Asymptotic viscoacoustic diffraction tomography of ultrasonic laboratory data: a tool for rock properties analysis

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SUMMARY
This paper presents an application of 2.5-D asymptotic viscoacoustic diffraction tomography to ultrasonic data recorded during a physically scaled laboratory experiment. This scaled experiment was used to test the reliability of our method when applied to a real data set to estimate the attenuation factor \( Q \).

Diffraction tomography relies on ray theory to compute the Green functions in a smooth background medium and on the Born approximation to linearize the relation between the scattered wavefield and the velocity and \( Q \) perturbations. The perturbations are inferred from the data by an iterative (linear) quasi-Newtonian algorithm.

The inversion formula was specifically developed to account for the acquisition geometry designed in this study. The derivation of the Hessian operator shows that, for this acquisition, the velocity and \( Q \) perturbations are theoretically decoupled.

The processing of the data was split into two steps: first, we applied the tomography to the data without deconvolving them. Second, we designed a post-processing procedure for the tomographic images to remove the source signature and to estimate the absolute values of the velocity and \( Q \). At the conclusion of the first step, both the velocity and the \( Q \) tomographic images allowed one to delineate the gross contour of the target. We obtained an excellent match between the observed data and the viscoacoustic ray-Born synthetics. The match obtained with the viscoacoustic rheology was significantly better than for a purely acoustic one.

In the second step, the post-processing allowed us to recover the shape of the target. We estimated the absolute values of the velocity and \( Q \), although we had no quality control with regard to these results (the rheological properties of the material used in this study were unknown). The results suggest that the uncertainty of the velocity measurement is lower than that for \( Q \).

The application presented in this study suggests that the procedure that we designed (experimental set-up, tomography, post-processing) can be useful for estimating rock properties in the framework of a laboratory experiment. Generalization of the method to other acquisition configurations such as surface seismic data requires further work.

Key words: asymptotic method, attenuation, diffraction tomography, rock properties, ultrasonic seismic data.

INTRODUCTION
During the last decade, interest in the attenuation of seismic waves has increased. This interest has been motivated by improvements in the characterization and monitoring of hydrocarbon reservoirs (Sheriff 1975) and, more generally, of the earth models that can be obtained when accounting for the anelasticity of rocks (Randall 1976). These improvements concern the lithologic description of the Earth, the analysis of its physical state and the degree of saturation of the rocks.

Different techniques have been used in the field and in the laboratory to study the attenuation of acoustic waves propagating through rocks (Toksöz & Johnston 1981).

In the field, attenuation measurements were derived from a wide range of seismic signals such as direct and refracted
compressional and shear waves, surface waves, reflected waves and full-wave acoustic well logs (e.g. McDonal et al. 1958; Tullos & Reid 1969; Ben-Menahem 1965; Kudo & Shima 1970; Hamilton 1972; Hauge 1981; Braile 1977; Spencer et al. 1979; Wilcock et al. 1992).

In the laboratory, the methods generally used for measuring attenuation can be separated into the following categories: free vibrations, forced vibrations, observation of stress–strain curves and wave propagation methods (see Toksöz & Johnston 1981 for a review). The method presented in this paper belongs to the category of wave propagation methods.

The use of wave propagation experiments in the lower ultrasonic frequency range is of particular interest since the loss parameters involved closely parallel those measured in field experiments. In general, wave propagation experiments can be classified according to whether they make use of pulse-echo or through-transmission methods. These methods assume that the amplitude of a seismic wave (generally considered to be a plane wave) decays exponentially with distance or time and that one can correct for losses other than intrinsic attenuation.

The extraneous losses include beam spreading, coupling losses, diffraction losses and wedging losses.

For the pulse-echo technique, the attenuation is found by measuring the amplitude decay of multiple reflections from a free surface. The advantage of this method is the use of the absolute amplitude, which allows one to analyse easily the behaviour of the pulse decay (e.g. Peselnick & Zietz 1959).

Transmission experiments can be separated into different categories depending on the sample size and the locations of the transmitter and receiver transducers. In most cases, the transducers are located at opposite ends of the sample. These techniques are fraught with experimental and interpretational difficulties. At long wavelengths (relative to the transducer diameter), diffraction losses may become important. Because of beam spreading, side-wall reflections and mode conversion may occur, interfering with the direct wave. Diffraction may cause problems even with pulse transmission techniques. Energy loss can also occur in the transducer itself, at the boundary between transducer and sample, and in the electronic measurement system. Transducer properties are generally known; however, the other losses are impossible to calculate theoretically and many problems are tackled empirically (Truell & Oates 1963; Truell et al. 1969).

In terms of processing, inversion methods applied to seismic attributes or to the full waveform were developed to recover both elastic parameters and attenuation (e.g. Dietrich & Bouchon 1992; Bruzostowski & McMechan 1992; Wilcock et al. 1992; Blanch et al. 1995; Liao & McMechan 1996, 1997; Vasco et al. 1996; Matheny & Nowack 1997; Ribodetti & Virieux 1998; Causse et al. 1999). The full-waveform inversion methods were generally based on a plane wave decomposition modelling of the wavefield assuming a laterally homogeneous earth (Minkoff & Symes 1997; Dietrich & Bouchon 1985), or on finite difference-based modelling (Liao & McMechan 1996). The main drawback of these methods is their computational cost, particularly when iterative optimization requiring the forward problem to be solved a large number of times is used.

The method presented in this paper is close in spirit to the full-waveform inversion method but relies on the asymptotic ray theory to solve the forward problem and thus is less computationally expensive. The method is an extension to the viscoacoustic case of the original elastic method of Jin et al. (1992), who introduced an attractive asymptotic method for inverting seismic reflection data. Since the paper of Jin et al. (1992), several applications to 2-D and 3-D data sets have been presented for the acoustic case (Lambaré et al. 1992; Forgues 1996; Thierry et al. 1999a, b). An extension of the method to the viscoelastic case, together with synthetic validation tests, was presented by Ribodetti (1998) and Ribodetti & Virieux (1998).

