Can we quantitatively image complex structures with rays?

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ABSTRACT
Ray-based prestack depth migration fails to image quantitatively complex structures when a single arrival—for example, the first or the strongest one—is taken into account. In this paper, we investigate whether accounting for multiple arrivals in ray-based preserved amplitude prestack depth migration allows one to improve quantitative imaging of complex media.

The asymptotic ray-Born migration/inversion, originally designed to process one single arrival, is extended to the case of multiple arrivals by accounting for the cross-contributions of all the source and receiver ray-paths. Multiple arrivals in the folded ray fields are computed by a dynamic ray tracing based on a wavefront construction technique.

With an application to the complex Marmousi model, we demonstrate that ray-Born inversion can provide a reliable quantitative estimation of the relative impedance perturbation even in the complex deep part of the model, for which the amplitudes were underestimated drastically when only a single arrival was used, either the first or the strongest one.

This 2-D case study shows that complex structures can be imaged quantitatively with rays. Future studies will require the optimizing of the implementation of ray-Born migration/inversion with multiple arrivals before considering 3-D applications.

INTRODUCTION
Seismic imaging of complex structures characterized by strong lateral variations in the velocity field remains a challenge. The 2-D synthetic Marmousi velocity model (Bourgeois et al., 1991) has become a reference for testing imaging methods in complex structures (e.g., Geoltrain and Brac, 1993; Eittrich and Gajewski, 1996; Audebert et al., 1997; Nichols, 1996). Estimation of the velocity macromodel is certainly the most difficult task, and numerous papers still address the problem of recovering the Marmousi velocity macromodel (e.g., Liu, 1997; Billette et al., 1998). On the other hand, once the velocity macromodel is defined, prestack depth migration is generally considered as a well-established procedure for imaging complex structures. For the 2-D Marmousi case study, the most convincing results were obtained from one-way paraxial prestack depth migration (Lailly et al., 1991, Ehinger et al., 1996), suggesting that, in 2-D, this method is the most effective one.

However, one-way paraxial prestack depth migration suffers some limitations. First, this migration is not really quantitative. This migration is equivalent only to the first iteration of a gradient-type minimization in the seismic linear inverse problem (Lailly, 1983; Tarantola, 1984), and further iterations thus are required for us to have confidence in the quantitative content of migrated images. Second, the extension to 3-D of one-way paraxial prestack depth migration schemes reveals difficulties. As with all finite-difference prestack depth migration schemes, paraxial prestack depth migration requires regularly sampled data, and applications to 3-D data thus require trace interpolation and homogenization at the surface. Moreover, applications to 3-D real-sized data sets still are limited by the present computing power.

With the actual challenge of 3-D prestack depth migration, alternative migration approaches have been revived actively. Among them, ray-based approaches offer an interesting compromise between accuracy and central processing unit (CPU) efficiency. Three-dimensional applications are feasible even with lateral velocity variations of the macromodel (Thierry et al., 1999a; Operto et al., 1997; Tura et al., 1997). In addition, much theoretical work has been done to recast ray-based migration in the general frame of linearized seismic inverse problem theory. These studies led to asymptotic approximations of the inverse operator (Beylkin, 1985; Bleistein, 1987; Jin et al., 1992), which allow one to obtain, in a single iteration,

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quite accurate quantitative estimation of the high-frequency part of the model.

Despite all these advantages, ray-based migration has been penalized by the difficult numerical computation of ray-based parameter maps (among others, traveltime, amplitude, and slowness vector maps). The first breakthrough was certainly the computation of first-arrival traveltimes by a finite-difference solution of the eikonal equation (Podvin and Lecomte, 1991). Nevertheless, preserved amplitude migration as well as migration-based AVO analysis requires us to compute additional parameters such as amplitude and slowness vectors. Interpolation of the ray fields computed by dynamic ray tracing provides a convenient solution to the problem. First approaches were proposed, for example by Lambaré et al. (1992), but the concept of wavefront construction, proposed by Vinje et al. (1993, 1996a, and b) rapidly appeared to be the most powerful approach. The initial version of wavefront construction (Vinje et al., 1992) suffered from a drastic undersampling of the ray field in caustic regions, and the accuracy of later arrivals was affected. The method was improved by Lambaré et al. (1996) and Lucio et al. (1996), who introduced a uniform ray-density criterion allowing one to compute accurate multivalued maps of all the necessary ray-based parameters, and providing the required inputs for imaging complex structures.

First applications of preserved amplitude migration using dynamic ray tracing were limited to noncomplex structures (Lambaré et al., 1992; Thierry et al., 1999a; Tura et al., 1997; Operto et al., 1997). In this case, the ray field is assumed to be single valued, which allows implementation of CPU-efficient algorithms based on interpolation of the continuous maps that are the output of ray tracing (Thierry et. al., 1999a, b). Nevertheless, the theoretical and practical extension of the method to complex structures remained an open question.

For imaging complex structures, the failure of kinematic migration using the first-arrival traveltime was demonstrated by Geoltrain and Brac (1993). This failure arose from the fact that both secondary arrivals and amplitudes of the arrivals are not taken into account. True or preserved amplitude migrations also were tested, using the first or the strongest arrival (Ettrich and Gajewski, 1996; Thierry et al., 1999b; Audebert et al., 1997). When using the first-arrival traveltime, it appeared that taking the amplitude into account was not sufficient for improving the image in the reservoir zone of the Marmousi model. Using the strongest arrival improved the image significantly (Ettrich and Gajewski, 1996; Thierry et al., 1999b), but the amplitudes remained drastically underestimated in the reservoir zone. These studies led to the conclusion that all the arrivals should be migrated to image complex structures properly.

