

Asymptotic normality in the maximum entropy models on graphs with an increasing number of parameters



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ABSTRACT

Maximum entropy models, motivated by applications in neuron science, are natural generalizations of the β -model to weighted graphs. Similar to the β -model, each vertex in maximum entropy models is assigned a potential parameter, and the degree sequence is the natural sufficient statistic. Hillar and Wibisono (2013) have proved the consistency of the maximum likelihood estimators. In this paper, we further establish the asymptotic normality for any finite number of the maximum likelihood estimators in the maximum entropy models with three types of edge weights, when the total number of parameters goes to infinity. Simulation studies are provided to illustrate the asymptotic results.

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1. Introduction

In neuron networks, neurons in one region of the brain may transmit a continuous signal using sequences of spikes to a second receiver region. The coincidence detectors in the second region capture the absolute difference in spike times between pairs of neurons projecting from the first region. There may be three possible types of timing differences: zero or nonzero indicator; countable number of possible values; any nonnegative real value. Exploring how the transmitted signal in the first region can be recovered by the second is a basic question in the analysis of neuron networks. Maximum entropy models provide a possible solution to this question for the above three possible weighted edges. For detailed explanations, see [11]; for their wide applications in the biological studies as well as other disciplines such as economics and physics, see [11,8,1,23,25] and references therein. Maximum entropy models (sometimes with different names) also appear in other fields of network analysis, e.g., community detection and social network analysis. For example, see [7,2,3,16,26,18].

In the maximum entropy models, the degree sequence is the exclusively natural sufficient statistics on the exponential family distributions and fully captures the information of an undirected graph. Its study primarily focuses on understanding the generating mechanisms of networks. When network edge takes dichotomous values (“0” or “1”), the maximum entropy model becomes the β -model (a name given by Chatterjee, Diaconis and Sly [7]), an undirected version of the p_1 model for directed graphs by Holland and Leinhardt [12]. Rinaldo, Petrović and Fienberg [18] derived necessary and sufficient conditions for the existence and uniqueness of the maximum likelihood estimate (MLE). As the number of parameters goes to infinity, Chatterjee, Diaconis and Sly [7] proved that the MLE is uniformly consistent; Yan and Xu [24] further derived its asymptotical normality. When the maximum entropy models involve the finite discrete, infinite discrete or continuous weighted edges, Hillar and Wibisono [11] have obtained the explicit conditions for the existence and uniqueness of the MLE and proved that the MLE is uniformly consistent as the number of parameters goes to infinity.

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Statistical interests are involved with not only the consistency of estimators but also its asymptotic distributions. The latter can be used to construct the confidence interval on parameters and performed the hypothesis testing. In the asymptotic framework considered in this paper, the number of network vertices goes to infinity and the number of parameter is identical to the dimension of networks (i.e., the number of vertices). Instead of studying a more complicated situation on linear combinations of all MLEs, we describe the central limit theorems for the MLEs through the asymptotic behavior of a finite number of the MLEs, although the total number of parameters goes to infinity. With this point, we aim to establish the asymptotic normality of the MLEs when edges take three types of weights as in [11]. A key step in our proofs applies a highly accurate approximate inverse of the Fisher information matrix by Yan and Xu [24].

The remainder of this article is organized as follows. In Section 2, we lay out the asymptotic distributions of the MLEs in the maximum entropy models with the finite discrete, infinite discrete and continuous weighted edges in Sections 2.1–2.3, respectively. Simulation studies are given in Section 3. Section 4 concludes with summary and discussion. All proofs are relegated to [Appendices](#).

2. Asymptotic normalities

We first give a brief description on the maximum entropy models. Consider an undirected graph \mathcal{G} with no self-loops on n vertices labeled by “1, . . . , n ”. Let a_{ij} be the weight of edge (i, j) taking values from the set Ω , where Ω could be a finite discrete, infinite discrete or continuous set. Define $d_i = \sum_{j \neq i} a_{ij}$ as the degree of vertex i , and $\mathbf{d} = (d_1, \dots, d_n)^T$ is the degree sequence of \mathcal{G} . Let \mathcal{S} be a σ -algebra over the set Ω of all possible values of a_{ij} , $1 \leq i < j \leq n$. Assume there is a canonical σ -finite probability measure ν on (Ω, \mathcal{S}) . Let $\nu^{\binom{n}{2}}$ be the product measure on $\Omega^{\binom{n}{2}}$. The maximum entropy models assume that the density function of the symmetric adjacent matrix $\mathbf{a} = (a_{ij})_{i,j=1}^n$ with respect to $\nu^{\binom{n}{2}}$ has the exponential form with the degree sequence as natural sufficient statistics,¹ i.e.,

$$p_{\theta}(\mathbf{a}) = \exp(-\theta^T \mathbf{d} - z(\theta)), \quad (1)$$

where $z(\theta)$ is the normalizing constant,

$$z(\theta) = \log \int_{\mathcal{S}^{\binom{n}{2}}} \exp(-\theta^T \mathbf{d}) \nu^{\binom{n}{2}}(d\mathbf{a}) = \log \prod_{1 \leq i < j \leq n} \int_{\mathcal{S}} \exp(-(\theta_i + \theta_j)a_{ij}) \nu(da_{ij}),$$

and for fixed n , the parameter vector $\theta = (\theta_1, \dots, \theta_n)^T$ belongs to the natural parameter space (p. 1, [5])

$$\Theta = \{\theta \in \mathbb{R}^n : z(\theta) < \infty\}.$$

The parameters $\theta_1, \dots, \theta_n$ can be interpreted as the strength of each vertex that determines how strongly the vertices are connected to each other. The probability distribution (1) implies that the edges (i, j) for all $1 \leq i < j \leq n$ are mutually independent. Since the sample is one realization of a graph, the density function in (1) is also the likelihood function. We can see that the solution to $-\nabla z(\theta) = \mathbf{d}$ is the maximum likelihood estimator (MLE) of θ .

We now consider the asymptotic distributions of the MLEs as the number of parameters goes to infinity. Let $V_n = (v_{ij})_{i,j=1,\dots,n}$ be the Fisher information matrix of the parameters $\theta_1, \dots, \theta_n$. It can be written as

$$V_n = \frac{\partial^2 z(\theta)}{\partial \theta \partial \theta^T}.$$

For three common types of weights as introduced in Section 1, V_n is the diagonal dominant matrix with nonnegative entries. This property is crucially used in the proof of the central limit theorem on the MLE.

2.1. Finite discrete weights

When network edges take finite discrete weights, we assume $\Omega = \{0, 1, \dots, q-1\}$ with q a fixed integer. In this case, ν is the counting measure and the edge weights a_{ij} are independent multinomial random variables with the probability:

$$P(a_{ij} = a) = \frac{e^{a(\theta_i + \theta_j)}}{\sum_{k=0}^{q-1} e^{k(\theta_i + \theta_j)}}, \quad a = 0, 1, \dots, q-1.$$

This model is a direct generalization of the β -model that only consider the dichotomous edges. The normalizing constant is

$$z(\theta) = \sum_{1 \leq i < j \leq n} \log \sum_{a=0}^{q-1} e^{-(\theta_i + \theta_j)a},$$

¹ Following Hillar and Wibisono [11], we use $-\theta$ in the parameterization (1) instead of the classical θ since it will simplify the notations in the later presentation.

and the parameter space is $\Theta = \mathbb{R}^n$. Let $\widehat{\theta} = (\widehat{\theta}_1, \dots, \widehat{\theta}_n)^T$ be the MLE of $\theta = (\theta_1, \dots, \theta_n)^T$. The likelihood equations are

$$d_i = \sum_{j=1:j \neq i}^n \sum_{a=0}^{q-1} \frac{ae^{a(\widehat{\theta}_i + \widehat{\theta}_j)}}{\sum_{k=0}^{q-1} e^{k(\widehat{\theta}_i + \widehat{\theta}_j)}}, \quad i = 1, \dots, n, \tag{2}$$

which is identical to the moment estimating equations. The fixed point iteration algorithm by Chatterjee et al. [7] or the minorization–maximization algorithm by Hunter [13] can be used to solve the above system of equations or analogous problems in the next two subsections. Following Hillar and Wibisono [11], we assume that $\max_{1 \leq i \leq n} |\theta_i|$ is bounded by a constant when considering the asymptotic distribution of the MLE. The central limit theorem for the MLE is stated as follows, whose proof is given in Appendix A.

Theorem 1. *In the case of finite discrete weights, the diagonal entries of V_n has the following representation:*

$$v_{ii} = \sum_{j=1:j \neq i}^n \frac{\sum_{0 \leq k < l \leq q-1} (k-l)^2 e^{(k+l)(\theta_i + \theta_j)}}{\left(\sum_{a=0}^{q-1} e^{a(\theta_i + \theta_j)}\right)^2}, \quad i = 1, \dots, n.$$

Assume that $\max_{1 \leq i \leq n} |\theta_i|$ is bounded by a fixed constant. Then for any fixed $r \geq 1$, the vector $(v_{11}^{1/2}(\widehat{\theta}_1 - \theta_1), \dots, v_{rr}^{1/2}(\widehat{\theta}_r - \theta_r))$ is asymptotically standard multivariate normal as $n \rightarrow \infty$.