The method is cast in the general framework of the least-squares inverse theory (Tarantola 1987) but exploits an asymptotic approximation of the forward operator in order to formulate the inversion through analytical expressions. The forward problem is linearized using the Born approximation. High-frequency asymptotic ray theory is used to compute traveltime, geometrical amplitude and attenuation of the scattered ray paths associated with each source–receiver pair. Diffraction tomography is solved by iterative least-squares minimization of the misfit between the observed and the computed scattered wavefields using a quasi-Newtonian algorithm. In the quasi-Newtonian algorithm, the high-frequency approximation allowed us to derive a local approximation of the Hessian operator. First, compared to more traditional gradient algorithms, the approximated Hessian allows us to reduce the number of iterations required to obtain a satisfactory fit between the observed and the computed data. Second, it allows us to analyse carefully the resolution of the diffraction tomography, which depends on the bandwidth of the source and the source–receiver configuration.

In this paper, the diffraction tomography formula was specifically developed for the acquisition geometry designed for the laboratory experiment. The data set used in this study was collected during a 2.5-D ultrasonic experiment carried out in a water tank (Valero 1997). We simulated a fixed-offset experiment with a circular acquisition geometry providing a complete angular coverage of the target. The target to be imaged is a section of a polyvinyl chloride (PVC) cylinder set vertically in a water tank. The original motivation for this experiment was to test the reliability of the method in recovering the attenuation factor $Q$ when applied to real data recorded during a physically scaled experiment, which allows good control of the experimental parameters (acquisition geometry and properties of the background medium). The future prospect of this work is to be able to measure velocity and attenuation in rock samples and to compare these experimental measurements with those obtained on the scale of field seismic reflection experiments.

The viscoacoustic tomography provided both velocity and $Q$ images of the target. We verified the efficiency of the inversion by comparing the observed and the computed data. An excellent match between the two data sets was obtained after six iterations. The absolute values of the velocity and $Q$ factor in the target were derived from a post-processing procedure applied to the tomographic images.

This paper is organized as follows. In the first part, we present the experimental set-up. In the second part, we present the data set collected during the experiment. In the third part, we derive the diffraction tomography formula for the acquisition geometry considered in this study. In the fourth part, we present the results of the tomography. Finally, in the fifth part, we present the post-processing of the tomographic images in order to estimate the absolute values of the velocity and attenuation.
EXPERIMENTAL SET-UP

The ultrasonic experiment was carried out in a water tank to simulate geophysical tomography. This water tank has dimensions of $2.0 \text{ m} \times 1.40 \text{ m} \times 1.50 \text{ m}$ and is equipped with computer-based control and data acquisition systems (National Instrument, Austin, TX) (Valero 1997). 5000 l of water was used as a constant-velocity background medium. We used one hydrophone for the source and another for the receiver. The wavebands of the source and receiver hydrophones were between 1 and 100 kHz and between 0.1 Hz and 160 kHz respectively (see Valero 1997 for more details). The source and receiver configuration is depicted in Fig. 1(a). Source and receiver are located on a horizontal plane and rotate together around a fixed vertical axis. The radius of the circle described by the source and receiver trajectories is 0.469 m. The offset between the source and the receiver is kept constant during acquisition (fixed-offset acquisition) with an angle $\Delta \phi_0$ of 15° between the source and the receiver radius. The source and receiver hydrophones are at $Z = 0.67$ m depth beneath the water level. The initial angle $\phi$ was fixed to 35°. The angular step $\Delta \phi_s$ between two consecutive source positions was 5°, which results in 72 seismograms per common-offset gather.

The dominant frequency of our signals is 80 kHz, corresponding to a wavelength of 0.019 m in water (Fig. 2). Waveforms are digitized with a sampling interval of $2 \times 10^{-6} \text{ s}$.

The object to be imaged consists of a PVC cylinder set vertically in the tank (Fig. 1b). The diameter of the PVC cylinder is 0.053 m. Note that the axis of the cylinder does not coincide with the axis of the source–receiver system (Fig. 1b). Because of the acquisition [source and receiver lie in a common $(X, Y, Z)$ plane] and target geometry (shape of the cylinder is invariable along $Z$), our experiment is '2.5-D'. Thus, the target is a section of the cylinder located in the horizontal plane defined by the source and receiver positions (Fig. 1b). The target zone is discretized with a uniform mesh of $301 \times 301$ nodes. The mesh spacing is $dx = dy = 0.0005 \text{ m}$.

PRE-PROCESSING OF DATA AND BACKGROUND MEDIUM ANALYSIS

The processing used in this study is diffraction tomography. Therefore, it is applied to the scattered wavefield. To separate the scattered wavefield from the total wavefield, the acquisition is performed in two steps (Lo et al. 1988). First, we put the cylinder inside the water tank, scanning the source and receiver

![Figure 1](image-url)
around it and measuring the total wavefield (Fig. 3a). After removing the object, we repeat the same scanning procedure in order to obtain the incident wavefield (Fig. 3b). The difference between these two data sets is the wavefield scattered by the object (Fig. 3c). This dual-experiment method also helps to eliminate interference from the experimental set-up such as walls of the tank and the hydrophone itself.

The reflector at around 0.87 ms is the reflection from the water surface (Figs 3a and b). Note that the traveltimes of this reflector increase between the angular bands $150^\circ - 220^\circ$ and $330^\circ - 360^\circ$. This shows that the source–receiver system deepens during the experiment. Nevertheless, the traveltime delay or advance of the horizontal scattered ray path connecting the source and the receiver to the cylinder caused by the deepening of the source–receiver system should be negligible compared to the total traveltime of the scattered ray path. A close-up of the scattered wavefield $\delta P$ is displayed in Fig. 4. For pre-processing, a trapezoidal bandpass filter $25-30-125-130$ kHz was applied to the data. We did not apply spiking deconvolution to the data. Instead we applied a post-processing procedure to the tomographic images in order to remove the source signature as explained in the section ‘Post-processing of images’.