In fact, several studies already demonstrated that ray-based migration could provide images in the Marmousi model almost as good as one-way paraxial migration images. For example, in the method proposed by Bevc (1995), the image is decomposed in several artificial layers. The data are redatumed at the top of each layer and migrated kinematically in the layer using first-arrival traveltimes. When using sufficiently thin layers, no multipathing arises during an individual step, but the overall sequence produces all required multipathings from the surface to the target. Thus, the combined final image is quite good. Unfortunately, Bevc’s strategy seems difficult to extend to 3-D with the present computers and acquisition geometries. It remains, however, that the results obtained by Bevc (1995) demonstrated the potential of ray-based processing for imaging complex structures if the full information contained in the rays is processed properly.

Theory of preserved amplitude migration had been developed originally for the caustic-free case (Beylkin, 1985). Since Beylkin’s paper, there have been several studies for the extension to the classical ray fields (Raksha, 1988; ten Kroode et al., 1994; ten Kroode and Smit, 1997; Nolan and Symes, 1996; Nolan, 1996; Operto et al., 1998; Xu et al., 1998). The conclusion of ten Kroode et al. (1994) and ten Kroode and Smit (1997) was that preserved amplitude migration could be generalized to complex structures in the case of multichannel data, provided that ray geometries satisfy certain conditions. In practice, the conditions are violated for very specific ray trajectories which potentially are encountered in well-to-well experiments but which remain very unlikely for surface data. Some applications to real data of these theoretical works were presented by Verdel and ten Kroode (1997) for a well-to-well application. Nolan and Symes (1996), Nolan (1996), and Xu et al. (1998) generalized the theory to the general acquisition geometries and showed that significant artifacts could occur for several common cases (individual common-shot or common-offset gathers, 3-D marine acquisitions, etc.).

The title of our paper is inspired by the paper of Geoltrain and Brac (1993), “Can we image complex structures with first-arrival traveltime?” Whereas Geoltrain and Brac (1993) tested whether complex structures can be imaged with first-arrival traveltime (i.e., using only part of the information contained in rays), we investigated in this paper whether complex structures can be imaged using the total ray-field information. Theory relies on ray-Born migration/inversion (Jin et al., 1992; Thierry et al., 1999a) which is extended to account for multiple arrivals. As an illustrative example, we present an application to the Marmousi data set. To assess the efficiency of the method, we show a direct comparison between the migrated images obtained with the first, the strongest, and all the arrivals and the exact perturbation Marmousi model. The results show that accounting for multiple arrivals in the migration allows us to image quantitatively the complex zone of the Marmousi model.

RAY-BORN APPROXIMATION WITH MULTIPLE ARRIVALS

Consider the scalar wave equation. The reference Green’s function $G_0(\mathbf{x}, t; s)$ (s denotes the source position, x the receiver position, and t the time) is the solution of

$$
\left( \frac{1}{c_0^2(x)} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) G_0(\mathbf{x}, t; s) = \delta(\mathbf{x} - s) \delta(t),
$$

where $c_0(\mathbf{x})$ denotes the wave-propagation velocity in a reference model. The Green’s function satisfies the reciprocity condition, $G_0(\mathbf{x}, t; s) = G_0(s, t; \mathbf{x})$.

Our convention for the Fourier transform is

$$
f(\omega) = \int_{-\infty}^{+\infty} dt f(t) e^{i\omega t}, \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega f(\omega) e^{-i\omega t}.
$$