Notice that $\widehat{\theta}_1, \dots, \widehat{\theta}_r$ are asymptotically mutually independent by the above theorem. This is due to that the maximum entropy models imply the mutually independent edges in a graph. It further implies that V_n^{-1} is an approximate diagonal matrix shown in Proposition 1 such that $\widehat{\theta}_1, \dots, \widehat{\theta}_r$ are asymptotically mutually independent.

2.2. Continuous weights

When network edges take continuous weights, $\Omega = [0, \infty)$, ν is the Lebesgue measure on $[0, \infty)$ and the normalizing constant is

$$z(\theta) = - \sum_{1 \leq i < j \leq n} \log(\theta_i + \theta_j).$$

Therefore, the corresponding natural parameter space is

$$\Theta = \{(\theta_1, \dots, \theta_n) \in \mathbb{R}^n: \theta_i + \theta_j > 0 \text{ for } i \neq j\}.$$

The edge weights a_{ij} ($1 \leq i < j \leq n$) are independently distributed by exponential distributions with density

$$p(a) = (\theta_i + \theta_j) \exp(-(\theta_i + \theta_j)a), \quad a > 0, \quad 1 \leq i < j \leq n,$$

whose expectation is $(\theta_i + \theta_j)^{-1}$. The likelihood equations are

$$d_i = \sum_{j \neq i} \frac{1}{\widehat{\theta}_i + \widehat{\theta}_j}, \quad i = 1, \dots, n.$$

The asymptotic distribution of the MLE is stated as follows, whose proof is given in Appendix B.

Theorem 2. *Let $L_n = \min_{i \neq j} (\theta_i + \theta_j) > 0$ and $M_n = \max_{i \neq j} (\theta_i + \theta_j)$. In the case of continuous weights, the diagonal entries of V_n are:*

$$v_{ii} = \sum_{j=1:j \neq i}^n \frac{1}{(\theta_i + \theta_j)^2}, \quad i = 1, \dots, n.$$

If $M_n/L_n = o\{n^{1/16}/(\log n)^{1/8}\}$, then for any fixed $r \geq 1$, the vector $(v_{11}^{1/2}(\widehat{\theta}_1 - \theta_1), \dots, v_{rr}^{1/2}(\widehat{\theta}_r - \theta_r))$ is asymptotically standard multivariate normal as $n \rightarrow \infty$.

2.3. Infinite discrete weights

When edges take infinite discrete weights, we assume that $\Omega = \{0, 1, \dots\}$. In this case, ν is the counting measure, the normalizing constant is

$$z(\theta) = \sum_{1 \leq i < j \leq n} \log \sum_{a=0}^{\infty} \exp(-(\theta_i + \theta_j)a) = \sum_{1 \leq i < j \leq n} -\log(1 - \exp(-(\theta_i + \theta_j))),$$

and the natural parameter space is

$$\Theta = \{(\theta_1, \dots, \theta_n) \in \mathbb{R}^n: \theta_i + \theta_j > 0 \text{ for } i \neq j\}.$$

The edge weights a_{ij} are independent geometric random variables with probability mass function:

$$\mathbb{P}^*(a_{ij} = a) = (1 - \exp(-\theta_i - \theta_j)) \exp(-(\theta_i + \theta_j)a), \quad 1 \leq i < j \leq n.$$

The likelihood equations are

$$d_i = \sum_{j \neq i} \frac{1}{\exp(\hat{\theta}_i + \hat{\theta}_j) - 1}, \quad i = 1, \dots, n,$$

which are identical to the moment estimating equations.

Theorem 3. Let $L_n = \min_{i \neq j}(\theta_i + \theta_j) > 0$ and $M_n = \max_{i \neq j}(\theta_i + \theta_j)$. In the case of infinite discrete weights, the diagonal entries of V_n are

$$v_{ii} = \sum_{j=1; j \neq i}^n \frac{e^{\theta_i + \theta_j}}{(e^{\theta_i + \theta_j} - 1)^2}, \quad i = 1, \dots, n.$$

If $e^{17M_n}/L_n^3 = o\{n^{1/2}/\log n\}$, then for any fixed $r \geq 1$, the vector $(v_{11}^{1/2}(\hat{\theta}_1 - \theta_1), \dots, v_{rr}^{1/2}(\hat{\theta}_r - \theta_r))$ is asymptotically standard multivariate normal as $n \rightarrow \infty$.

Remark 1. By Theorems 1–3, we have: (1) for any fixed r , $\hat{\theta}_1, \dots, \hat{\theta}_r$ are asymptotically independent; (2) as $n \rightarrow \infty$, the convergence rate of $\hat{\theta}_i$ is $1/v_{ii}^{1/2}$. If M_n and L_n are constants, then this convergence rate is in the magnitude of $n^{-1/2}$; otherwise it is between $O\{1/(n^{1/2}L_n)\}$ and $O\{1/(n^{1/2}M_n)\}$ when edges take continuous weights, between $O\{(e^{M_n} - 1)/[n^{1/2}e^{-M_n/2}]\}$ and $O\{(e^{L_n} - 1)/[n^{1/2}e^{-L_n/2}]\}$ when edges take infinite discrete weights. To compare with the convergence rate in the continuous and infinite discrete cases, we consider a special case $M_nL_n = 1$. Since $e^{M_n/2}(e^{M_n} - 1) \gg M_n = L_n^{-1}$ and $(e^{L_n} - 1)e^{L_n/2} \sim L_n = M_n^{-1}$ when M_n is large enough, the former is faster than the latter. This can be understood that a lower convergence rate can be incurred if the parameter vector is more quickly close to the boundary of the mean parameter space by noting that $\mathbb{E}(d_i) = \sum_{j \neq i}(\theta_i + \theta_j)^{-1}$ in the continuous case and $\mathbb{E}(d_i) = \sum_{j \neq i}(e^{\theta_i + \theta_j} - 1)^{-1}$ in the infinite discrete case.

Remark 2. In contrast with the conditions (i.e., $M_n^2/L_n = o(n^{1/2}/(\log n)^{1/2})$ in the continuous case and $e^{5M_n}(e^{L_n/2} - 1)^{-1/2} = o(n^{1/2}/(\log n)^{1/2})$ in the infinite discrete case) guaranteeing the consistency of the MLE by Hillar and Wibisono [11], the ones for asymptotic normality seems much more strict. The simulations in the next section suggest there may be space for improvement. On the other hand, the consistency and asymptotic normality for the MLE in the finite discrete case requires the assumption that all parameters are bounded by a constant. This assumption may not be best possible. We will investigate these problems in the future.

Remark 3. The three theorems in this section only describe the joint asymptotic distribution of the first r estimators $\hat{\theta}_1, \dots, \hat{\theta}_r$ with a fixed constant r . Actually, the starting point of subscripts is not essential. These three theorems hold for any fixed r MLEs. Since the usual counting subscript starts from 1, we only show the case presented in the theorems. Our proofs can be directly extended to the case of any r fixed MLEs $\hat{\theta}_{i_1}, \dots, \hat{\theta}_{i_r}$ without any difficulty. Another interesting problem is investigating the asymptotic distribution of the linear combination $\sum_{i=1}^n c_i \hat{\theta}_i$ on all the MLEs or a linear combination with a growing number of the MLEs as pointed by one referee. Are there results similar to Propositions 2–4(2)? We will investigate this problem in future work.

Remark 4. According to Theorems 1–3, an approximate $100(1 - \alpha)\%$ confidence interval for $\theta_i - \theta_j$ is $\hat{\theta}_i - \hat{\theta}_j \pm Z_{1-\alpha/2}(1/\hat{v}_{ii} + 1/\hat{v}_{jj})^{1/2}$, where \hat{v}_{ii} and \hat{v}_{jj} are the natural estimates of v_{ii} and v_{jj} by replacing all $\theta_1, \dots, \theta_n$ by their MLEs, and Z_β denotes the 100β percentile point of the standard normal distribution. To test whether $\theta_i = \theta_j$ at level α , the hypothesis can be rejected if $|\hat{\theta}_i - \hat{\theta}_j| > Z_{1-\alpha/2}(1/\hat{v}_{ii} + 1/\hat{v}_{jj})^{1/2}$. The confidence intervals for contrasts and the hypothesis test for the equality of two parameters can be generalized to multiple parameters. For example, one can use the test statistic

$$(\hat{\theta}_1 - \hat{\theta}_2, \hat{\theta}_2 - \hat{\theta}_3, \hat{\theta}_3 - \hat{\theta}_4) \begin{pmatrix} \frac{1}{\hat{v}_{11}} + \frac{1}{\hat{v}_{22}} & -\frac{1}{\hat{v}_{22}} & 0 \\ -\frac{1}{\hat{v}_{22}} & \frac{1}{\hat{v}_{22}} + \frac{1}{\hat{v}_{33}} & -\frac{1}{\hat{v}_{33}} \\ 0 & -\frac{1}{\hat{v}_{33}} & \frac{1}{\hat{v}_{33}} + \frac{1}{\hat{v}_{44}} \end{pmatrix}^{-1} \begin{pmatrix} \hat{\theta}_1 - \hat{\theta}_2 \\ \hat{\theta}_2 - \hat{\theta}_3 \\ \hat{\theta}_3 - \hat{\theta}_4 \end{pmatrix}$$

to test whether $\theta_1 = \theta_2 = \theta_3 = \theta_4$, which asymptotically follows the chi-square distribution with the degree of freedom 3.