In Fig. 4, we superimpose the traveltimes of the first reflected arrival corresponding to the reflection from the external edge of the cylinder. The secondary arrivals correspond to the two reflections from inside the cylinder and from the opposite external edge of the cylinder and to multiples from inside the cylinder. These latter arrivals will not be processed properly by the inversion since we used the first-order Born approximation. The arrival labelled $a$ is a free-surface reflection of the scattered wavefield. The arrival labelled $b$ is a reflection from the bottom of the water tank plus a reflection from the water level.

Note that the reflections from the water level were not fully eliminated by the dual procedure because we did not account for the change of water level when inserting the cylinder and its prop into the water. In contrast, the reflections from the edges of the cylinders were correctly eliminated, suggesting that the source and receiver positions were accurately reproduced during the two experiments.

In our experiment, the background model is composed of water. In the frame of an ultrasonic experiment for which very short wavelengths are considered, an accurate estimation of the water velocity is important. In fact, the localization of the structures in the image is sensitive to the accuracy of the background model (that is, the accuracy of the water velocity.

Figure 2. Amplitude spectrum of the data.

Figure 3. Illustration of the dual acquisition procedure. (a) Full wavefield recorded when the cylinder is set in the water tank. (b) Full wavefield recorded without the cylinder. (c) Difference between sections of (a) and (b). This data set represents the wavefield scattered by the cylinder. The three sections are plotted with the same gain.

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We used a velocity of 1520 m s\(^{-1}\) and a \(Q\) factor of 210 000 in the water, corresponding to a temperature of 34 °C and a frequency of 100 kHz (Fujii & Masui 1993; Toksöz & Johnston 1981).

**THEORY OF 2.5-D VISCOACOUSTIC RAY–BORN INVERSION**

To recover the spatially dependent parameters of the viscoacoustic medium (velocity and attenuation), we propose an extension of the single-parameter acoustic ray–Born algorithm of Thierry et al. (1999b). The theory is based on a combination of the ray theory to compute the asymptotic Green functions and the Born approximation to linearize the relation between the scattered waveform and a model of small perturbations.

It is well known that the introduction of \(Q\) in a modelling scheme is considered to be equivalent to treating velocity as a complex quantity (Chang & McMechan 1996). The complex velocity \(c\) is defined by Toksöz & Johnston (1981) as

\[
\frac{1}{c} = \frac{1}{c} \left( 1 + i \frac{\text{sign}(\omega)}{2Q} \right),
\]

where \(c\) is the velocity, \(Q\) is the attenuation factor and \(\omega\) is the angular frequency.

In contrast to the algorithms of Jin et al. (1992) and Thierry et al. (1999a), the inversion is implemented in the frequency domain in order to integrate easily the attenuation into the inversion. A frequency-domain inversion allowed us to set up, on the one hand, an analytical kernel for the forward problem and, on the other hand, an approximate analytical kernel for the linearized inversion.

From the Born approximation, we obtain a solution of the linearized forward problem for a viscoacoustic medium (Ribodetti & Virieux 1998),

\[
\delta P(s, r, \omega) = \frac{i \omega}{2} \int_{\mathcal{M}} G_0(s, x, \omega) G_0(x, r, \omega) K(x, \omega) \left( \frac{\delta c(x)}{c^0(x)} - i \frac{\text{sign}(\omega)}{2Q(x)} \right) dx,
\]

where the integration domain \(\mathcal{M}\) is over the diffracting points \(x\). Here \(s\) and \(r\) denote the source and receiver respectively and \(\delta P\) denotes the wavefield scattered by the perturbation model \((\delta c, \delta Q)\). \(G_0(s, x, \omega)\) and \(G_0(x, r, \omega)\) are the asymptotic Green functions computed in the background medium for the ray paths connecting the source \(s\) to the scatter point \(x\) and from the scatter point \(x\) to the receiver \(r\) respectively. In the frame of asymptotic ray theory for a viscoacoustic medium, it takes the form

\[
G_0(s, x, \omega) = \mathcal{A}(s, x) \mathcal{F}(\omega) \exp \left[ -|\omega| z(s, x) \right] \exp \left[ i \omega \mathcal{T}(s, x) \right],
\]

where \(\mathcal{A}\) and \(\mathcal{F}\) are the geometrical amplitude and traveltine associated with the ray connecting \(s\) to \(x\), \(\mathcal{F}(\omega)\) is the source term related to the dimension of propagation and \(z\) is the attenuation integrated along the ray path (Aki & Richards 1980). \(K(x, \omega)\) is the vector of components,

\[
\left( -\frac{2}{c^0(x)} \left( 1 + i \text{sign}(\omega) \right) \frac{1}{2Q(x)} \right)^2;
\]

\[
\frac{1}{c^0(x)Q^0(x)} \left( \frac{1}{2Q(x)} - i \text{sign}(\omega) \right),
\]

where \(c_0\) and \(Q_0\) parametrize the properties of the background medium. Note that the terms of the complex vector \(K\) depend only on the rheology of the background medium, not on the source–receiver configuration. This contrasts with other multi-parameter inversion schemes, e.g. acoustic (Forgues 1996), elastic (Jin et al. 1992) and viscoelastic inversions (Ribodetti & Virieux 1998). Indeed, for multi-parameter inversion, the analogue of \(K\) is the radiation diagram which depends on the source–receiver aperture. This dependence allows one to decouple the different parameters with the help of data redundancy. Although the aperture dependence does not exist in the viscoacoustic case, we will show with the analysis of the Hessian operator that the velocity and the \(Q\) perturbations are decoupled in the case of the acquisition designed into this experiment.