Ray theory provides us with a high-frequency asymptotic approximation of the Green’s functions, $G_0 \approx G_1$ (Cervený et al.,
functions in equation (7) by their asymptotic expression, equa-
ted by the linear Born approximation. Replacing the Green’s
where \( n \) denotes the index of the ray branch, \( N \) the total num-
ber of ray branches, \( \mathbf{A} \) the amplitude, \( T \) the traveltime, \( S \) the
signature (the wavelet) of the Green’s function, \( \omega \) the angular
frequency, and \( a \) the KMAH index (Cerveny et al., 1977; 
Chapman, 1985). The KMAH index summarizes the phase
shifts associated with triplications in the ray field. It is an
integer, initially set to zero, and incremented along the ray by
one each time a caustic is crossed (e.g., Chapman, 1985). We
define by ray branch a family of rays whose properties vary
monotonously in a smooth velocity model and that have the
same KMAH index.
In 2-D, \( S \) and \( A \) are given by
\[
S(\omega) = \frac{1}{\sqrt{-i\omega}} \quad \text{and} \quad A(\mathbf{x}, \mathbf{s}) = \sqrt{\frac{c_0(\mathbf{x})}{8\pi J(\mathbf{x}, \mathbf{s})c_0(\mathbf{s})}}, \tag{4}
\]
where \( J(\mathbf{r}, \mathbf{s}) \) denote the geometric spreading associated to the
2-D asymptotic Green’s function, i.e.,
\[
J(\mathbf{x}, \mathbf{s}) = \frac{\partial L(\mathbf{x})}{\partial \psi(\mathbf{s})}, \tag{5}
\]
with \( \partial L(\mathbf{r}) \) the width of the elementary normal section of the
ray tube, and \( \psi(\mathbf{s}) \) the initial normal angular aperture
(Thierry et al., 1999b). At a caustic point, \( \partial L(\mathbf{r}) \) goes to zero,
and consequently the amplitude grows to infinity, which is a
very well known artifact of ray theory. All the parameters in-
volved in the asymptotic Green’s function, equation (3), can be
estimated along the rays by integration of ordinary differential
equations known as the ray and paraxial ray equations (Farra
and Madariaga, 1987).
To introduce the Born approximation, we consider a pertur-
bation \( \delta m(\mathbf{x}) \) of the square slowness of the reference model
\( 1/c_0^2(\mathbf{x}) \) and the associated perturbation of Green’s function,
\( \delta G \):
\[
\begin{align*}
1/c_0^2(\mathbf{x}) &= \frac{1}{1/c_0^2(\mathbf{x}) + \delta m(\mathbf{x})} \\
G(\mathbf{r}, \omega; \mathbf{s}) &= G_0(\mathbf{r}, \omega; \mathbf{s}) + \delta G(\mathbf{r}, \omega; \mathbf{s}). \tag{6}
\end{align*}
\]
The Born approximation is a linear approximation around the
reference model \( c_0 \) of the relation connecting the data pertur-
bation \( \delta G \) to the model perturbation \( \delta m(\mathbf{x}) \),
\[
\delta G(\mathbf{r}, \omega; \mathbf{s}) \approx B(\mathbf{r}, \omega; \mathbf{s})[\delta m(\mathbf{x})] = \omega^2 \int d\mathbf{x} \delta m(\mathbf{x}) G_0(\mathbf{r}, \omega; \mathbf{x}) G_0(\mathbf{x}, \omega; \mathbf{s}), \tag{7}
\]
where \( B \) denotes the forward linear Born operator.
Born approximation allows us to simulate reflected or scat-
tered arrivals as soon as the perturbations remain small. Nei-
ther multiples reflections nor refracted arrivals can be simu-
lated by the linear Born approximation. Replacing the Green’s
functions in equation (7) by their asymptotic expression, equa-
tion (3), gives the ray-Born approximation. We get for the per-
turbation of the asymptotic Green’s function,
\[
\delta G(\mathbf{r}, \omega; \mathbf{s}) \approx B(\mathbf{r}, \omega; \mathbf{s})[\delta m(\mathbf{x})] = \int d\mathbf{x} \delta m(\mathbf{x}) \sum_{n=1}^{N(\mathbf{x}, \mathbf{s})} \sum_{l=1}^{1} E_{nl}(\mathbf{r}, \omega; \mathbf{s}) \\
= \mathcal{K}(\omega) \int d\mathbf{x} \delta m(\mathbf{x}) \sum_{n=1}^{N(\mathbf{x}, \mathbf{s})} \sum_{l=1}^{1} E_{nl}(\mathbf{r}, \omega; \mathbf{s}) \\
\times A_{nl}(\mathbf{r}, \mathbf{x}) e^{i\omega T_{nl}(\mathbf{r}, \mathbf{x})}, \tag{8}
\]
where \( B \) denotes the forward linear ray-Born operator, and \( \mathcal{K}, \)
\( A, \) and \( T \) denote respectively the signature (the wavelet), the
amplitude, and the phase of the ray-Born operator associated to
the receiver ray branch \( n \) and to the source ray branch \( l \):
\[
\begin{align*}
A_{nl}(\mathbf{r}, \mathbf{x}) &= A_n(\mathbf{r}) A_l(\mathbf{x}) \\
T_{nl}(\mathbf{r}, \mathbf{x}) &= T_n(\mathbf{r}, \mathbf{x}) + T_l(\mathbf{x}, \mathbf{s}) - \frac{\pi}{\omega} \text{sign}(\omega) (\alpha_n(\mathbf{r}, \mathbf{x}) \\
&\quad + \alpha_l(\mathbf{x}, \mathbf{s})) \\
\mathcal{K}(\omega) &= i\omega.
\end{align*} \tag{9}
\]
Since we have introduced high-frequency approximations of
the Green’s functions, \( \delta G \) yields only the high-frequency con-
tent of the perturbation of the Green’s function \( \delta G \). Physically,
It represents the singularities in \( \delta G \) associated with the re-
dected or scattered arrivals. Rakesh (1988) demonstrated that the
ray-Born operator \( B \) is a Fourier integral operator (FIO) (pro-
vided that there were no grazing rays and no direct rays) even in
the case of triplicated ray fields (for the caustic-free case, it
was demonstrated by Beylkin in the 1980s). Because the ray-Born
operator \( B \) is representative only of the most singular part of
the Born operator, \( B \), we can expect only to recover singulari-
ties of the square slowness perturbation \( \delta m \) (the discontinu-
ties at reflectors or diffractors) (Beylkin, 1985).

RAY-BORN INVERSION WITH MULTIPLE ARRIVALS

Consider now a 2-D multichannel data set in which traces are
parameterized by independent source and receiver positions
(\( \mathbf{s}, \mathbf{r} \)). The linear relation in equation (8) can be in-
verted within the general frame of inverse problem theory (Tarantola,
1987). Using a weighted \( l^2 \) norm to measure the discrepancies
between observed and calculated data, we define the objective
function as
\[
C[\delta m] = \frac{1}{2} \int d\omega \int ds \int d\mathbf{r} Q(\mathbf{r}) \delta G_{\text{obs}} - \delta G_{\text{cal}}[\delta m]^2, \tag{10}
\]
where \( \delta G_{\text{obs}} \) is the observed reflected-scattered data, and \( Q \) is
a weighting factor equivalent to the covariance matrix in the
data space of the inverse problem theory (Tarantola, 1987). \( \delta G_{\text{cal}} \)
is the data calculated for perturbation \( \delta m \) using the linear
forward ray-Born approximation, equation (8).
The expression of the solution \( \delta m \) minimizing the objective
function is given by
\[
\delta m = (B^T Q B)^{-1} B^T Q \delta G_{\text{obs}}, \tag{11}
\]
where $^\dagger$ denotes the adjoint operator, $-B^\dagger Q B_s$ is the gradient of the cost function, and the operator, $B^\dagger Q B$, is the Hessian. In general seismic applications, the Hessian is a very large matrix impossible to invert numerically and thus, iterative local minimizations as gradient-type methods were proposed (Beydoun and Mendes, 1989). Any improvement in the estimation of the inverse Hessian accelerates convergence of the local algorithm.

In this study, following Jin et al. (1992), we propose an asymptotic approximation for the inverse Hessian in the case of multiple arrivals. Application to the Marmousi data set demonstrates that these approximations are relevant even when the local optimization is limited to a single iteration.