3. Simulations

In this section, we will evaluate the asymptotic results for maximum entropy models on weighted graphs with continuous and infinite discrete weights through numerical simulations. The simulation results for finite discrete weights are similar to the binary case, which has been shown in [24], so we do not repeat it here. Firstly, we study the consistency of the estimation. We plot the estimated $\hat{\theta}$ vs. θ to evaluate the accuracy. Secondly, by Theorems 2 and 3, $\hat{v}_{ii}^{1/2}(\hat{\theta}_i - \theta_i)$ and $(\hat{\theta}_i + \hat{\theta}_j - \theta_i - \theta_j)/(1/\hat{v}_{ii} + 1/v_{ii})^{1/2}$ are asymptotically normally distributed, where \hat{v}_{ii} is the estimator of v_{ii} by replacing θ_i with $\hat{\theta}_i$. The quantile–quantile (QQ) plots of $\hat{v}_{ii}^{1/2}(\hat{\theta}_i - \theta_i)$ are shown. We also report the 95% coverage probabilities for certain $\theta_i - \theta_j$, as well as the probabilities that the maximum likelihood estimator does not exist in the case of discrete weights. The parameter settings in simulation studies are listed as follows. For continuous weights, let $\theta_i = \tilde{M} + i\tilde{M}^2/n$, $i = 1, \dots, n$ such that $L_n \approx \tilde{M}$, $M_n \approx \tilde{M}^2$ and $M_n/L_n \approx \tilde{M}$; for discrete weights, let $\theta_i = 0.1 + i\tilde{M}/n$, $i = 1, \dots, n$ such that $L_n \approx 0.1$, $M_n \approx \tilde{M}$, and $e^{M_n}/L_n \approx 10e^{\tilde{M}}$. Here, we suppress the subscript n of \tilde{M} in order to conveniently display the notations in the figures. A variety of \tilde{M} are chosen: $\tilde{M} = 1, \log(n), n^{1/2}, n$ for continuous weights; $\tilde{M} = 0, \log(\log n), (\log n)^{1/2}, \log n$ for discrete weights.

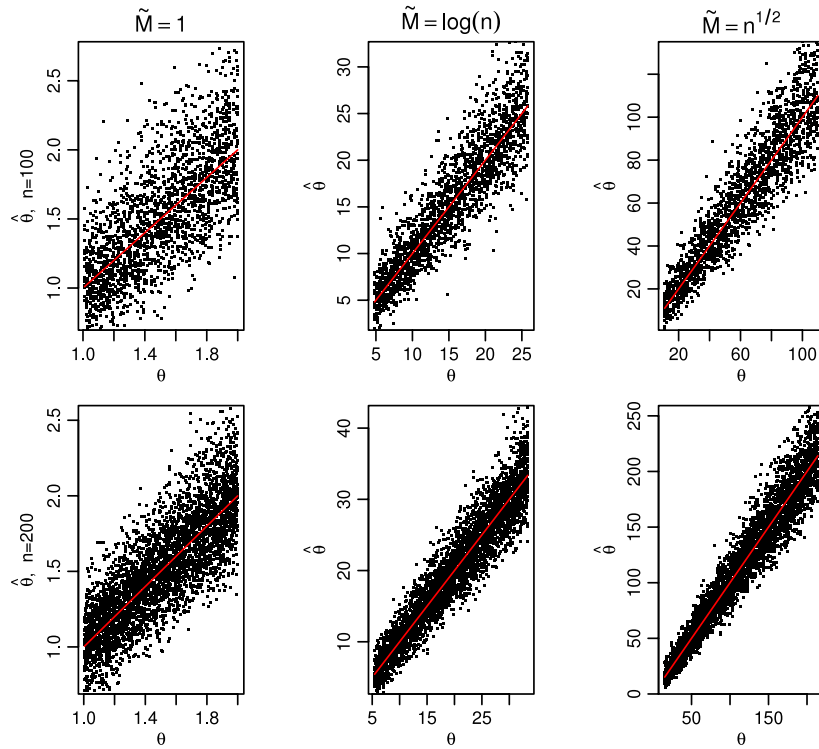
The plots of $\hat{\theta}_i$ vs. θ_i are shown in Fig. 1. We used $\tilde{M} = 1$ in this figure for the case of discrete weights instead of $\tilde{M} = 0$ in order to make θ_i vary (when $\tilde{M} = 0$, all the θ_i , $1 \leq i \leq n$ equal to 0.1). The red lines correspond to the case that $\theta = \hat{\theta}$. For each sub-figure, the first and second rows represent $n = 100$ and $n = 200$, respectively. The first, second and third columns represent $\tilde{M} = 1, \log(n), n^{1/2}$ for continuous weights and $\tilde{M} = 1, \log \log(n), (\log n)^{1/2}$ for discrete weights, respectively. From this figure, we can see that as n increases, the estimators become more close to the true parameters. As \tilde{M} increases, $\max_i |\hat{\theta}_i - \theta_i|$ becomes much larger, indicating that controlling the increasing rate of M_n (or decreasing rate of L_n) is necessary. For continuous weights, when $\tilde{M} = n^{1/2}$, $\max_i |\hat{\theta}_i - \theta_i|$ are very large, exceeding 30; for discrete weights, when $\tilde{M} = (\log(n))^{1/2}$, the points of $\hat{\theta}$ vs. θ diverge, indicating that $\hat{\theta}_i$ may not be the consistent estimate of θ_i in this case. Therefore, the conditions to guarantee the consistency results in Theorems 2 and 3 seem to be reasonable.

The QQ plots in Figs. 2 and 3 are based on 5000 repetitions for each scenario. The horizontal and vertical axes are the empirical and theoretical quantiles, respectively. The red lines correspond to $y = x$. The coverage frequencies are reported in Table 1. When $\tilde{M} = \log n$, the MLEs for the case of discrete weights do not exist with 100% frequencies. Therefore, the QQ plots for this case are not available. In Fig. 2, when $\tilde{M} = 1, \log n, n^{1/2}$, the sample quantiles coincide with the theoretical ones very well (the plot of the case of $\tilde{M} = 1$ is similar to that of $\tilde{M} = \log n$ and is not shown here). On the other hand, when $\tilde{M} = n$, the sample quantile of $\hat{v}_{11}^{1/2}(\hat{\theta}_1 - \theta_1)$ evidently deviates from the theoretical one. In this case, the estimated variances \hat{v}_{ii} of d_i are very small, approaching to zero. For example, when $n = 200$ and $\tilde{M} = n$, the estimated \hat{v}_{ii} is in the magnitude of $10^{-6} \sim 10^{-8}$, where the central limit theorem cannot be expected according to the classical large sample theory. In Fig. 3, the approximation of asymptotic normality is good when $\tilde{M} = 0$ and $\log(\log n)$; while there are notable derivations for $\hat{v}_{nn}^{1/2}(\hat{\theta}_n - \theta_n)$ when $\tilde{M} = (\log n)^{1/2}$.