If we define the total amplitude, the total traveltime and the total attenuation as

\[
\mathcal{A}(r, x, s) = \mathcal{A}(r, x) \mathcal{F}(x, s),
\]

\[
\mathcal{F}(r, x, s) = \mathcal{F}(r, x) + \mathcal{F}(x, s),
\]

\[
\tau(r, x, s) = \tau(r, x) + \tau(x, s),
\]

Figure 4. Close-up of the scattered wavefield shown in Fig. 3(c).
we can rewrite eq. (2) as
\[
\delta P(s, r, \omega) = \mathcal{H}(\omega) \int_{\mathbb{R}} \mathcal{A}(r, x, s) \exp[i\omega \mathcal{F}(r, x, s)]
\times \exp[-\frac{\rho(r, x, s)}{\rho} \mathcal{K}(x, \omega)] \left( \begin{array}{c} \delta \mathcal{Q}(x) \\ \delta \mathcal{Q}(x) \end{array} \right) dx ,
\] (6)
where in 2.5-D,
\[
\mathcal{H}(\omega) = \omega^2 \mathcal{F}(\omega)^2 = \frac{1}{\sqrt{-i\omega}} \omega^2
\] (7)
(Thierry 1997). In compact form, we note the forward problem as
\[
\delta P = \mathcal{G}_\omega \mathcal{F},
\] (8)
where
\[
\mathcal{G}_\omega = \left( \begin{array}{c} \delta \mathcal{Q}(x) \\ \delta \mathcal{Q}(x) \end{array} \right).
\]
Following the approach of Jin et al. (1992), the velocity and \( \mathcal{Q} \) perturbations are obtained by iterative least-squares minimization of the weighted misfit between the observed and the computed wavefields using a quasi-Newtonian algorithm. The iterative inversion is linear since the background model is kept constant over iterations.

The solution of the quasi-Newtonian inversion is given by
\[
\mathcal{F}_\omega = \left[ \mathcal{G}_\omega^\dagger \mathcal{G}_\omega \right]^{-1} \mathcal{G}_\omega^\dagger (\delta \mathcal{P}_{\text{obs}} - \delta \mathcal{P}_{\text{synth}}),
\] (9)
where \( \delta \mathcal{P}_{\text{obs}} \) and \( \delta \mathcal{P}_{\text{synth}} \) are the observed and the computed scattered wavefields respectively and the subscript \( \dagger \) denotes the adjoint operator. At the first iteration, \( \delta \mathcal{P}_{\text{synth}} = 0 \) since the background model is homogeneous and the perturbations are zero. \( \mathcal{G}_\omega \) is a weighting of the cost function.

**Approximation of the Hessian**

In eq. (9), the term to be inverted, \( \mathcal{G}_\omega^\dagger \mathcal{G}_\omega \), is the Hessian. The Hessian is an operator mapping the model space to itself and thus represents a huge matrix whose dimension equals the square of the number of scatterers in the image. Due to its huge dimension, it is difficult to invert it numerically.

The asymptotic (local) approximation of the Hessian for the acoustic case is given by Thierry et al. (1999b). It follows for the viscoacoustic case and for the source–receiver configuration of concern that
\[
\mathcal{G}_\omega^\dagger \mathcal{G}_\omega \approx \int_{\phi} d\phi \int_{\omega_+} d\omega_+ \int_{\omega_-} d\omega_- \int_{\omega_0} d\omega_0 \int_{\omega_0} d\omega_0 \mathcal{D}(r, x_0, s, \omega) \mathcal{K}_+(x_0, \omega) \mathcal{K}_-(x_0, \omega)
\times \exp[-i\omega \mathcal{F}(r, x_0, s) \cdot (x - x_0)]
\approx \int_{\phi} d\phi \int_{\omega_+} d\omega_+ \int_{\omega_-} d\omega_- \int_{\omega_0} d\omega_0 \mathcal{D}(r, x_0, s, \omega) \mathcal{K}_+(x_0, \omega) \mathcal{K}_-(x_0, \omega)
\times \exp[-i\omega \mathcal{F}(r, x_0, s) \cdot (x - x_0)],
\] (10)
where
\[
\mathcal{D}(r, x_0, s, \omega) = \mathcal{A}(r, x_0, s) \mathcal{H}(\omega)^2 \exp[-2|\omega|\mathcal{F}(r, x_0, s)].
\] (11)

We remind the reader that \( \mathcal{K}(r, x, s, \omega) = \mathcal{F}(r, x, s, \omega) \mathcal{K}(x, \omega) \) is the two-way traveltime of the scattered ray path \( s \rightarrow x \rightarrow s \) (Fig. 5).

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**Figure 5.** (a) Illustration of the double coverage provided by the acquisition geometry. (b) Close-up of (a) together with the parameters involved in the tomography formalism. Note that the two reflected ray paths at scatter point \( x \) do not have the same aperture (thus, the two vectors \( q \) shown on the figure do not have the same norm).

Integration in the Hessian formula is carried out over a set of parameters allowing us to describe the data space. In the case of this constant-offset experiment, the data set can be parameterized by two parameters, the angle \( \phi \) and the angular frequency \( \omega \). \( \phi \) is the angle between the \( y \)-axis and the segment \( (O, s) \) and allows us to describe the position of any source–receiver pair (Fig. 1). In the following, we shall replace the pair of arguments \( (s, r) \) by the single argument \( \phi \). For reasons explained later, we split the integral over the frequency of the Hessian (eq.10) into an integral over positive and negative frequencies,
\[
\mathcal{G}_\omega^\dagger \mathcal{G}_\omega \approx \int_{\phi} d\phi \int_{\omega_+} d\omega_+ \int_{\omega_-} d\omega_- \int_{\omega_0} d\omega_0 \mathcal{D}(r, x_0, s, \omega) \mathcal{K}_+(x_0, \omega) \mathcal{K}_-(x_0, \omega)
\times \exp[-i\omega \mathcal{F}(r, x_0, s) \cdot (x - x_0)]
\] (12)
The expressions for the Fourier integral of the 2-D Dirac function in polar and Cartesian coordinates are respectively

\[
\delta(x-x_0) = \frac{1}{(2\pi)^2} \int |d\Psi| |k| \exp\left[-ik(x-x_0)\right],
\]

where \(\Psi \in [0; 2\pi]\) and \(|k| \in [0; +\infty]\), and

\[
\delta(x-x_0) = \frac{1}{(2\pi)^2} \int |d\Psi| |k| \exp\left[-ik(x-x_0)\right],
\]

where the two components of \(k\) vary between \(-\infty\) and \(+\infty\).