**Approximation of the Hessian**

Jin et al. (1992) proposed to introduce a special weight $Q$ such that the Hessian matrix becomes asymptotically diagonal. Jin’s method allows us to reconcile the direct inversion approach of Beylkin (Beylkin, 1985) with the stochastic approach of inverse problems (Tarantola, 1987). Let us now develop Jin’s approach in case of triplicated ray fields. Consider the Hessian

$$
B^\dagger Q B(x, x_0) = \int ds \int dr \int d\omega B^\dagger(r, \omega, s, x) \times Q B(r, \omega, s, x_0)
$$

$$
= \int d\omega \int ds \int dr \sum_{n=1}^{N(x,r)} \sum_{l=1}^{N'(r, x_0)} \frac{L_n(x_0)}{L(x,x_0)}
\times D_{nl}^\dagger^{\prime}(r, \omega, s, x, x_0).
$$

with

$$
D_{nl}^\dagger^{\prime}(r, \omega, s, x, x_0) = |K(\omega)|^2 A_{nl}(r, x, s)
\times Q A_{nl}^\prime(r, x_0, s) e^{i(\omega T_{nl}(r, x_0) - T_{nl}(r, x, s))}.
$$

Physically, the traveltime injectivity condition means that any locally coherent event in the data cube, defined at the surface by a shot-and-receiver position, a shot-and-receiver slope, and a two-way traveltime (Billette and Lambaré, 1998), can be associated unambiguously with a couple of ray segments (Nolan and Symes, 1996; Xu et al., 1998) (Figure 1). Here, a couple of ray segments is parameterized in depth by the position of the point diffractor $x$, the angles $\beta_1$ and $\beta_2$, and the one-way traveltimes $T_s$ and $T_r$ (Figure 1). If traveltime injectivity condition holds, migration will focus any reflected/diffracted event at its correct position in depth (if the exact-frequency macromodel is used).

The theory of multiarrival migration was extended to general acquisition geometries such as single-channel gathers (individual common-shot or common-offset gathers) or 3-D marine acquisitions (with limited coverage in azimuth) (Nolan and Symes, 1996; Nolan, 1996). In these cases, the traveltime injectivity condition has to be adapted, and a more general imaging condition can be derived (Nolan and Symes, 1996; Nolan, 1996). Physical principle remains the same, i.e., any locally coherent event in the data set must be interpreted unambiguously as a couple of ray segments. For general acquisitions geometries, it is not always possible to obtain, from the slopes of a locally coherent event, the horizontal components of slowness vectors at the surface for both the shot and receiver positions $(p^s, p^r)$. For example, in the common-shot case, slopes at the shot position $p^s$ are not available. When there is no triplication, this limited information may be sufficient, but in case of triplications, some ambiguity may appear when associating couples of ray segments to local coherent events in the data. As a consequence, imaging condition may break down much more easily in case of triplicated ray fields for single-channel data than for multi-channel data. Artifacts in the migrated images can arise in these cases (Nolan and Symes, 1996; Nolan, 1996; Xu et al., 1998). In case of multichannel data, violation of the traveltime injectivity condition occurs only for some tortuous ray trajectories (ten Kroode et al., 1994; ten Kroode and Smit, 1997; Nolan and Symes, 1996; Nolan, 1996), which generally can be omitted.
Let us suppose that the traveltime injectivity condition is satisfied. In the high-frequency approximation, the singular part of the Hessian operator is obtained with a local analysis around the diagonal terms, $x = x_0$ for $n' = n$ and $\ell' = \ell$.

The Hessian operator can be approximated up to smooth operators by

$$B^T Q B(x, x_0) \approx \int d\omega \int ds \int d\ell \sum_{n=1}^{N(x, r)} \sum_{\ell=1}^L Q |\mathcal{K}(\omega)|^2 \times A^2_{nl}(r, x_0, s) e^{-\omega \sqrt{T_{nl}(r, x_0, s)} (x-x_0)},$$

where we used local approximations for the amplitude and phase terms (Miller et al., 1987; Jin et al., 1992)

$$A_{nl}(r, x_0, s) \approx A^2_{nl}(r, x_0, s)$$

and

$$T_{nl}(r, x_0, s) - T_{nl}(r, x_0, s) \approx -\sqrt{T_{nl}(r, x_0, s)} \cdot (x-x_0).$$

The next step consists of unfolding the multivalued ray fields by making a change of variables in equation (15) from $(s, r)$ to $(\beta_r, \beta_s)$ where $\beta_r$ and $\beta_s$ are the takeoff angles of the rays ($x_0 \rightarrow s$) and ($s \rightarrow r$), respectively (Figure 2) (Verdel and Kroode, 1997).

These new variables $(\beta_r, \beta_s)$ unfold the integral path in equation (15) (Figure 3), as long as there are no grazing rays at the surface. We obtain for the Hessian operator

$$B^T Q B(x, x_0) \approx \int d\omega \int ds \int d\beta_s d\beta_r$$

$$\times Q D(\beta_r, x_0, \beta_s, \omega) e^{-\omega \sqrt{T_{nl}(r, x_0, \beta_s)} (x-x_0)},$$

with

$$D(\beta_r, x_0, \beta_s, \omega) = |\mathcal{K}(\omega)| \left| \frac{\partial (r, s)}{\partial (\beta_r, \beta_s)} \right| (x_0) A^2(\beta_r, x_0, \beta_s).$$

The Jacobian $|\partial (r, s)/\partial (\beta_r, \beta_s)|$ vanishes at caustics and becomes infinite for grazing rays at surface. It also vanishes at zero offset when $\beta_s = \beta_r$.