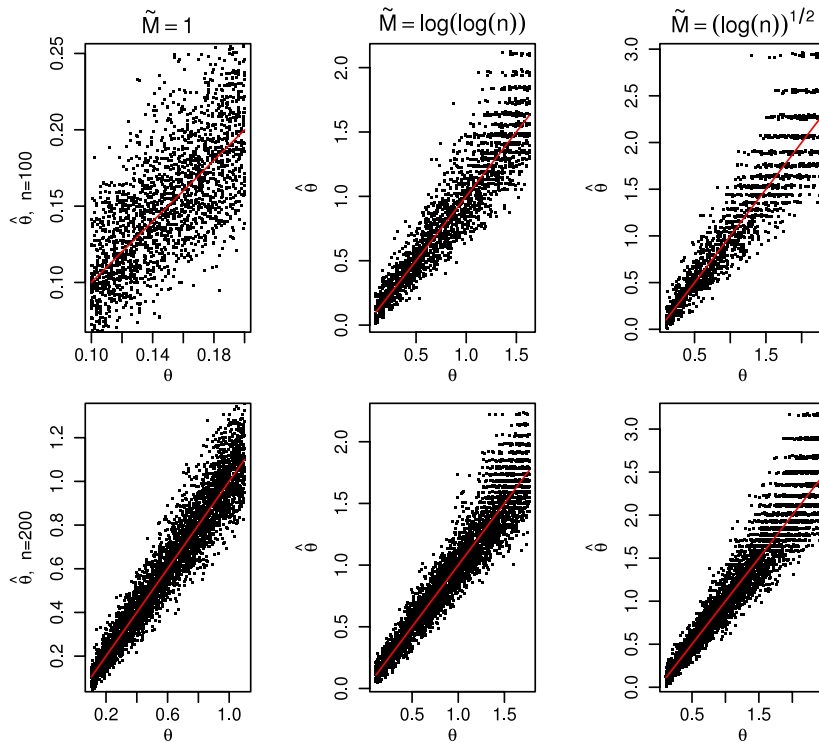
In both cases of continuous and discrete weights of Table 1, the length of estimated confidence intervals increases as \tilde{M} becomes larger when n is fixed. In the case of continuous weights, when $\tilde{M} = 1, \log(n)$, the length of estimated confidence intervals decreases as n increases; but when $\tilde{M} = n^{1/2}$ and n , it instead becomes larger. This is because v_{ii} (between $n(2\tilde{M})^{-2}$ and $n/(4\tilde{M}^4)$) goes to zero as n increases when $\tilde{M} \geq n^{1/2}$, leading to a larger confidence interval. In particular, when $\tilde{M} = n$, some of them exceed 10000, indicating an extremely inaccurate estimate, although the corresponding coverage probabilities are close to 95%. In the case of discrete weights, when $\tilde{M} = 0, \log(\log n)$ and $n \geq 100$, the coverage frequencies are close to the nominal level; when $\tilde{M} = (\log n)^{1/2}$, the coverage frequencies of pair $(n - 1, n)$ are higher than the nominal level; when $\tilde{M} = \log n$ that greatly exceeds the condition of Theorem 3, the MLE almost does not exist. These phenomena further suggest that controlling increasing rate of M_n or decreasing rate of L_n in Theorems 2 and 3 is necessary.

4. Summary and discussion

Investigating the asymptotic theories for the network models are open and challenging problems, especially when the number of parameters increases with the size of network. One reason is that network data are not a standard type of data. In a traditional statistical framework, the number of parameters is fixed and the number of samples goes to infinity. In the asymptotic scenario considered in this paper, the sample is only one realization of a random graph and the number of parameters is identical to that of vertices. However, the MLE in some simple undirected models with the degree sequence as the exclusively natural sufficient statistics (i.e., the maximum entropy models) have been derived. As the number of parameters goes to infinity, we obtain the asymptotic normality of the MLE in the maximum entropy models for a class of weighted edges, the proofs of which are in help of the approximated inverses of the Fisher information matrix. We expect that the methods of our proofs can be applied to other high-dimensional cases in which the Fisher information matrix are nonnegative and diagonally dominant or other similar cases. For example, Perry and Wolfe [17] introduced a family of null models for network data in which the entries of the upper triangle matrix of \mathbf{a} are assumed independent Bernoulli random variables with success probabilities $\exp[\theta_i + \theta_j + \varepsilon(\theta_i, \theta_j)]$ for $1 \leq i < j \leq n$, where $\varepsilon(\theta_i, \theta_j)$ are smooth functions on parameters θ_i and θ_j . By making some assumptions of the second derivative of ε , the Fisher information matrix of the parameters also shares the similar properties like those in the maximum entropy models.



(a) Continuous weights.



(b) Discrete weights.

Fig. 1. The plots of $\hat{\theta}_i$ vs. θ_i for $1 \leq i \leq n$ based on 20 samples. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

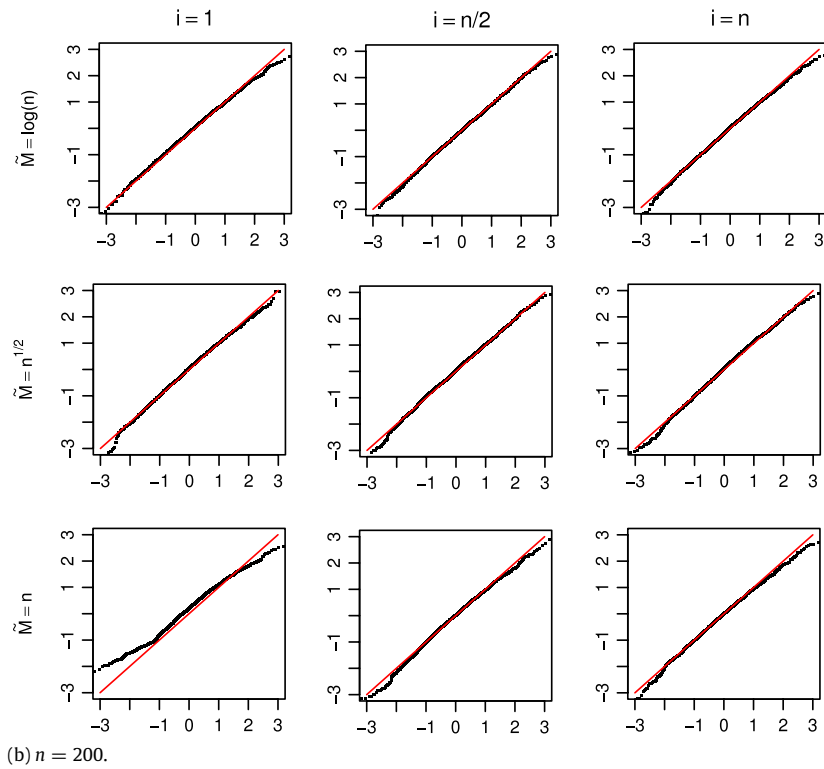
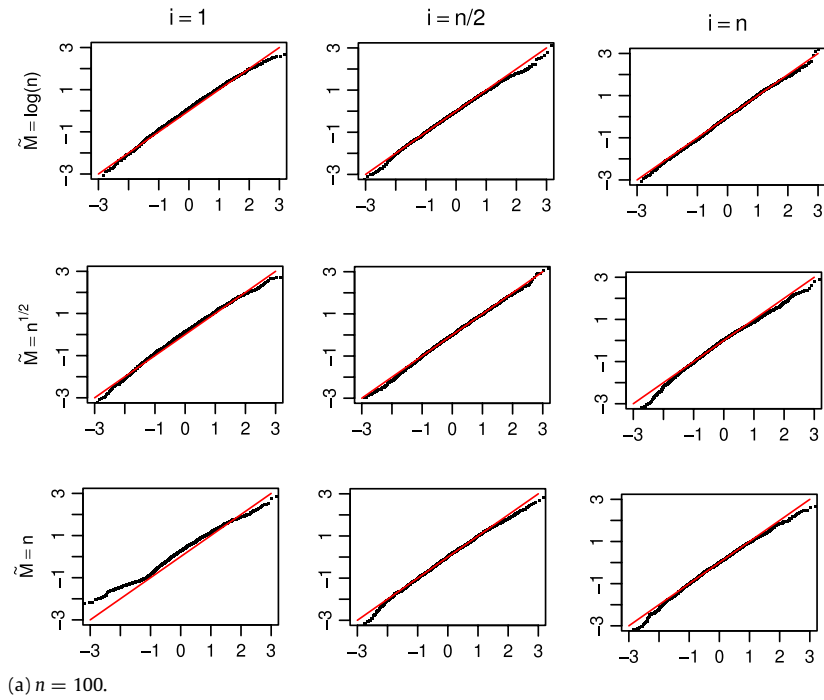


Fig. 2. The QQ plots of $\hat{v}_i^{1/2}(\hat{\theta}_i - \theta_i)$ in the case of continuous weights. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Finally, we shed some light on why the consistency and asymptotic normality of the MLE can be achieved in the maximum entropy models, even though the dimension of parameters increases with the size of network and the sample is only one realization of a random graph. First, in an undirected random graph, it lurks with $n(n - 1)/2$ random variables, which are higher order than the number of parameters. Second, the Fisher information of each parameter are combinations of $n - 1$