One can see the similarity of the phases for the Hessian (eq. 10) and the Dirac function (eqs 13 and 14). In order to formulate the Hessian as a weighted Dirac function, one has to apply two changes of variables from the two data subspaces, \((\phi, \omega^-)\) and \((\phi, \omega^+)\), to the model space, \((\Psi, |k|)\):

\((\phi, \omega^-) \rightarrow (\Psi, |k|)\) or, equivalently, \((\phi, \omega^-) \rightarrow (k)\),

\((\phi, \omega^+) \rightarrow (\Psi, |k|)\) or, equivalently, \((\phi, \omega^+) \rightarrow (k)\).

The integral over frequency in the Hessian formula (eq. 10) was previously split into two integrals (eq. 12) in order to have a one-to-one mapping in the two changes of variables. Indeed, the angular coverage of the acquisition design, \(0 < \phi < 2\pi\), allows one to cover all the directions of the \(k\) vector parameterized by the angle \(\Psi\) (Fig. 5b). The two subdata sets \((\phi, \omega^-)\) and \((\phi, \omega^+)\) then each map the data space \(k\) once within the limit of the bandwidth of source (Fig. 6). Note that this contrasts with a seismic surface data set whose angular coverage provided by the source–receiver configuration is much more narrow. In the ideal case of an infinite constant-offset surface acquisition, the \(\Psi\) coverage is between 0 and \(\pi\) and is one half of the angular coverage of this experiment. Then, in this case, both negative and positive frequencies are required to map the model space \((k)\) once. Fig. 7 shows the spatial Fourier-domain resolution provided by the source–receiver configuration. The resolution is mainly governed by the limited bandwidth of the source. Note also that the contour of the resolved area in Fig. 7 is not rigorously a circle because \(|k| = \omega |q| \approx \omega(1/V_0)\), \(p_+ + p_-\), varies with the aperture of the scattered ray path (which in turn varies with \(\phi\) since the target is not centred on the circle described by the source, Fig. 5). Nevertheless, at the scale of this experiment, the effect of the variable aperture of the scattered ray paths is negligible compared to the bandwidth of the source.

After applying the two changes of variables, the Hessian is given by

\[
\frac{\partial^2}{\partial \phi \partial \omega}(x, x_0) = \frac{1}{(2\pi)^2} \int |d\Psi| |k| \exp\left[-ik(\phi, \omega, x_0) \cdot (x-x_0)\right]
\]

\[
\times \int (\frac{\partial (\phi, \omega^-)}{\partial k})(2\pi) K_{\omega^-}^{-1}(x_0) K_{\omega^-}(x_0) \frac{\partial}{\partial \omega}(\phi, x_0, \omega^-)
\]

\[
+ \frac{1}{(2\pi)^2} \int |d\Psi| |k| \exp\left[-ik(\phi, \omega, x_0) \cdot (x-x_0)\right]
\]

\[
\times \int (\frac{\partial (\phi, \omega^+)}{\partial k})(2\pi) K_{\omega^+}^{-1}(x_0) K_{\omega^+}(x_0) \frac{\partial}{\partial \omega}(\phi, x_0, \omega^+).
\]

Figure 6. Illustration of the double coverage of the model space \((k_x, k_y)\) by the two data subspaces \((\omega^+, \phi)\) and \((\omega^-, \phi)\). Angle \(\phi\) is defined in Fig. 1.

(a) Coverage provided by the positive frequencies. (b) Coverage provided by the negative frequencies.

Figure 7. Coverage of the tomography in the spatial Fourier domain.
is the Jacobian of the changes of variables. The expression for the Jacobian is change of variable never vanishes for any scatterer \( x \) and any angle \( \phi \). Indeed, this assumption can be easily checked given the simple acquisition geometry and the fact that the background model is homogeneous.

We choose \( \mathcal{J} \) to make the amplitude term of the two integrals of the Hessian (eq. 15) reduce to the matrices \([1/(2\pi)^2]^{\mathbf{K}_-^\dagger \mathbf{K}_-}\) and \([1/(2\pi)^2]^{\mathbf{K}_+^\dagger \mathbf{K}_+}\), respectively:

\[
\mathcal{J} = \frac{1}{(2\pi)^2} \mathcal{F}(\phi, x_0, \omega^-) \frac{\partial k}{\partial k} \frac{\partial k}{\partial \phi}
\]

\[
= \frac{1}{(2\pi)^2} \mathcal{F}(\phi, x_0, \omega^+) \frac{\partial k}{\partial k} \frac{\partial k}{\partial \phi}.
\]

Substituting the explicit expression of \( \mathcal{J} \) in the Hessian formula (eq. 15) gives

\[
\mathcal{G} \mathcal{H}(x, x_0) = \frac{1}{\mathcal{A}} \int \frac{d\mathbf{K}^\dagger}{d\mathbf{K}} (x_0) \mathbf{K}_- (x_0) \times \exp \left[ -i \mathbf{k} \cdot (x - x_0) \right]
\]

\[
+ \int \frac{d\mathbf{K}^\dagger}{d\mathbf{K}} (x_0) \mathbf{K}_+ (x_0) \times \exp \left[ -i \mathbf{k} \cdot (x - x_0) \right].
\]

As \( \mathbf{K}(x_0) \) depends only on the properties of the background model, the Hessian can be approximated by a weighted Dirac function,

\[
\mathcal{G} \mathcal{H}(x, x_0) \approx \left[ \mathbf{K}_-^\dagger (x_0) \mathbf{K}_- (x_0) + \mathbf{K}_+^\dagger (x_0) \mathbf{K}_+ (x_0) \right] \cdot \delta (x - x_0).
\]

As mentioned earlier, the final expression for the Hessian (eq. 18) is an approximation because of the limited bandwidth of the source. The bounds of integration over \( k \) are infinite in the Dirac expression (eq. 13), while they are limited by the bandwidth of the source in the expression of the Hessian (eq. 17). The exact asymptotic Hessian is not a weighted Dirac function but a filtered one, where the filter defines the resolution of the image displayed in Fig. 7.