The kernel of the integral operator, equation (17), involves an amplitude $D$ and a phase $\sqrt{T_n} (x - x_0)$ term. It reminds us of the integral expression of the 2-D Dirac delta function,

$$\delta(x - x_0) = \frac{1}{(2\pi)^2} \int \frac{dk}{|k|^2} e^{-ik \cdot (x-x_0)}.$$  

The analogy between the asymptotic form of the asymptotic Hessian, equation (17), and the Dirac delta function, equation (19), can be used for obtaining an analytic form of the inverse Hessian. Jin et al. (1992) proposed choosing the weighting function $Q$ such that the Hessian operator reduces to a weighted Dirac delta function (which corresponds to a diagonal Hessian matrix) (Jin et al., 1992; Thierry et al., 1999b). In the case of a multichannel data set with triplicated ray fields, we propose

$$Q(\beta_r, x_0, \beta_s, \omega) = \frac{1}{D(\beta_r, x_0, \beta_s, \omega)} \left| \frac{\partial (k, \theta)}{\partial (\beta_r, \beta_s, \omega)} \right| (x_0),$$

where $k = \omega \sqrt{T_n}(\beta_r, x_0, \beta_s) = \omega q$ with $q = p_s + p_r$ and $\theta = \beta_s - \beta_r$ (Figure 2).

The approximate Hessian becomes

$$B^T Q B(x, x_0) \approx \int d\omega \int ds \int d\beta_s d\beta_r \left| \frac{\partial (k, \theta)}{\partial (\beta_r, \beta_s, \omega)} \right| e^{ik \cdot (x-x_0)}$$

$$\approx \int d\theta \int \frac{dk}{|k|^2} e^{ik \cdot (x-x_0)}$$

$$\approx \delta(x - x_0) \int_{\theta_{\min}}^{\theta_{\max}} d\theta = \delta(x - x_0) (\theta(x_0))_{\max} - \theta_{\min},$$

where $\theta(x_0)_{\max} - \theta_{\min}$ denotes the difference between the maximum and minimum values of $\theta(x_0)$, obtained for

![Fig. 2. Integration space changes from $(s, r)$ to $(\beta_r, \beta_s)$. $p_s$, $p_r$ are the slowness vectors at $x$ associated with the rays connecting the source and receiver to the scatterer, respectively.](image)

![Fig. 3. Unfolding of the integration path over data by change of variable $s \rightarrow \beta_s$. (a) Folded ray-field "illuminating" point $x$ for a homogeneous media with a smooth, low-velocity heterogeneity. (b) Curve $\beta_s$ as a function of source position. The integration path over $s$ is folded. It is unfolded when changing integration coordinate to $\beta_s$.](image)
the considered acquisition geometry. The singular part of the Hessian matrix can be inverted easily.

**Final migration formula with multiple arrivals**

Replacing the Hessian by its asymptotic approximation (23), we get the final migration formula,

\[
\delta m(\mathbf{x}) \approx \frac{1}{[\theta(\mathbf{x})]_{\min}} \int d\omega \int ds \int dr \sum_{n=1}^{N(x,x)} \sum_{i=1}^{L(x,x)} \mathcal{E}_{nl}(\mathbf{r}, \omega, s, \mathbf{x}) \times \mathcal{Q} \mathcal{B}_{nl}(\mathbf{r}, \omega, s, \mathbf{x}) \delta G_{obs}(\mathbf{r}, \omega, s, \mathbf{x}).
\]  

(24)

Using the expression of \( B_{nl} \) deduced from equations (8) and the expression of \( Q \), equation (20), we get

\[
\delta m(\mathbf{x}) \approx \frac{1}{[\theta(\mathbf{x})]_{\min}} \sum_{s} \sum_{r} \sum_{n=1}^{N(x,x)} \sum_{i=1}^{L(x,x)} \mathcal{E}_{nl}(\mathbf{r}, \omega, s, \mathbf{x}) \times \mathcal{Q} \mathcal{B}_{nl}(\mathbf{r}, \omega, s, \mathbf{x}) \delta G_{obs}(\mathbf{r}, \omega, s, \mathbf{x}).
\]  

(25)

where \( \mathcal{H}_{nl}(\mathcal{B}_{nl}(\mathcal{G}_{obs}(\mathbf{r}, t, s))) \) is the multivalued maps of Jin et al. (1992) and Thierry et al. (1999a) to the case where \( \mathcal{B}_{nl} = \mathcal{G}_{obs}(\mathbf{r}, t, s) \) is the sum of KMAH index for the rays \( s \rightarrow \mathbf{x} \) and \( r \rightarrow \mathbf{x} \). The amplitude of the quantitative migration operator is given by

\[
\mathcal{E}_{nl}(\mathbf{r}, \omega, s, \mathbf{x}) = \frac{2 \Delta s}{\pi \bar{c}^2(\mathbf{x})} \left( \frac{\theta_{ct}}{2} \right) \left| \left( \frac{\partial \mathbf{x}}{\partial s} \right)_{i} \right| \left( \frac{\partial \mathbf{x}}{\partial r} \right)_{n}.
\]  

(26)

where \( \Delta s \) and \( \Delta r \) are the source and receiver spacings, respectively, and where we used equality,

\[
\left( \frac{\partial \mathbf{k}}{\partial (\mathbf{r}, s, \omega)} \right)_{nl} = \frac{4 [\omega]_{n} \bar{c}^2(\mathbf{x})}{\left( \frac{\theta_{ct}}{2} \right)} \left( \frac{\partial \mathbf{x}}{\partial s} \right)_{i} \left( \frac{\partial \mathbf{x}}{\partial r} \right)_{n}.
\]  

(27)

demonstrated in the appendix.

The final formula, (25), is an extension of the caustic-free formula of Jin et al. (1992) and Thierry et al. (1999a) to the case of triplicated ray fields. We see that all cross-contributions associated with triplications from the source and receiver points have to be added to the image. Migration/inversion formula is not a mean over these cross-contributions but a summation.

**PRACTICAL ASPECTS**

The numerical application of formula (25) requires the computation of multivalued maps of amplitude, traveltimes, KMAH index, and angles of rays. Such maps can be computed accurately by the wavefront construction code developed by Lambaré et al. (1996).