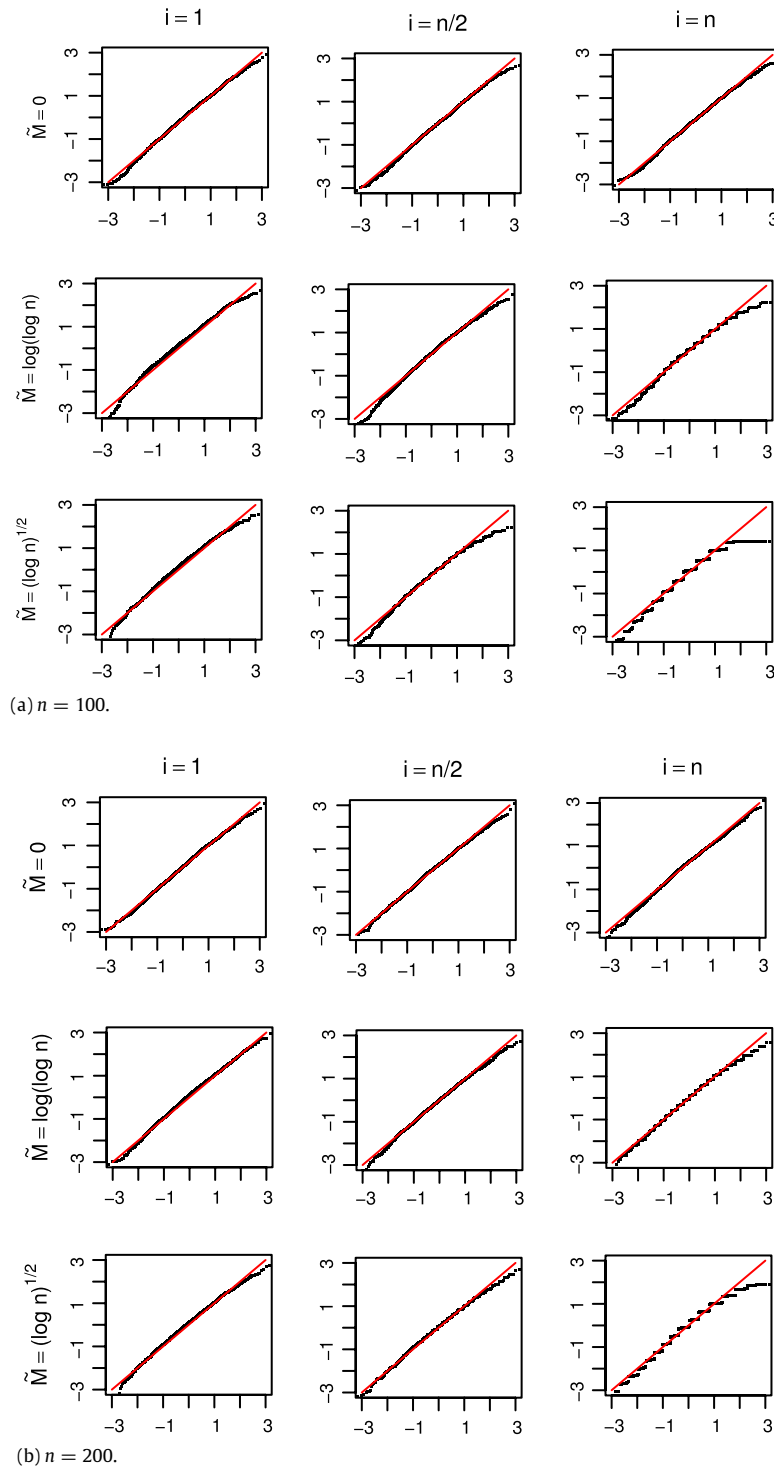


Fig. 3. The QQ plots of $\hat{v}_i^{1/2}(\hat{\theta}_i - \theta_i)$ in the case of discrete weights. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

variances of $n - 1$ random variables. Under some conditions, it goes to infinity as n increases. Third, the assumption of independent edges avoid the degeneracy problem, unlike Markov dependent exponential random graphs [9]. The model degeneracy problems of the exponential random graphs have received wide attention (e.g., [22,21,10,14,6,19]). Moreover, considering the case that the number of parameters is fixed, Shalizi and Rinaldo [20] demonstrated that exponential random graph models are projective in the sense of that the same parameters can be used for the full network and for any of its

Table 1

Estimated coverage probabilities of $\theta_i - \theta_j$ for pair (i, j) as well as the probabilities that the MLE does not exist (in parentheses), multiplied by 100, and the length of confidence intervals (in square brackets).

Weighted random graphs with continuous weights					
n	(i, j)	$\tilde{M} = 1$	$\tilde{M} = \log n$	$\tilde{M} = n^{1/2}$	$\tilde{M} = n$
50	(1, 50)	95.55[2.00]	95.40[11.83]	95.62[28.08]	96.35[730.54]
	(25, 26)	95.63[2.39]	95.15[17.93]	95.57[48.59]	95.95[1889.10]
	(49, 50)	95.25[2.80]	95.64[24.03]	95.25[69.17]	96.15[2932.88]
100	(1, 100)	95.43[1.39]	95.25[9.84]	94.75[30.29]	95.45[1409.94]
	(50, 51)	94.75[1.67]	95.60[16.20]	95.25[62.13]	95.05[5059.72]
	(99, 100)	95.55[1.95]	94.85[22.29]	95.25[91.70]	96.45[8060.81]
200	(1, 200)	95.51[0.97]	95.45[8.21]	95.45[33.93]	95.35[2750.85]
	(100, 101)	95.10[1.17]	95.05[14.28]	95.05[81.45]	95.35[13 958.43]
	(199, 200)	95.36[1.37]	94.67[20.10]	95.39[123.43]	95.59[22 556.82]
Weighted random graphs with discrete weights					
n	(i, j)	$\tilde{M} = 0$	$\tilde{M} = \log(\log n)$	$\tilde{M} = (\log n)^{1/2}$	$\tilde{M} = \log(n)$
50	(1, 50)	95.55[0.16](0)	94.37[0.56](1.35)	95.04[0.71](51.55)	(100)
	(25, 26)	95.10[0.16](0)	96.45[1.30](1.35)	97.27[2.01](51.55)	(100)
	(49, 50)	95.95[0.16](0)	97.52[2.23](1.35)	100.00[3.65](51.55)	(100)
100	(1, 100)	95.17[0.11](0)	94.45[0.37](0.05)	95.46[0.46](16.75)	(100)
	(50, 51)	95.15[0.11](0)	95.75[0.99](0.05)	95.34[1.45](16.75)	(100)
	(99, 100)	94.95[0.11](0)	95.85[1.74](0.05)	98.91[3.00](16.75)	(100)
200	(1, 200)	95.15[0.08](0)	94.68[0.26](0)	94.77[0.31](1.60)	(100)
	(100, 101)	94.85[0.08](0)	95.55[0.75](0)	95.57[1.09](1.60)	(100)
	(199, 200)	95.45[0.08](0)	95.62[1.33](0)	97.51[2.28](1.60)	(100)

subnetworks simultaneously, essentially only for those models with the assumption of dyadic independence, under which the consistency of the MLE is available.

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Appendix A

For fixed r , the central limit theorem for the vector (d_1, \dots, d_r) can be easily derived by noting that d_1, \dots, d_r are asymptotically independent. In view of that d_i is the sufficient statistic on θ_i , $\hat{\theta}_i$ may be approximately represented as a function of d_i . If this can be done, then the asymptotic distribution for the MLE may follow. In order to establish the relationship between $\hat{\theta}_i$ and d_i , we will approximate the inverse of a class of matrices. We say an $n \times n$ matrix $V_n = (v_{ij})$ belongs to a matrix class $\mathcal{L}_n(m, M)$ if V_n is a symmetric nonnegative matrix satisfying

$$v_{ii} = \sum_{j=1, j \neq i}^n v_{ij}; \quad M \geq v_{ij} = v_{ji} \geq m > 0, \quad i \neq j.$$

Yan and Xu [24] have proposed to use $\tilde{S}_n = \text{diag}(1/v_{11}, \dots, 1/v_{nn}) + v_{..}^{-1} \mathbf{1}_n \mathbf{1}'_n$ to approximate V_n^{-1} , where $\mathbf{1}_n$ is a vector of n entries whose values are all of 1 and $v_{..} = \sum_{i=1}^n v_{ii}$, and obtained an upper bound on the approximate errors. Here we use a simpler matrix $S_n = \text{diag}(1/v_{11}, \dots, 1/v_{nn})$ to approximate V_n^{-1} . Let $\|A\| = \max_{i,j} |a_{ij}|$ for a general matrix $A = (a_{ij})$. It is clear that $\|A + B\| \leq \|A\| + \|B\|$ for two matrices A and B , and $\|\tilde{S}_n - S_n\| = v_{..}^{-1} \leq (mn(n - 1))^{-1}$. By Proposition A1 in [24], we have:

Proposition 1. *If $V_n \in \mathcal{L}_n(m, M)$, then for $n \geq 3$, the following holds:*

$$\begin{aligned} \|W_n := V_n^{-1} - S_n\| &\leq \|V_n^{-1} - \tilde{S}_n\| + \|\tilde{S}_n - S_n\| \\ &\leq \frac{M(nM + (n - 2)m)}{2m^3(n - 2)(n - 1)^2} + \frac{1}{2m(n - 1)^2} + \frac{1}{mn(n - 1)}. \end{aligned}$$

Note that $d_i = \sum_{j \neq i} a_{ij}$ are sums of $n - 1$ independent multinomial random variables. By the central limit theorem for the bounded case ([15], p. 289), $v_{ii}^{-1/2} \{d_i - \mathbb{E}(d_i)\}$ is asymptotically standard normal if v_{ii} diverges. Following Hillar and Wibisono [11], we assume that $\max_i |\theta_i|$ is bounded by a constant in this appendix. For convenience, we assume that $\max_i |\theta_i| \leq L/2$ with L a fixed constant. Thus, $\max_{i,j} |\theta_i + \theta_j| \leq L$. Since

$$e^{2k(\theta_i + \theta_j)} \leq e^{(k+(k-1))(\theta_i + \theta_j) + L}, \quad 1 \leq k \leq q - 1,$$

we have

$$\sum_{k=0}^{q-1} e^{2k(\theta_i + \theta_j)} \leq \sum_{0 \leq k \leq q-1} e^{(k+l)(\theta_i + \theta_j)} e^L.$$

