l (\pi_{i,n+1} - \kappa^*)$, we let

\[
\kappa_{i,n+1}^{*} = \frac{\kappa^{*}}{\pi_{i,n+1}}
\]

\[
\kappa_{i,n+1}^{*} = \frac{\kappa^{*}}{\pi_{i,n+1}} - \frac{\kappa^{*}}{\pi_{i-1,n+1}} \quad \text{for } i = 2, \ldots, n
\]

\[
\psi_{ij} = \frac{\bar{d}_i}{\pi_{i,n+1}} \quad \text{for } i < j \leq n
\]

\[
\gamma_{ij} = 0 \quad \text{for } i < j \leq n + 1.
\]

For any $i$ with $\bar{d}_k \kappa^* > \bar{d}_l (\pi_{i,n+1} - \kappa^*)$, we let

\[
\kappa_{i,n+1}^{*} = \frac{\kappa^{*}}{\pi_{i,n+1}}
\]

\[
\kappa_{i,n+1}^{*} = \frac{\kappa^{*}}{\pi_{i,n+1}} - \frac{\kappa^{*}}{\pi_{i-1,n+1}} \quad \text{for } i = 2, \ldots, n
\]

\[
\psi_{ij} = \frac{\bar{d}_i}{\pi_{i,n+1}} \quad \text{for } i < j \leq n
\]

\[
\gamma_{ij} = 0 \quad \text{for } i < j \leq n + 1.
\]

It is straightforward to verify that the above solution is feasible to the dual problem because it ensures constraints (A.8)-(A.11) to hold as equality. Moreover, its associated objective value is

\[
(\sum_{i=1}^{n} \kappa_i - T) \kappa^* + \sum_{i=1}^{n} \min(\bar{d}_i \kappa^*, \bar{d}_l (\pi_{i,n+1} - \kappa^*))
\]

This completes the proof. \qed

**Proof of Theorem 6**

Since $\pi_{i,n+1} \geq \gamma \geq \kappa\psi$, $\pi_{i,n+1} - \kappa\psi$ must be positive and decreasing in $i$. Following the same proof as that of Theorem 3, we have

\[
G(\psi) \geq \left( \sum_{i=1}^{n} \mu_i - T \right) \kappa\psi + L_{\psi} \sum_{i=1}^{n} \min(\varphi \kappa\psi, (1 - \varphi)(\pi_{i,n+1} - \kappa\psi))
\]
\[ \geq \left( \sum_{i=1}^{n} \mu_i - T \right) \kappa_{\psi^*} + L_{\psi} \sum_{i=1}^{n} \min(\varphi \kappa_{\psi^*}, (1 - \varphi)(\pi_{i,n+1} - \kappa_{\psi^*})) \]
\[ \geq \left( \sum_{i=1}^{n} \mu_i - T \right) \kappa_{\psi^*} + L_{\psi} \sum_{i=1}^{n} \min(\varphi \kappa_{\psi^*}, (1 - \varphi)(\pi_{i,n+1} - \kappa_{\psi^*})) \]
\[ = G(\psi^*) \]

This completes the proof. \( \square \)

**Proof of Lemma 8**

Proof: We introduce a new variable \( z \) to denote \( \sum_{j=1}^{m} a_j x_j \). By assumption, for any feasible solution \( \mathbf{x} \), \( \sum_{j=1}^{m} a_j x_j \in [0, a_m] \). Then, problem (39) can be reformulated as

\[
\min_{z \in [0, a_m]} \ opt(z), \tag{A.12}
\]

where, for any given \( z \in [0, a_m] \),

\[
opt(z) = \max_{\mathbf{x}} \sqrt{\sum_{j=1}^{m} a_j^2 x_j - z^2 - bz} \tag{A.13}
\]

s.t. \( \sum_{j=1}^{m} x_j = 1 \)
\( \sum_{j=1}^{m} a_j x_j = z \)
\( x_j \geq 0, \text{ for } j = 1, \cdots, m. \)

When \( z \) is fixed, the objective function of problem (A.13) is strictly increasing in \( \sum_{j=1}^{m} a_j^2 x_j \). Thus, any optimal solution to problem (A.13) is also optimal to the following problem

\[
\max_{\mathbf{x}} \sum_{j=1}^{m} a_j^2 x_j \tag{A.14}
\]

s.t. \( \sum_{j=1}^{m} x_j = 1 \)
\( \sum_{j=1}^{m} a_j x_j = z \)
\( x_j \geq 0, \text{ for } j = 1, \cdots, m, \)

and vice versa.

We now solve problem (A.14) for any given \( z \in [0, a_m] \). The problem is a linear program with two linear constraints, besides the nonnegativity constraints. Thus, there exists an optimal solution, denoted by \( \mathbf{x}(z) \), which has at most two non-zero variables. Then suppose that the two non-zero variables are \( x_i(z) > 0 \) and \( x_k(z) \geq 0 \). And \( x_j(z) = 0 \) for all \( j \neq i, k \). Without loss of generality, we assume that \( i \leq k \). From the constraints of problem (A.14), we must have \( x_k(z) = 1 - x_i(z) \), and

\[
z = a_i x_i(z) + a_k x_k(z) = a_i x_i(z) + a_k (1 - x_i(z)) = a_k - (a_k - a_i) x_i(z).
\]
It follows that
\[ x_i(z) = \frac{a_k - z}{a_k - a_i} > 0, \quad x_k(z) = \frac{z - a_i}{a_k - a_i} \geq 0. \]
Therefore, the optimal objective value of (A.14) is given by
\[ a_i^2 x_i(z) + a_k^2 x_k(z) = a_k z + a_i (z - a_k) \leq a_k z \leq a_m z \]
where the first inequality holds because \( z - a_k \leq 0 \) and the second holds because \( a_k \leq a_m \) and \( z \geq 0 \).
That is, the optimal objective value of (A.14) is bounded above by \( a_m z \), which is attainable when \( i = 1 \) and \( k = m \). This shows that \( x_1(z) = 1 - \frac{z}{a_m}, \ x_m(z) = \frac{z}{a_m} \). Therefore, for any given \( z \in [0, a_m] \),
\[ \text{opt}(z) = \sqrt{a_m z - z^2 - bz}. \]
By Lemma 7, \( z^* = \frac{am}{2} \left[ 1 - \frac{b}{\sqrt{1+b^2}} \right] \) maximizes \( \text{opt}(z) \) in \([0, a_m]\). And \((x(z^*), z^*)\) is an optimal solution to problem (39). The lemma follows by noticing that \( x_1(z^*) = \frac{1}{2} + \frac{b}{2\sqrt{1+b^2}}, \ x_m(z^*) = \frac{1}{2} - \frac{b}{2\sqrt{1+b^2}}, \) and \( x_j(z^*) = 0 \) for \( 1 < j < m \). \( \Box \)