Wavefront construction was proposed initially by Vinje et al. (1992). This approach is based on the decomposition of the ray field in elementary quadrangular cells defined by successive wavefronts and adjacent rays. Rays are propagated with constant traveltime steps, while a ray-density criterion is applied at each sampled wavefront to control the size of the cells. The ray density initially proposed by Vinje et al. (1992) was based simply on the distance between adjacent rays at the top of each cell. It was not to able to ensure the accuracy of the wavefront sampling in zones with caustics, and the wavefront was cut in such regions. The ray-density criterion was improved later (Sun, 1992; Vinje et al., 1996a, 1996b) by taking into account, in addition to the distance in (x), the angular distance between the direction of adjacent rays at the top of the cell. Even such a criterion provides badly sampled ray fields in caustic regions (Lambaré et al., 1996). Lambaré et al. (1996) proposed a new ray-density criterion based on the curvature of the ray field both in the (x) space and in the slowness (p) space. Their uniform ray-density criterion ensures the CPU efficiency as well as the precision of the ray field sampling, even in case of triplications (Lambaré et al., 1996; Thierry et al., 1999b). 2-D (Lambaré et al., 1996) and 3-D (Lucio et al., 1996) codes were developed for smooth-velocity macromodels defined by cubic cardinal B-splines.

Both codes already have been used for 2-D (Thierry et al., 1999b; Ribodetti and Virieux, 1996) and 3-D migration/inversion (Thierry et al., 1999a; Tura et al., 1997; Operto et al., 1997), and were designed to achieve CPU efficiency and low RAM storage. [The 2-D migration/inversion code developed by Thierry et al. (1999b) was a testing platform for a fast 3-D migration/inversion code able to run on a workstation (Thierry et al., 1999a)]. Both codes rely on linear interpolations of maps of ray-based parameters which cannot be extended easily to multivalued maps.

The fast 2-D ray-Born migration/inversion code already was applied to the complex Marmousi data set (Thierry et al., 1999b), but only a single arrival, either the first or the strongest one, was used. From the results obtained, it was also clear that the use of a single arrival (either the first or the strongest one) is not sufficient for achieving quantitative imaging in the region of the hydrocarbon trap.

In this paper, we want to demonstrate that taking into account all the arrivals allows us to image quantitatively a complex model with strong triplications. We use interpolations neither for the ray-Born inverse operator nor for the ray-based parameter maps. We calculate multivalued maps for all ray-shooting positions at surface, and they are stored in random access memory (RAM). It is clear that for any 3-D application, interpolations will be unavoidable and specific interpolation schemes will have to be developed. A solution could be to interpolate the ray field discretized by the wavefront construction rather than the multivalued maps.

**APPLICATION TO MARMOUSI**

The Marmousi model is a well-known 2-D synthetic acoustic model (Figure 4). It often is considered as a reference for testing imaging in complex area (e.g., Audubert et al., 1997). It was built by the Institut Français du Pétrole for simulating a 2-D marine seismic acquisition (Bourgeois et al., 1991). Data were computed by finite differences of the acoustic equation, and the model was given by dense velocity and density grids sampled in X and Z to 4 m. A hydrocarbon trap is located just under the complex structure at X = 5500 m and about Z = 2500 m.

The spectrum of the source signature is a trapezoid (0, 10, 35, 55) Hz. The time sampling is 4 ms. There are 240 shots, with 96 receiver groups each. The shot-and-receiver spacing is 25 m. The shot and receiver depths are respectively 8 and 12 m. The nearest offset is 200 m, and the first shot is located at X = 3000 m with the streamer on the left.
Ray-Born inversion requires data to be deconvolved from the source signature and multiples (water-bottom reverberation at the free surface). A deterministic deconvolution was applied for the source signature and for the source-and-receiver ghosts (Bourgeois et al., 1991). Finally, a predictive deconvolution (using the Seismic Unix software package) was applied to attenuate multiples.

The Marmousi model is in fact an acoustic model parameterized both by velocity and density. Our inverse formula was developed for the scalar wave equation, and it required a slight adaptation (Thierry et al., 1999b). It is well known that at near offsets, reflection/diffraction is essentially sensitive to the impedance perturbation \( I = \rho c \). In this case, the ray-Born approximation, equation (8), must be modified by replacing the 2-D amplitude of the Green’s function, \( A(x, s) \) in expression (4), by the amplitude

\[
A_{\text{Ray-Born}}(x, s) = \sqrt{\rho(s)\rho_0(x)} A^{2D}(x, s),
\]

and the perturbation of model, \( \delta m(x) \), by

\[
\delta m(x) = \frac{-2}{\rho_0^2(x) c_0^2(x)} \delta I(x).
\]

The inversion formula for the relative perturbation of impedance can be derived straightforwardly from equations (25) and (26) just by taking into account equations (28) and (29).

Our 2-D ray-tracing code (Lambaré et al., 1996) requires smooth-velocity models. For our applications, the background velocity model was obtained by low-pass filtering of the exact velocity model by a Gaussian filter. The cutoff frequency of the Gaussian filter is given by the correlation length \( \tau \) such that

\[
f(x) = \frac{1}{2\pi\tau} \exp\left(-\frac{x^2}{\tau^2}\right).
\]

The smoothed model then was projected on the cardinal cubic B-splines basis (Operto et al., 1997). Following Versteeg (1993) and Thierry et al. (1999b), we chose \( \tau = 76 \) m and a spacing between the B-spline knots of 76 m. This macromodel corresponds approximatively to the smoothest macromodel acceptable for accurate imaging of the Marmousi model using one-way paraxial depth migration (Versteeg, 1993). Figure 5 shows the smoothed velocity model as well as the associated ray field provided by wavefront construction for the ray-shooting position \( X = 6000 \) m. Note that we stopped computing rays as soon as they started to propagate upward (after the grazing point). Even so, there are many triplications in the ray fields, particularly in the reservoir zone. When the lateral extent of the ray field is limited to 3000 m on both sides of the ray-shooting position (this seems reasonable given the acquisition geometry), there are as many as five arrivals.