The velocity and the factor \( Q \) perturbations are decoupled in the inverse problem if the matrix \( \left[ \mathbf{K}_-^\dagger (x_0) \mathbf{K}_- (x_0) + \mathbf{K}_+^\dagger (x_0) \mathbf{K}_+ (x_0) \right] \) is not singular.

In compact form, we can write the expression for \( \mathbf{K}(x, \omega) \) (eq. 4) as

\[
\mathbf{K}(x, \omega) = (a + i \operatorname{sign}(\omega) b; c + i \operatorname{sign}(\omega) d),
\]

where

\[
a = -\frac{1}{\omega_0} \left( 1 - \frac{1}{4 \omega_0^2} \right),
\]

\[
b = \frac{2}{\omega_0^2 \Phi_0},
\]

\[
c = + \frac{1}{2 \omega_0^2 \Phi_0^3}
\]

and

\[
d = - \frac{1}{\omega_0^2 \Phi_0^3}.
\]

We then have

\[
\mathbf{K}_+ (x) = (a + ib; c + id),
\]

\[
\mathbf{K}_- (x) = (a - ib; c - id)
\]

and

\[
R = \mathbf{K}^\dagger \mathbf{K} = \mathbf{K}_-^\dagger \mathbf{K}_- + \mathbf{K}_+^\dagger \mathbf{K}_+ = \begin{pmatrix} A & B \\ B & C \end{pmatrix}.
\]

The inversion of the Hessian matrix now reduces to the inversion of a \( 2 \times 2 \) symmetric matrix. This matrix can be diagonalized by an equivalent matrix with a conditioning number less than or equal to one (Forgues 1996).

**Final formula**

After diagonalization of the Hessian (eq. 18), the perturbations \( f(x) \) are obtained analytically by a weighted diffraction stack of the data misfit \( \Delta \mathcal{P}(\phi, \omega) = \delta \mathcal{P}_{\text{obs}}(\phi, \omega) - \delta \mathcal{P}_{\text{synth}}(\phi, \omega) \).

\[
f(x) = -\frac{1}{2} \mathcal{R}^{-1} \int d\phi \mathcal{A}(x, \phi) \int d\alpha \mathcal{K}^\dagger (x, \omega) \frac{1}{\mathcal{A}(x, \omega)} \mathcal{A}(x, \omega) \]

\[
\times \exp \left[ -|\alpha|^2 \mathcal{A}(\phi, \omega) \right] \exp \left[ -io \mathcal{A}(\phi, \omega) \right] \Delta \mathcal{P}(\phi, \omega) .
\]

Note that the perturbations are divided by 2 to account for the double coverage provided by the acquisition geometry.

This equation can be discretized

\[
f(x) = -\frac{1}{2} \mathcal{R}^{-1} \sum_{\phi} \Delta \phi \mathcal{A}(x, \phi) \sum_{\omega} \Delta \omega \frac{1}{\mathcal{A}(x, \omega)} \mathcal{K}^\dagger (x, \omega) \frac{1}{\mathcal{A}(x, \omega)} \mathcal{A}(x, \omega) \]

\[
\times \exp \left[ -|\omega|^2 \mathcal{A}(\phi, \omega) \right] \exp \left[ -io \mathcal{A}(\phi, \omega) \right] \Delta \mathcal{P}(\phi, \omega) .
\]

Again the summation over the frequencies is split into two summations over positive and negative frequencies,

\[
f(x) = -\frac{1}{2} \mathcal{R}^{-1} \sum_{\phi} \Delta \phi \frac{1}{\mathcal{A}(x, \phi)} \mathcal{K}^\dagger (x, \phi) \frac{1}{\mathcal{A}(x, \phi)} \mathcal{A}(x, \phi) \]

\[
\times \left( \sum_{\omega} \Delta \omega \frac{1}{\mathcal{A}(x, \omega)} \mathcal{K}^\dagger (x, \omega) \frac{1}{\mathcal{A}(x, \omega)} \mathcal{A}(x, \omega) \right) \exp \left[ io \mathcal{A}(\phi, \omega) \right] \exp \left[ -io \mathcal{A}(\phi, \omega) \right] \Delta \mathcal{P}(\phi, \omega) .
\]

All the complex terms in the summation over positive frequencies are the conjugates of those in the summation over
negative frequencies. Then, the summation can be computed over positive frequencies only,

$$f(x) = -R^{-1} \sum_{\phi} \sum_{\omega} |v| \left| \frac{\partial k}{\partial v} \right| \exp[i(\phi + \omega)]$$

$$\times |Re(1) K(v, x) \exp [-i\omega T(x)] | dP(v, x).$$

where $Re$ denotes the real part of a complex number.

INVERSION RESULTS

The frequency range used in the inversion is between 25 and 130 kHz. Six iterations were computed to obtain the final velocity and $Q$ images. The CPU time on one processor of a SUN E450 computer required to compute the six iterations (forward and inverse problems) was 4 hr 35 min.

The recovered velocity and $Q$ images are shown in Fig. 8 at the first iteration and in Fig. 9 at the sixth iteration. The shape and dimensions of the recovered object are difficult to assess accurately because the source signature and the multiples pollute the images. Nevertheless, the gross contours of the
cylinder both on the velocity and the $Q$ images are clearly identified in Figs 8 and 9. The effect of the iterations is to update the amplitude of the perturbations while the shape and the localization of the object do not vary over the iterations.

