Supplementary Material for "Asymptotics of score test in the generalized β-model for networks"

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One lemma used in the proof of Theorem 2

We say an $n\times n$ matrix $V_n=(v_{ij})$ belongs to a matrix class $\mathcal{L}_n(m,M)$ if V_n is a symmetric nonnegative matrix satisfying

$$v_{ii}=\sum_{j=1;j\neq i}^{n} v_{ij};\quad M\ge v_{ij}=v_{ji}\ge m>0,\quad i\neq j.$$

Yan et al. (2015) proposed a simple matrix $S_n=\mathrm{diag}(1/v_{11},\ldots,1/v_{nn})$ to approximate the inverse of V_n and obtained the upper bounds on approximation errors below.

Lemma 1. *If $V_n\in\mathcal{L}_n(m,M)$, then for $n\ge 3$, the following holds:*

$$\|W_n:=V_n^{-1}-S_n\|\le \frac{M(nM+(n-2)m)}{2m^3(n-2)(n-1)^2}+\frac{1}{2m(n-1)^2}+\frac{1}{mn(n-1)},$$

where $\|A\|=\max_{i,j}|a_{ij}|$ denotes a matrix norm for a general matrix $A=(a_{ij})$.

Calculation of equation (2)

The variance of $\sum_i c_i\tilde{d}_i^2$ can be divided into two parts:

$$Var(\sum_i c_i\tilde{d}_i^2)=Cov(\sum_i c_i\tilde{d}_i^2,\sum_j c_j\tilde{d}_j^2)=\sum_i c_i^2 Var(\tilde{d}_i^2)+2\sum_{1\le i<j\le n} c_ic_jCov(\tilde{d}_i^2,\tilde{d}_j^2).\quad (1)$$

We will calculate the first part:

$$Var(\tilde{d}_i^2)=Cov((\sum_{\alpha=1}^{n}\tilde{a}_{i\alpha})^2,(\sum_{\zeta=1}^{n}\tilde{a}_{i\zeta})^2)=Cov(\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\tilde{a}_{i\alpha}\tilde{a}_{i\beta},\sum_{\zeta=1}^{n}\sum_{\eta=1}^{n}\tilde{a}_{i\zeta}\tilde{a}_{i\eta}).$$

Note that the random variables $\tilde{a}_{ij}$ for $1\le i<j\le n$ are mutually independent. There are only two cases for which $Cov(\tilde{a}_{i\alpha}\tilde{a}_{i\beta},\tilde{a}_{i\zeta}\tilde{a}_{i\eta})$ is not equal to zero: (1) $\alpha=\beta=\zeta=\eta\neq i$;

(2) $\alpha = \zeta, \beta = \eta$ or $\alpha = \eta, \beta = \zeta$. By calculation, we have

$$Var(\tilde{d}_i^2) = 2v_{ii}^2 + \sum_{j \neq i} u_{ij}. \tag{2}$$

The second part of (1) can be calculated as follows.

$$Cov(\tilde{d}_i^2, \tilde{d}_j^2) = Cov((\sum_{\alpha=1}^{n} \tilde{a}_{i\alpha})^2, (\sum_{\zeta=1}^{n} \tilde{a}_{j\zeta})^2) = Cov(\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \tilde{a}_{i\alpha}\tilde{a}_{i\beta}, \sum_{\zeta=1}^{n}\sum_{\eta=1}^{n} \tilde{a}_{j\zeta}\tilde{a}_{j\eta}).$$

In the above, the only case for $Cov(\tilde{a}_{i\alpha}\tilde{a}_{i\beta}, \tilde{a}_{j\zeta}\tilde{a}_{j\eta})$ not equal to 0, is $\alpha = \beta = j$ and $\zeta = \eta = i$. Then

$$Cov(\tilde{d}_i^2, \tilde{d}_j^2) = E(\tilde{d}_{ij}^4) - (E(\tilde{d}_{ij}^2))^2 = u_{ij}. \tag{3}$$

Combing (2) and (3) into (1), it yields equation (2).

References

Yan T., Zhao Y., and Qin H. (2015). Asymptotic normality in the maximum entropy models on graphs with an increasing number of parameters. *J. Multivariate Anal.* **133**, 61–76.