For comparing results of migration, we generated the exact perturbation model by subtracting the background model from the exact model and by band-pass filtering in time the resultant model with a filter with bandwidth similar to the source bandwidth (Figure 6). This perturbation model can be compared directly with the migrated sections.

Ray-Born inversion already has been applied to the Marmousi data set (Thierry et al., 1999b). In this former study, a single arrival was considered, either the first or the strongest one, for the macromodel smoothed with \( \tau = 76 \) m (Figure 7). In terms of quantitative imaging, even if using the strongest arrival significantly improved the image in the complex deep part of the model, the amplitude of the recovered impedance perturbation remained underestimated drastically (Figures 7 and 8). The incomplete amplitude estimation is explained by the fact that the ray field is so folded that the propagation of seismic energy cannot be approximated with a single arrival.

When using a single arrival, which has many advantages in terms of CPU efficiency, a compromise may be sought by using a smoother-background velocity model. Indeed, a smoother-velocity macromodel leads to fewer folded ray fields (Figure 9) and, in this case, propagation of seismic energy can be approximated with a single arrival. Migrated sections using the first and the strongest arrival for a macromodel smoothed with \( \tau = 200 \) m are shown on Figure 10). These two images obviously are very closed because very few triplications develop in the macromodel (Figure 10c). A surprising result is that the amplitudes of the perturbations in the complex zone are recovered better than on the images obtained with a macromodel smoothed with \( \tau = 76 \) m (compare Figures 7 and 10 and Figures 8 and 11; all the sections are plotted with the same gain).
FIG. 5. (a) The velocity model obtained by smoothing the exact Marmousi model with a Gaussian filter with $\tau = 76$ m. (b) Same as (a) with superimposed ray field for a source located at $x = 6000$ m and $z = 10$ m. Note that the ray field is folded severely in the complex deep zone.

FIG. 6. Exact relative impedance perturbation. The perturbation model was filtered according to the source-signature bandwidth.
Fig. 7. (a) Relative impedance perturbation obtained by ray-Born migration/inversion using the first-arrival traveltime and the background model of Figure 5. (b) Relative impedance perturbation obtained by ray-Born migration/inversion using the strongest arrival and the background model of Figure 5. (c) Differences between sections (a) and (b).
This suggests that in the case of a smoother macromodel ($\tau = 200$ m), the contribution of all the arrivals in the data are “averaged” by the single-arrival migration. In compensation, the excessive smoothness of the macromodel degrades spatial resolution and the positioning of reflectors (compare resolution of Figures 7 and 10).

These two examples illustrate, in fact, the compromise in migration based on the single-arrival hypothesis between the need for an accurate macromodel for spatial resolution and positioning and the need for a very smooth macromodel for quantitative imaging of complex media. The conclusion of this comparative test is that accounting for multiple arrivals in an accurate macromodel is the only reliable strategy to image quantitatively complex structures with asymptotic methods.

To check the improvement provided by the use of all the arrivals, we applied the multipathing ray-Born migration/inversion formula, equation (25), with the velocity model smoothed with $\tau = 76$ m. Figure 12(a) shows the migrated image obtained by using all the arrivals, and Figure 13 shows the comparison between logs extracted from the exact perturbation model and from the migrated section. There were as many as 25 combinations of source-and-receiver raypaths. With our nonoptimized code, the application requires 23 hours on a SUN Sparc 20 workstation. Because the RAM of the workstation was limited to 120 Mb, we had to split the image into six horizontal layers. From a purely structural point of view, the quality of the migrated image obtained with multiple arrivals [Figure 12(a)] is comparable to that of the image obtained with the strongest arrivals. It may be sufficient to increase the gain in Figure 7(b) to view the structure better. The main improvement obtained when accounting for all the arrivals is a better estimation of the amplitude of the model perturbations. This conclusion is illustrated first in the differences between the migrated images obtained by using all the arrivals and the first or the strongest ones [Figures 12(b, c)]. These differences contain pieces of reflectors located in areas of caustics and visible in Figure 7 with underestimated amplitude. Accounting for the multiple arrivals allowed us to improve the amplitude of these reflectors. This conclusion is supported by examining some logs of relative impedance perturbations. The log located at $x = 6200$ m, where the ray fields are folded highly, shows that estimation of the perturbations associated with the target were improved significantly when we accounted for multiple arrivals, but the logs located at $x = 3700$ m and $8000$ m were not modified (compare the logs in Figures 8 and 13). Note, however, that the perturbations in the complex deep part of the model still are not recovered perfectly [Figure 13 ($x = 6200$ m)]. This can result from some nonlinear effects which are not taken into account by the linear Born approximation or from the limited accuracy of ray theory in case of highly triplicated ray fields. The fact that perturbations are estimated fairly well in the deep part of the model for the logs located at $x = 3700$ m and $8000$ m (i.e., where fewer caustics develop) lends more support to the second interpretation (Figure 13).

**Discussion and Conclusion**

With the Marmousi case study, we have shown that ray-Born migration/inversion allows us to image quantitatively a complex model with triplicated ray fields. Results do not seem
to be affected too dramatically by nonlinear effects not taken into account by the linear Born approximation. Moreover, amplitude of perturbations seems quite robust with respect to the smoothing of the velocity macromodel, which is quite important when we think about the difficulty of having precise-velocity macromodels.

The study was limited to migration/inversion for a single acoustic parameter. Results could be extended to the elastic case (Jin et al., 1992; Forgues, 1995) for $P$-P and $P$-S wavefields. But rather than the extension of the theory to the multiparameter and elastic cases, the difficulties are associated with the high-sensitivity of multiparameter ray-Born migration/inversion (both for the acoustic and elastic cases) to imprecisions in the velocity macromodel (Forgues, 1995). Postprocessing of common image gathers by residual moveout (Tura et al., 1998) seems definitely unavoidable, even if it still has to be tested.