To verify the efficiency of the iterative inversion, we compared the observed data and the ray–Born synthetics and displayed the residuals between the two sets of seismograms (Figs 10 and 11). A good fit between the observed and the

Figure 10. (a) Observed scattered wavefield. (b) Viscoacoustic ray–Born synthetics computed from the perturbations obtained for iteration 1. (c) Residuals.

Figure 11. (a) Observed scattered wavefield. (b) Viscoacoustic ray–Born synthetics computed from the perturbations obtained at iteration 6. (c) Residuals.
predicted seismograms was obtained. This fit was significantly improved until the sixth iteration, thus validating the efficiency of the iterative procedure.

For comparison, we also computed the image of the velocity perturbations in the acoustic case (Fig. 12). The acoustic and viscoacoustic velocity perturbation images are very close. Nevertheless, the viscoacoustic synthetics fit the observed scattered wavefield significantly better than the acoustic one (Figs 13 and 14). Note that, in the acoustic case, we computed up to 12 iterations to verify that the better fit obtained with the viscoacoustic tomography was not due to a faster convergence over the iterations (in fact, the same number of iterations was required with acoustic and viscoacoustic tomography to obtain the best-fitting synthetics). Indeed, the better fit obtained when accounting for attenuation cannot prove that the viscoacoustic model is more rheologically significant than the acoustic one since the viscoacoustic rheology is parametrized by one more parameter than that for the acoustic one. Nevertheless, the excellent match between the observed and computed seismograms together with the result of the theoretical study of the Hessian operator, which showed that the velocity and $Q$ parameters are decoupled in the inversion, gives us confidence in the validity of the viscoacoustic model.

Note also that both the acoustic and viscoacoustic inversions were able to fit artificially several coherent noises such as multiples, although the first-order Born approximation, used to solve the forward problem, does not account for these signals. The fit of these noises is better when using the viscoacoustic parametrization than the acoustic one due to the additional parameter $Q$. After migration these noises can be observed on the migrated sections (reflector labelled A on Fig. 8) as coherent reflectors with significant amplitudes. These artificial reflectors are classically attenuated in migrated sections by stacking multichannel data characterized by strong data redundancy. In this study, we used a small volume of common-offset data (single-channel data) which precludes efficient attenuation of the coherent noises in the migrated sections by stacking the data. The poor redundancy of the data also explains why the inversion is able to fit signals that are not properly

---

**Figure 12.** Velocity perturbations obtained at iteration 12 for an acoustic inversion.

**Figure 13.** (a) Observed scattered wavefield. (b) Acoustic ray–Born synthetics computed from the perturbations obtained for iteration 12. (c) Residuals.
taken into account by the forward modelling (i.e. multiples, coherent noises). The coherent noises after migration can locally alter the quantitative estimation of the properties of the cylinder where they interfere with the image of the cylinder. Nevertheless, the overall good continuity of the velocity and $Q$ images of the cylinder suggests that the estimation of these two parameters is reliable along most of the cylinder contour.

POST-PROCESSING OF IMAGES: GEOMETRY AND MEDIUM PROPERTIES ANALYSIS

We designed a post-processing procedure in order to remove the source signature from the tomographic velocity and $Q$ images and to estimate the absolute value of the velocity and attenuation in the cylinder (Fig. 15). The post-processing is formulated as an inverse problem where the tomographic images constitute the data space. More precisely, the data set is composed of several traces extracted from the images. These traces go through the centre of the cylinder section and intersect it with different azimuths (see upper right panel in Fig. 15). The model space is composed of a boxcar family with a fixed amplitude and width and with a variable position (Fig. 15). These boxcars mimic the radius of a cylinder. The amplitude of the boxcar represents the amplitude of the velocity or $Q$ perturbations. The width of the boxcar represents the thickness of the cylinder section. To build the predicted data the boxcars are converted from space to time, using the velocity of the background medium, and are convolved with the source wavelet. To estimate the source wavelet we stack several traces centred on the direct arrival. Then, the convolved boxcar is converted back from time to space for comparison with the trace extracted from the tomographic image. The inverse problem is solved by a systematic exploration of the model space for each azimuth, keeping the best-fitting model in a least-squares sense. We tested 21 radii with a step of 0.0005 m. The smallest radius was 0.021 m. Once all the azimuths were explored, the best-fitting convolved and impulsive boxcars were put together in the $(X, Y)$ plane to build 2-D synthetic images of the cylinder section (the two bottom panels of Fig. 15).

The results of this post-processing are shown in Fig. 16 for velocity and Fig. 17 for the attenuation factor $Q$. The shape of the recovered cylinder section is acceptable for the velocity and $Q$ synthetic images (Figs 16 and 17). Nevertheless, we note that the recovered shape of the cylinder is slightly oval. This may result from a combination of inaccuracies in the experimental set-up (inaccurate measurement of the centre of the source–receiver system, inaccurate measurement of the radius of the circle described by the source–receiver system, the source–receiver system not remaining in a horizontal plane during the experiment) rather than from the processing itself.

On the attenuation synthetic image, the post-processing failed locally to fit the exact shape owing to the sensitivity of the attenuation to the amplitude. To estimate the absolute value of the velocity and attenuation factor in the cylinder, we applied the post-processing for several values of the velocity and $Q$ perturbations (namely, for several amplitudes of the boxcars in Fig. 15). We computed the $L_2$ misfit for each tested
amplitude. The misfit as a function of the velocity and $Q$ perturbations are displayed in Fig. 18. A minimum is easily identified on the velocity curve (Fig. 18a). On the attenuation curve, the minima are less easily identified with low values of $\delta Q$, suggesting that the uncertainty for the estimation of $Q$ is more important than for velocity.

**CONCLUSIONS**

We have derived the formulae for an asymptotic viscoacoustic diffraction tomography that uses an iterative quasi-Newtonian formalism. We have shown, with the help of an analysis of the Hessian operator, that for the acquisition geometry designed in...
this study, the velocity and the attenuation factor parameters were decoupled in the asymptotic inversion.