Therefore,

$$\begin{aligned} \frac{\frac{1}{2} \sum_{k \neq l} e^{(k+l)(\theta_i + \theta_j)}}{\left(\sum_{a=0}^{q-1} e^{a(\theta_i + \theta_j)}\right)^2} &= \frac{\frac{1}{2} \sum_{k \neq l} e^{(k+l)(\theta_i + \theta_j)}}{\sum_{k \neq l} e^{(k+l)(\theta_i + \theta_j)} + \sum_{k=0}^{q-1} e^{2k(\theta_i + \theta_j)}} \\ &\geq \frac{\sum_{k \neq l} e^{(k+l)(\theta_i + \theta_j)}}{2(1 + e^L) \sum_{k \neq l} e^{(k+l)(\theta_i + \theta_j)}} \geq \frac{1}{2(1 + e^L)}. \end{aligned} \tag{3}$$

Recall the definition of v_{ii} in Theorem 1. It shows that $v_{ii} \geq (n - 1)/(2(1 + e^L))$. If L is a constant, then $v_{ii} \rightarrow \infty$ for all i as $n \rightarrow \infty$. If r is a fixed constant, one may replace the statistics d_1, \dots, d_r by the independent random variables $\tilde{d}_i = d_{i,r+1} + \dots + d_{in}$, $i = 1, \dots, r$ when considering the asymptotic behaviors of d_1, \dots, d_r . Therefore, we have the following proposition.

Proposition 2. Assume that $\max_i |\theta_i| \leq L$ with L a constant. Then as $n \rightarrow \infty$:

- (1) For any fixed $r \geq 1$, the components of $(d_1 - \mathbb{E}(d_1), \dots, d_r - \mathbb{E}(d_r))$ are asymptotically independent and normally distributed with variances v_{11}, \dots, v_{rr} , respectively.
- (2) More generally, $\sum_{i=1}^n c_i (d_i - \mathbb{E}(d_i)) / \sqrt{v_{ii}}$ is asymptotically normally distributed with mean zero and variance $\sum_{i=1}^{\infty} c_i^2$ whenever c_1, c_2, \dots are fixed constants and the latter sum is finite.

Part (2) follows from part (1) and the fact that

$$\lim_{r \rightarrow \infty} \limsup_{t \rightarrow \infty} \text{Var} \left(\sum_{k=r+1}^n c_k \frac{d_k - \mathbb{E}(d_k)}{\sqrt{v_{kk}}} \right) = 0$$

by Theorem 4.2 of Billingsley [4]. To prove the above equation, it suffices to show that the eigenvalues of the covariance matrix of $(d_i - \mathbb{E}(d_i)) / v_{ii}^{1/2}$, $i = r + 1, \dots, n$ are bounded by 2 (for all $r < n$). This comes from the well-known Perron–Frobenius theory: if A is a symmetric positive definite matrix with diagonal elements equaling to 1, and with negative off-diagonal elements, then its largest eigenvalue is less than 2. We will only use part (1) to prove Theorem 1.

Before proving Theorem 1, we show three lemmas below. By direct calculations,

$$v_{ij} = \frac{\sum_{0 \leq k < l \leq q-1} (k - l)^2 e^{(k+l)(\theta_i + \theta_j)}}{\left(\sum_{a=0}^{q-1} e^{a(\theta_i + \theta_j)}\right)^2}, \quad i, j = 1, \dots, n; i \neq j$$

and $v_{ii} = \sum_{j \neq i} v_{ij}$. On the other hand, it is easy to see that

$$\frac{\frac{1}{2} \sum_{k \neq l} (k - l)^2 e^{(k+l)(\alpha_i + \alpha_j)}}{\left(\sum_{a=0}^{q-1} e^{a(\alpha_i + \alpha_j)}\right)^2} \leq \frac{1}{2} \max_{k \neq l} (k - l)^2 \leq \frac{q^2}{2}. \tag{4}$$

In view of inequality (3), if $\max_{i,j} |\theta_i + \theta_j| \leq L$ with L a constant, then $V_n \in L_n(m, M)$ with m and M constants. Applying Proposition 1, we have

Lemma 1. Assume that $\max_i |\theta_i| \leq L/2$ with L a fixed constant. If n is large enough, then

$$\|V_n^{-1} - S_n\| \leq c_1(n - 1)^{-2}, \tag{5}$$

where c_1 is a constant only depending on L .

Lemma 2. Assume that $\max_i |\theta_i| \leq L/2$ with L a fixed constant. Let $U_n = \text{cov}[W_n\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]$. Then

$$\|U_n\| \leq \|V_n^{-1} - S_n\| + c_2(n - 1)^{-2}, \tag{6}$$

where c_2 is a constant only depending on L .

Proof. Note that

$$U_n = W_n V_n W_n^T = (V_n^{-1} - S_n) - S_n(I_n - V_n S_n),$$

and

$$\{S_n(I_n - V_n S_n)\}_{i,j} = \frac{(\delta_{ij} - 1)v_{ij}}{v_{ii}v_{jj}}.$$

By (3) and (4),

$$\{|S_n(I_n - V_n S_n)\}_{ij}\| \leq c_2(n - 1)^{-2},$$

where c_2 is a constant. Thus,

$$\|U_n\| \leq \|V_n^{-1} - S_n\| + \|S_n(I_n - V_n S_n)\| \leq \|V_n^{-1} - S_n\| + c_2(n - 1)^{-2}. \quad \square$$

In order to prove the below lemma, we need one theorem due to Hillar and Wibisono [11].

Theorem 4. Assume that $\max_i |\theta_i| \leq L/2$ with $L > 0$ a fixed constant. Then for sufficiently large n , with probability at least $1 - 2/n$, the MLE $\hat{\theta}$ exists and satisfies

$$\max_i |\hat{\theta}_i - \theta_i| \leq c_3 \sqrt{\frac{\log n}{n}}$$

where c_3 is a constant that only depends on L .

Lemma 3. If $\max_i |\theta_i| \leq L/2$ with $L > 0$ a fixed constant, then for $i = 1, \dots, r$ with r a fixed constant,

$$\theta_i - \hat{\theta}_i = [V_n^{-1}\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + o_p(n^{-1/2}).$$

Proof. Let E_n be the event that the MLE $\hat{\theta}$ exists and F_n be the event that $\lambda_n := \max_i |\hat{\theta}_i - \theta_i| \leq c(\log n)^{1/2}/n^{1/2}$. Derivations in what follows are on the event $E_n \cap F_n$. Let

$$\mu(t) = \sum_{a=0}^{r-1} \frac{ae^{at}}{\sum_{k=0}^{r-1} e^{kt}}.$$

It is easy to verify that

$$\mu'(t) = \frac{\sum_{0 \leq k < l \leq r-1} (k-l)^2 e^{(k+l)t}}{\left(\sum_{a=0}^{r-1} e^{at}\right)^2}$$

and

$$\mu''(t) = \left[\frac{\frac{1}{2} \sum_{k \neq l, a} (k-l)^2 (k+l-2a) e^{(k+l+a)t}}{\left(\sum_{a=0}^{r-1} e^{at}\right)^3} \right],$$

such that

$$|\mu''(t)| \leq (r - 1)^3. \tag{7}$$

Applying Taylor's expansions to $\mu(\hat{\theta}_i + \hat{\theta}_j)$ at the point $\theta_i + \theta_j$, for $i = 1, \dots, n$, we have

$$\begin{aligned} d_i - E(d_i) &= \sum_{j \neq i} (\mu(\hat{\theta}_i + \hat{\theta}_j) - \mu(\theta_i + \theta_j)) \\ &= \sum_{j \neq i} [\mu'(\theta_i + \theta_j)(\mu(\hat{\theta}_i + \hat{\theta}_j) - \mu(\theta_i + \theta_j))] + h_i, \end{aligned}$$

where $h_i = \frac{1}{2} \sum_{j \neq i} \mu''(\hat{\gamma}_{ij}) [(\hat{\theta}_i + \hat{\theta}_j) - (\theta_i + \theta_j)]^2$, and $\hat{\gamma}_{ij} = t_{ij}(\theta_i + \theta_j) + (1 - t_{ij})(\hat{\theta}_i + \hat{\theta}_j)$, $0 < t_{ij} < 1$. Writing the above expressions into a matrix, it yields,

$$\mathbf{d} - \mathbb{E}(\mathbf{d}) = V_n(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + \mathbf{h},$$

or equivalently,

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} = V_n^{-1}(\mathbf{d} - \mathbb{E}(\mathbf{d})) + V_n^{-1}\mathbf{h}, \quad (8)$$

where $\mathbf{h} = (h_1, \dots, h_n)^T$. By (7), $|h_i| \leq \frac{1}{2}(n-1)(r-1)^3 \hat{\eta}_{ij}^2$, where $\hat{\eta}_{ij} = (\hat{\theta}_i + \hat{\theta}_j) - (\theta_i + \theta_j)$. Therefore, by Lemma 1,

$$\begin{aligned} |(V_n^{-1}\mathbf{h})_i| &= |(S_n\mathbf{h})_i| + |(W_n\mathbf{h})_i| \\ &\leq \max_i \frac{|h_i|}{v_{ii}} + \|W\| \sum_i |h_i| = O\left(\frac{\log n}{n}\right). \end{aligned}$$