For the extension of ray-Born migration/inversion to triplicated ray fields, the two key points are to take into account all the cross-contributions of source-and-receiver multiple ray-paths and to process the entire multichannel data set in one go. Applications to individual common-offset or individual common-shot gather may provide spurious artifacts in the individual migrated images (Nolan and Symes, 1996; Nolan, 1996; Xu et al., 1998).

Artifacts of ray theory in a complex velocity model, such as shadow zones and infinite amplitude at caustics, do not alter the quality of the migrated image too much because they are smoothed by the multichannel stack. Several methods which were developed for avoiding these artifacts in asymptotic forward modeling [Maslov summation (Chapman, 1985; Piserchia et al., 1998); Gaussian beam summation (Červený et al., 1982, Hill, 1990)] should be thought of for improving asymptotic imaging in complex areas. At the present time, first attempts were not really convincing but may be improved (Xu and Lambèrè, 1998).

The theoretical analysis for imaging in complex media is a very powerful tool (ten Kroode et al., 1994; ten Kroode and Smit, 1997; Nolan and Symes, 1996; Nolan, 1996). It allows us to develop quantitative migration formula in complex models and helps us to understand difficulties encountered in common image-gather analysis in complex media. It was noticed already (Duquet et al., 1994) that common-offset image gathers and common-shot image gathers were affected by spurious artifacts in complex media. Xu et al. (1998) analyzed the problem in terms of asymptotic imaging and proposed a common-angle image strategy to avoid these artifacts. This common-angle imaging offers an opportunity for studies on amplitude variation with angle of incidence (AVA) and migration-based

![Figure 9](image-url)  
**Fig. 9.** (a) The velocity model obtained by smoothing the exact Marmousi model with $r = 200$ m. (b) Same as (a) with ray field superimposed for a source located at $x = 6000$ m and $z = 10$ m. The ray field is folded less severely than in Figure 5b because of the smoothness of the background model.
Fig. 10. (a) Relative impedance perturbation obtained by ray-Born migration/inversion using first-arrival traveltime and the background model of Figure 9. (b) Relative impedance perturbation obtained by ray-Born migration/inversion using the strongest arrival and the background model of Figure 9. (c) Difference between sections (a) and (b). The difference is insignificant because the ray field is not folded strongly (see Figure 9b).
velocity analysis (Symes, 1993; Chauris and Noble, 1998) in complex media.

The practical implementation of ray-Born migration/inversion in complex media largely must be improved before we consider 3-D applications (complex structures are most often 3-D). The CPU efficiency of existing 3-D ray-Born migration/inversion codes (Thierry et al., 1999a; Tura et al., 1997; Operto et al., 1997) rely on the use of very simple interpolations of the various maps computed by ray tracing. Such strategies cannot be extended straightforwardly to the case of multivalued ray fields. Because interpolations are the main strategy for improving CPU efficiency, new interpolation strategies must be designed. For example, it should be possible to interpolate the ray field sampled by wavefront construction rather than regular maps.

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Fig. 12. (a) Relative impedance perturbation obtained by ray-Born migration/inversion using all the arrivals and the background model of Figure 5. (b) Differences between sections obtained with all the arrivals (Figure 12a) and with the first-arrival traveltime (Figure 7a). (c) Difference between sections obtained with all the arrivals (Figure 12a) and with the strongest arrival (Figure 7b). Using all the arrivals improves the imaging of the complex deep zone.


Thierry, P., Lambaré, G., Podvin, P., and Noble, M., 1999a, 3-D preserved amplitude prestack depth migration on a workstation: Geophys., 64, 222–229.


In this appendix, we demonstrate that

\[
\left| \frac{\partial (k, \theta)}{\partial (r, s, \omega)} \right|_{nt} = \frac{4|\omega|}{c_0^2} \cos^2 \left( \frac{\theta_{nt}}{2} \right) \left| \frac{\partial \beta_s}{\partial s} \right| \left| \frac{\partial \beta_r}{\partial r} \right| n. \tag{A-1}
\]

For a given \( nt \), the Jacobian can be decomposed as

\[
\left| \frac{\partial (k, \theta)}{\partial (r, s, \omega)} \right| = \left| \frac{\partial (k, \theta)}{\partial (\beta_r, \beta_s, \omega)} \right| \left| \frac{\partial (\beta_r, \beta_s, \omega)}{\partial (r, s, \omega)} \right|. \tag{A-2}
\]

It is obvious that

\[
\left| \frac{\partial (\beta_r, \beta_s, \omega)}{\partial (r, s, \omega)} \right| = \left| \frac{\partial (\beta_r)}{\partial (r)} \right| \left| \frac{\partial (\beta_s)}{\partial (s)} \right|. \tag{A-3}
\]

Using

\[
k = \omega \nabla s T(\beta_r, x_n, \beta_s) = \omega \left( \frac{p_{r_1} + p_{r_2}}{p_{r_1} + p_{r_2}} \right), \tag{A-4}
\]

and \( \theta = \beta_s - \beta_r \), the first term on the right-hand side of equation (A-2) can be written as

\[
\left| \frac{\partial (k, \theta)}{\partial (\beta_r, \beta_s, \omega)} \right| = \left| \begin{array}{c}
\frac{\partial p_{r_1}}{\partial \beta_r} \\
\frac{\partial p_{r_2}}{\partial \beta_r} \\
\frac{\partial p_{r_1}}{\partial \beta_s} \\
\frac{\partial p_{r_2}}{\partial \beta_s} \\
1 \\
1
\end{array} \right| \left| \begin{array}{c} p_{r_1} + p_{r_2} \\
p_{r_1} + p_{r_2} \\
p_{r_1} + p_{r_2} \\
p_{r_1} + p_{r_2} \\
1 \\
1
\end{array} \right| = |q|^2, \tag{A-7}
\]

where \( q = p_r + p_s \). Using \( |q|^2 = \frac{1}{c_0^2} \cos^2 \left( \frac{\theta_{nt}}{2} \right) \) concludes the proof of equation (A-1).