Tomographic images of both velocity and attenuation factor \( Q \) allow us to identify the target clearly. Moreover, we obtained a very good fit between observed and viscoacoustic predicted synthetics, which gave us confidence in the viscoacoustic model. The fit obtained in the viscoacoustic case was significantly better than that obtained with an acoustic rheology. We also proposed a post-processing procedure in order to remove the source signature from the velocity and \( Q \) tomographic images and to estimate the absolute values of the velocity and the attenuation factor in the target.

Future work will concern (1) some improvements of the experimental set-up in order to verify more rigorously the estimation of the absolute amplitude of the parameters and then to improve more efficiently the signal-to-noise ratio; this will require a more accurate control of the source–receiver positions together with a better a priori knowledge of the rheological properties of the experimental material (unknown in this study); and (2) the generalization of the method to other ‘less ideal’ acquisition geometries.

The experimental set-up and the processing developed in this study could be used as a tool for the estimation of rock properties.

\[\text{Figure 16. (a) Convolved model for velocity perturbations obtained with the procedure described in Fig. 15. (b) Associated impulsional model. The exact cylinder contours are indicated by the dotted grey circles.}\]

\[\text{Figure 17. (a) Convolved model for } Q \text{ perturbations obtained with the procedure described in Fig. 15. (b) Associated impulsional model. The exact cylinder contours are indicated by the dotted grey circles.}\]
The wave propagation method introduced in this paper may present the following advantages for the analysis of rock properties.

1. Compared to wave propagation methods using transmitted waves, the results obtained with diffraction tomography may be extrapolated more easily to the scale of reflection seismic experiments because of the analogy of the propagation mode (namely, reflection) used in the two kinds of experiments.

2. The experimental set-up is particularly adaptable. The background medium is composed of water and it is easy to control its properties such as temperature. It is also easy to separate the wavefield scattered by the target from the other signals by performing a dual acquisition (with and without the target in the water tank).

3. The ranges of acquisition and sample geometries (shape and dimensions) that can be considered are very wide.

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Figure 18. (a) Misfit between the velocity perturbation images and the best-fitting convolved model as a function of the velocity perturbation. The best-fitting value is 400 m s$^{-1}$ corresponding to a value of 1920 m s$^{-1}$ in the cylinder. (b) Misfit between the $Q$ perturbation images and the best fitting convolved model as a function of the $Q$ perturbation. The best fitting value is $-5000$ corresponding to a $Q$ of 160 000 in the cylinder.
Appendix A: Derivation of the Jacobian

In this Appendix, we derive the Jacobian $\mathcal{J}$ for the change of variable from $(\phi, \omega)$ to $\mathbf{k}$.

$$\mathcal{J} = \left| \frac{\partial \mathbf{k}}{\partial (\phi, \omega)} \right|. \tag{A1}$$

Given that $\mathbf{k} = \omega \mathbf{q}$,

$$\mathcal{J} = \omega \left| \frac{\partial \hat{q}_x}{\partial \omega} \frac{\partial \hat{q}_y}{\partial \phi} - \frac{\partial \hat{q}_x}{\partial \phi} \frac{\partial \hat{q}_y}{\partial \omega} \right|. \tag{A2}$$

The components of $\mathbf{q}$ (Fig. 6) are decomposed as

$$q_x = p_s + p_x, \quad q_y = p_s + p_y, \tag{A3}$$

where $s$ and $r$ represent the source and the receiver respectively.

We define the angles $\theta_s$ and $\theta_r$ as the angles made by $p_s$ and $p_r$ with the $x$-axis respectively (Fig. A1). We then have

$$p_s = \frac{1}{c_0} \cos (\theta_s); \quad p_r = \frac{1}{c_0} \sin (\theta_r); \quad \theta_s = \frac{1}{c_0} \sin (\theta_s), \tag{A4}$$

where $c_0$ is the velocity of the background medium. $l$ is the length of the segments $(O, s)$ and $(O, r)$. $L_s$ and $L_r$ are the distance between the source and the receiver and the scatterer $x$ respectively (Fig. A1).

We have for the source

$$\frac{\partial \hat{p}_s}{\partial \theta_s} = \frac{\partial \hat{p}_s}{\partial \theta_s} \frac{\partial \theta_s}{\partial \omega} \tag{A5}$$

We now consider a small perturbation of angle $\phi$, $d\phi$, and the associated small perturbation of angle $\theta_s$, $d\theta_s$, and we define the angle $\zeta$ as indicated in Fig. A1. We have the two relations

$$\xi = \theta_s - \phi + \frac{d\theta_s + d\phi}{2}, \tag{A6}$$

and

$$\cos \zeta \approx \frac{L_s d\theta_s}{L d\phi} \tag{A7}$$

Figure A1. Parameters involved in the calculation of the Jacobian.
Combining these two relations, we have

\[
\frac{d\theta_s}{d\phi} \approx \frac{1}{L_s} \cos \left( \theta_s - \phi \right), \quad \text{(A8)}
\]

Similarly, we have for the receiver

\[
\frac{d\theta_r}{d\phi} \approx \frac{1}{L_r} \cos \left( \theta_r - \phi \right), \quad \text{(A9)}
\]

where \( \Delta \theta_{st} \) is the angle between the segments \((\mathbf{O}, \mathbf{s})\) and \((\mathbf{O}, \mathbf{r})\) (see Fig. 2).

As the norms of \( \mathbf{p}_s \) and \( \mathbf{p}_r \) are known \((=1/c_0)\), as well as the angles \( \theta_s \) and \( \theta_r \) (by geometrical construction), all the terms in eq. (A2) can be easily computed. The angles \( \theta_s \) and \( \theta_r \) can be computed by geometrical construction. Then, all the terms of eq. (A2) can be easily computed using eqs (A3), (A4), (A5), (A8) and (A9).