By Theorem 4, $P(E_n \cap F_n) \rightarrow 1$ as $n \rightarrow \infty$. It shows $(V^{-1}\mathbf{h})_i = o_p(n^{-1/2})$ for $i = 1, \dots, r$ with r a fixed constant. Consequently, we have $\theta_i - \hat{\theta}_i = [V_n^{-1}\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + o_p(n^{-1/2})$ for $i = 1, \dots, r$. \square

Proof of Theorem 1. By (8),

$$(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})_i = [S_n\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + [W_n\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + (V_n^{-1}\mathbf{h})_i.$$

By Lemmas 2 and 3, if $\max_i |\theta_i| \leq L/2$ with $L > 0$ a fixed constant, then

$$(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})_i = \frac{d_i - \mathbb{E}(d_i)}{v_{ii}} + o_p(n^{-1/2}).$$

Theorem 1 follows directly from Proposition 2, part (1). \square

Appendix B

In the case of continuous weights, the elements of V_n are

$$v_{ii} = \sum_{j \neq i} \frac{1}{(\theta_i + \theta_j)^2}, \quad i = 1, \dots, n; \quad v_{ij} = \frac{1}{(\theta_i + \theta_j)^2}, \quad i \neq j.$$

Note that V_n is also the covariance matrix of \mathbf{d} . Recall that $L_n := \min_{i \neq j} (\theta_i + \theta_j)$ and $M_n := \max_{i \neq j} (\theta_i + \theta_j)$. Therefore,

$$\frac{1}{M_n^2} \leq v_{ij} \leq \frac{1}{L_n^2}, \quad i \neq j, \quad \frac{(n-1)}{M_n^2} \leq v_{ii} \leq \frac{(n-1)}{L_n^2}, \quad i = 1, \dots, n. \quad (9)$$

Applying Proposition 1, we have:

Lemma 4. If n is large enough, then

$$\|V_n^{-1} - S_n\| \leq \frac{cM_n^2}{L_n^3(n-1)^2}, \quad (10)$$

where c is a constant.

Note that $d_i = \sum_{j \neq i} a_{ij}$ are sums of $n-1$ independent exponential random variables. It is easy to show that the third moment of the exponential random variable with rate parameter λ is $6\lambda^{-3}$. Then we have

$$\frac{\sum_{j \neq i} \mathbb{E}(a_{ij}^3)}{v_{ii}^{3/2}} = \frac{6 \sum_{j \neq i} (\theta_i + \theta_j)^{-1}}{v_{ii}^{1/2}} \leq \frac{6M_n/L_n}{(n-1)^{1/2}}.$$

If $M_n/L_n = o(n^{1/2})$, then the above expression goes to zero. This shows that the condition for the Lyapunov's central limit theorem, holds. Therefore, $v_{ii}^{-1/2}\{d_i - \mathbb{E}(d_i)\}$ is asymptotically standard normal if $M_n/L_n = o(n^{1/2})$. Similar to Proposition 2, we have the proposition below.

Proposition 3. If $M_n/L_n = o(n^{1/2})$, then as $n \rightarrow \infty$:

- (1) For any fixed $r \geq 1$, the components of $(d_1 - \mathbb{E}(d_1), \dots, d_r - \mathbb{E}(d_r))$ are asymptotically independent and normally distributed with variances v_{11}, \dots, v_{rr} , respectively.
- (2) More generally, $\sum_{i=1}^n c_i(d_i - \mathbb{E}(d_i))/\sqrt{v_{ii}}$ is asymptotically normally distributed with mean zero and variance $\sum_{i=1}^{\infty} c_i^2$ whenever c_1, c_2, \dots are fixed constants and the latter sum is finite.

Before proving Theorem 2, we show the following two lemmas. The proof of Lemma 5 is similar to that of Lemma 2 and we omit it.

Lemma 5. Let $U_n = \text{cov}[W_n\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]$. Then

$$\|U_n\| \leq \|V_n^{-1} - S_n\| + \frac{M_n^2}{L_n^2(n-1)^2}. \tag{11}$$

In order to prove the lemma below, we need one theorem due to Hillar and Wibisono [11].

Theorem 5. Let $k > 1$ be fixed. Then for sufficiently large n , with probability at least $1 - 3n^{-(k-1)}$, the MLE $\widehat{\boldsymbol{\theta}}$ exists and satisfies

$$\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\infty} \leq \frac{150M_n^2}{L_n} \sqrt{\frac{k \log n}{n}}.$$

Lemma 6. If $M_n/L_n = o\{n^{1/16}/(\log n)^{1/8}\}$, then for $i = 1, \dots, r$ with a fixed constant r ,

$$\theta_i - \hat{\theta}_i = [V_n^{-1}\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + o_p(n^{-1/2}). \tag{12}$$

Proof. By Theorem 5, if $M_n/L_n = o\{n^{1/16}/(\log n)^{1/8}\}$, then

$$\lambda_n = \max_{1 \leq i \leq n} |\hat{\theta}_i - \theta_i| = O_p \left\{ \frac{M_n^2}{L_n^2} \sqrt{\frac{\log n}{n}} \right\}. \tag{13}$$

For $i = 1, \dots, n$, direct calculations give

$$\begin{aligned} d_i - \mathbb{E}(d_i) &= \sum_{j \neq i} \left(\frac{1}{\hat{\theta}_i + \hat{\theta}_j} - \frac{1}{\theta_i + \theta_j} \right) = \sum_{j \neq i} \frac{\theta_i - \hat{\theta}_i + \theta_j - \hat{\theta}_j}{(\hat{\theta}_i + \hat{\theta}_j)(\theta_i + \theta_j)} \\ &= \sum_{j \neq i} \left[\frac{\theta_i - \hat{\theta}_i + \theta_j - \hat{\theta}_j}{(\theta_i + \theta_j)^2} \left(\frac{\theta_i + \theta_j}{\hat{\theta}_i + \hat{\theta}_j} - 1 \right) + \frac{\theta_i - \hat{\theta}_i + \theta_j - \hat{\theta}_j}{(\theta_i + \theta_j)^2} \right] \\ &= \sum_{j \neq i} \left[\frac{(\theta_i - \hat{\theta}_i + \theta_j - \hat{\theta}_j)^2}{(\theta_i + \theta_j)^2(\hat{\theta}_i + \hat{\theta}_j)} + \frac{\theta_i - \hat{\theta}_i + \theta_j - \hat{\theta}_j}{(\theta_i + \theta_j)^2} \right]. \end{aligned}$$

Writing the above expression for $i = 1, \dots, n$ into the form of a matrix, it yields

$$\mathbf{d} - \mathbb{E}(\mathbf{d}) = V_n(\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}) + \mathbf{h},$$

where $\mathbf{h} = (h_1, \dots, h_n)'$ and

$$h_i = \sum_{j \neq i} \frac{(\theta_i - \hat{\theta}_i + \theta_j - \hat{\theta}_j)^2}{(\theta_i + \theta_j)^2(\hat{\theta}_i + \hat{\theta}_j)} := \sum_{j \neq i} h_{ij}.$$

Equivalently,

$$\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}} = V_n^{-1}\{\mathbf{d} - \mathbb{E}(\mathbf{d})\} + V_n^{-1}\mathbf{h}. \tag{14}$$

In view of (9) and (13), we have

$$|h_{ij}| \leq \frac{2\lambda_n^2}{L_n^2(L_n - \lambda_n)}; \quad |h_i| \leq n \times \frac{2\lambda_n^2}{L_n^2(L_n - \lambda_n)}.$$

Note that $(S_n\mathbf{h})_i = h_i/v_{ii}$ and $(V_n^{-1}\mathbf{h})_i = (S_n\mathbf{h})_i + (W_n\mathbf{h})_i$. By direct calculation, we have

$$|(S_n\mathbf{h})_i| \leq \frac{2\lambda_n^2 M_n^2}{L_n^2(L_n - \lambda_n)},$$

and, by Lemma 4,

$$|(W_n \mathbf{h})_i| \leq \|W_n\| \times (n \max_i |h_i|) \leq O_p \left(\frac{M_n^6}{L_n^8} \times \frac{\log n}{n} \right).$$

If $M_n/L_n = o\{n^{1/16}/(\log n)^{1/8}\}$, then $|(V_n^{-1} \mathbf{h})_i| \leq |(S_n \mathbf{h})_i| + |(W_n \mathbf{h})_i| = o_p(n^{-1/2})$. This completes the proof. \square

Proof of Theorem 2. By (14),

$$(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})_i = [S_n\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + [W_n\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + (V_n^{-1} \mathbf{h})_i.$$

By Lemmas 5 and 6, if $M_n/L_n = o\{n^{1/16}/(\log n)^{1/8}\}$, then

$$\theta_i - \hat{\theta}_i = (d_i - \mathbb{E}(d_i))/v_{ii} + o_p(n^{-1/2}).$$

Theorem 2 follows directly from Proposition 3, part (1). \square

Appendix C

Note that $d_i = \sum_{j \neq i} a_{ij}$ is a sum of $n - 1$ geometric random variables. Recall the definitions of L_n and M_n , we have

$$\frac{e^{M_n}}{(e^{M_n} - 1)^2} \leq v_{ij} = \frac{e^{\theta_i + \theta_j}}{(e^{\theta_i + \theta_j} - 1)^2} \leq \frac{e^{L_n}}{(e^{L_n} - 1)^2}, \quad 1 \leq i < j \leq n.$$

This shows $V_n \in L_n(m, M)$ with m and M given by the lower and upper bounds in the above inequalities on v_{ij} . By the moment-generating function of the geometric distribution, it is easy to verify that

$$\mathbb{E}(a_{ij}^3) = \frac{1 - p_{ij}}{p_{ij}} + \frac{6(1 - p_{ij})}{p_{ij}^2} + \frac{6(1 - p_{ij})^2}{p_{ij}^3},$$

where $p_{ij} = 1 - e^{-(\theta_i + \theta_j)}$. By simple calculations, we have

$$\mathbb{E}(a_{ij}^3) = v_{ij} \left(6 + \frac{e^{\theta_i + \theta_j} - 1}{e^{\theta_i + \theta_j}} + \frac{6}{e^{\theta_i + \theta_j} - 1} \right).$$

It follows

$$\frac{\sum_{j \neq i} \mathbb{E}(a_{ij}^3)}{v_{ii}^{3/2}} \leq \frac{7 + 6(e^{L_n} - 1)^{-1}}{v_{ii}^{1/2}} \leq \frac{[7 + 6(e^{L_n} - 1)^{-1}](e^{M_n} - 1)}{n^{1/2} e^{M_n/2}}.$$

If $e^{M_n/2}/L_n = o(n^{1/2})$, then the above expression goes to zero. This shows that the condition for the Lyapunov's central limit theorem holds. Therefore, $v_{ii}^{-1/2}\{d_i - \mathbb{E}(d_i)\}$ is asymptotically standard normal under the condition $e^{M_n/2}/L_n = o(n^{1/2})$. Similar to Proposition 2, and Lemmas 1 and 2, we have the following proposition and Lemmas 7 and 8.

Proposition 4. If $e^{M_n/2}/L_n = o(n^{1/2})$, then as $n \rightarrow \infty$:

- (1) For any fixed $r \geq 1$, the components of $(d_1 - \mathbb{E}(d_1), \dots, d_r - \mathbb{E}(d_r))$ are asymptotically independent and normally distributed with variances v_{11}, \dots, v_{rr} , respectively.
- (2) More generally, $\sum_{i=1}^n c_i(d_i - \mathbb{E}(d_i))/\sqrt{v_{ii}}$ is asymptotically normally distributed with mean zero and variance $\sum_{i=1}^{\infty} c_i^2$ whenever c_1, c_2, \dots are fixed constants and the latter sum is finite.

Lemma 7. If n is large enough, then

$$\|V_n^{-1} - S_n\| \leq \frac{ce^{3M_n}}{L_n^4(n - 1)^2}, \tag{15}$$

where c is a constant.

Lemma 8. Let $U_n = \text{cov}[W_n\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]$. Then

$$\|U_n\| \leq \|V_n^{-1} - S_n\| + \frac{e^{L_n}(e^{M_n} - 1)^4}{(n - 1)^2(e^{L_n} - 1)^2 e^{2M_n}}. \tag{16}$$

In order to prove the lemma below, we need one theorem due to Hillar and Wibisono [11].

Theorem 6. If $e^{5M_n}/L_n^{1/2} = o\{(n/\log n)^{1/2}\}$, the MLE $\hat{\boldsymbol{\theta}}$ exists with probability approaching one and is uniformly consistent in the sense that

$$\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_\infty \leq O_p \left(\frac{(\exp(5M_n) - 1)^2}{\exp(5M_n)} \sqrt{\frac{12}{\exp(L_n/2) - 1}} \sqrt{\frac{2 \log n}{n}} \right) = o_p(1).$$

Lemma 9. If $e^{17M_n}/L_n^3 = o\{n^{1/2}/\log n\}$, then for $i = 1, \dots, r$ with a fixed constant r ,

$$\theta_i - \hat{\theta}_i = [V_n^{-1}\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + o_p(n^{-1/2}). \tag{17}$$

Proof. By Theorem 6, if $e^{17M_n}/L_n^3 = o\{n^{1/2}/\log n\}$, then

$$\lambda_n = \max_{1 \leq i \leq n} |\hat{\theta}_i - \theta_i| = O_p \left\{ \frac{(e^{5M_n} - 1)^2}{e^{5M_n}} \sqrt{\frac{\log n}{n(e^{L_n/2} - 1)}} \right\}.$$

Let $\hat{\gamma}_{ij} = \theta_i + \theta_j - \hat{\theta}_i - \hat{\theta}_j$. By Taylor's expansion, for any $i \neq j$,

$$\frac{1}{e^{\hat{\theta}_i + \hat{\theta}_j} - 1} - \frac{1}{e^{\theta_i + \theta_j} - 1} = \frac{e^{\theta_i + \theta_j}}{(e^{\theta_i + \theta_j} - 1)^2} \hat{\gamma}_{ij} + h_{ij},$$

where

$$h_{ij} = -\frac{e^{\theta_i + \theta_j + \alpha_{ij} \hat{\gamma}_{ij}} (1 + e^{\theta_i + \theta_j + \alpha_{ij} \hat{\gamma}_{ij}})}{(e^{\theta_i + \theta_j + \alpha_{ij} \hat{\gamma}_{ij}} - 1)^3} \hat{\gamma}_{ij}^2,$$

and $0 < \alpha_{ij} < 1$. It is easy to verify that

$$\mathbf{d} - \mathbb{E}(\mathbf{d}) = V_n(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \mathbf{h},$$

where $\mathbf{h} = (h_1, \dots, h_n)^T$ and $h_i = \sum_{j \neq i} h_{ij}$. Equivalently,

$$\boldsymbol{\theta} - \hat{\boldsymbol{\theta}} = V_n^{-1}\{\mathbf{d} - \mathbb{E}(\mathbf{d})\} + V_n^{-1}\mathbf{h}. \tag{18}$$

Since $\theta_i + \theta_j > 0$ and λ_n is sufficiently small, we have

$$|h_{ij}| \leq \frac{e^{2(M_n + \lambda_n)}(1 + e^{2(M_n + \lambda_n)})}{[e^{2(L_n - \lambda_n)} - 1]^3} \lambda_n^2, \quad |h_i| \leq \sum_{j \neq i} |h_{ij}| \leq (n - 1) \frac{e^{2(M_n + \lambda_n)}(1 + e^{2(M_n + \lambda_n)})}{[e^{2(L_n - \lambda_n)} - 1]^3} \lambda_n^2.$$

Note that $(S_n \mathbf{h})_i = h_i/v_{ii}$ and $(V_n^{-1} \mathbf{h})_i = (S_n \mathbf{h})_i + (W_n \mathbf{h})_i$. By direct calculation, we have

$$|(S_n \mathbf{h})_i| \leq \frac{(e^{M_n} - 1)^2}{e^{M_n}} \times \frac{e^{2(M_n + \lambda_n)}(1 + e^{2(M_n + \lambda_n)})}{[e^{2(L_n - \lambda_n)} - 1]^3} \lambda_n^2 = O \left\{ \frac{e^{15M_n} \log n}{nL_n^3} \right\},$$

and, by Lemma 7,

$$|(W_n \mathbf{h})_i| \leq \|W_n\| \times (n \max_i |h_i|) \leq O \left\{ \frac{e^{17M_n} \log n}{nL_n^3} \right\}.$$

If $e^{17M_n}/L_n^3 = o\{n^{1/2}/\log n\}$, then $|(V_n^{-1} \mathbf{h})_i| \leq |(S_n \mathbf{h})_i| + |(W_n \mathbf{h})_i| = o(n^{-1/2})$. This completes the proof. \square

Proof of Theorem 3. By (18),

$$(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})_i = [S_n\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + [W_n\{\mathbf{d} - \mathbb{E}(\mathbf{d})\}]_i + (V_n^{-1}\mathbf{h})_i.$$

By Lemmas 8 and 9, if $e^{17M_n}/L_n^3 = o\{n^{1/2}/\log n\}$, then

$$\hat{\theta}_i - \theta_i = (d_i - \mathbb{E}(d_i))/v_{ii} + o_p(n^{-1/2}).$$

Theorem 3 follows directly from Proposition 4. \square

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