Recognizing Ricci

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The names of the Italian mathematicians Ricci and Levi-Civita have been enshrined in the theory of general relativity since Einstein seized on the absolute differential calculus as the indispensable mathematical tool for expressing his uniquely determined gravitational equations. The physicist's long-standing indifference to mathematics changed abruptly as he struggled with the theory, methods, and notation of the calculus developed and refined by Gregorio Ricci-Curbastro, together with Tullio Levi-Civita at the University of Padua, before the end of the nineteenth century. While mathematicians and physicists are familiar with the Ricci tensor, by and large they know very little else about the mathematician for whom this symbol in differential geometry is named. This talk is a brief introduction to the story of his life.

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Who was Gregorio Ricci-Curbastro?

He was born and raised in Lugo di Romagna, a small papal town of about 22,000 inhabitants in the northeast corner of the Italian peninsula. The double-barreled last name seems to have been part of a longstanding tradition to differentiate among the different branches of the Ricci family. Ricci himself used only the first part of his name in the landmark 1900 tensor analysis paper he coauthored with Levi-Civita—and he has been known by that single name ever since.

His father, Antonio Ricci Curbastro, an engineer and substantial landowner, served for a time as the pope's *gonfaloniere*, Lugo's highest-ranking church official. In 1857, on the one occasion that Pius IX visited Lugo, he spent the night at the Ricci family palazzo. Two years later, in 1859, the country's long political struggle for independence erupted in Ricci's backyard. Before the Risorgimento, the peninsula's campaign for national unification, Italy consisted of a hodgepodge of states, from duchies, big and small, to the kingdoms of Sardinia and the Two Sicilies, to the states of the Church—all of them owned, swapped, and fought over by a succession of foreign rulers and popes. Then, between 1859 and 1861, Camillo Cavour, the prime minister of Piedmont, systematically annexed one region after another to transform "a geographical expression" into a single nation under one ruler. With the proclamation of the Kingdom of Italy in 1861, Victor Emmanuel II of the House of Savoy became the reigning monarch over twenty-two million people, four percent of whom had the right to vote. Only Venice, Rome, and Trieste remained beyond his grasp, until they too joined the new nation in 1866, 1870, and 1919 respectively.

Until the age of sixteen, Ricci was tutored privately at home in a wide range of subjects, including Italian and Latin, humanities and rhetoric; he excelled in

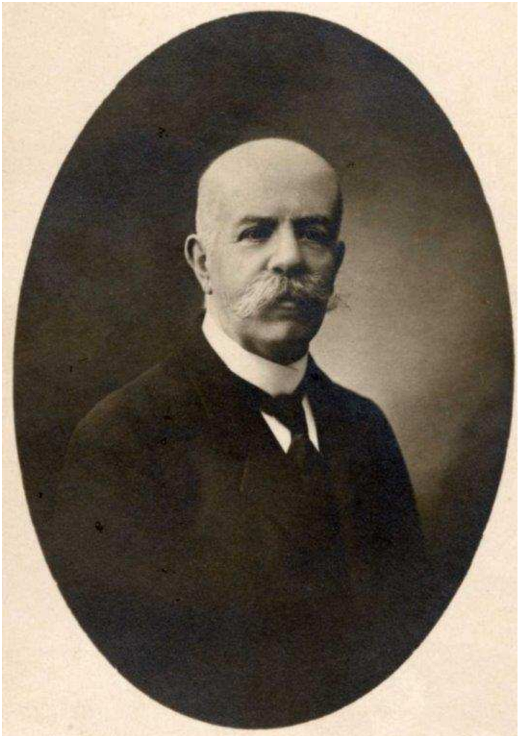

Fig. 1. Gregorio Ricci Curbastro (1853–1925). © Fondo Ricci Curbastro, Biblioteca Fabrizio Trisi, Lugo di Romagna.

translating long passages from Cicero, Virgil, and Homer into elegant Italian prose. Two years of private instruction in mathematics, physics, and astronomy rounded out his secondary school education. In 1869, following in the footsteps of his father, he enrolled at the Archiginnasio Romano in Rome, the last of the papal cities, which also offered instruction in medicine, law, and theology. A year later, he left Rome, with his bachelor's diploma in mathematics. This was a certificate that the school conferred on students who had successfully passed an examination in their field at the end of the first year's work.

Ricci was back home in Lugo when war between the armies of Napoleon III and Bismarck broke out in July 1870. It ended two months later in a crushing defeat for the French at the hands of the Prussians. Bismarck's victory provided Italy's king, Victor Emmanuel, with the pretext he needed to occupy Rome. The king's troops broke through the ancient walls of the city at Porta Pia on Sept. 20, bringing an end to papal rule in Rome and laying the groundwork for the city's annexation to the kingdom of Italy. Overnight, and with barely any armed resistance inside the walls, Rome had become the nation's capital.

When Ricci returned to Rome that fall to continue his university studies, he found that the collapse of the pope's temporal power there had dramatically changed the city's academic landscape. The Archiginnasio Romano had relocated to a nearby palazzo, along with those faculty and students who remained loyal to the pope. However, his parents fearing "further political disorder" [2] in the capital, summoned him home, where he studied on his own until 1872 when he enrolled as a second year student in mathematics at the University of Bologna. He spent exactly one year at Bologna, before enrolling at the Scuola Normale Superiore in Pisa the following year as a third-year transfer student.

Unlike the University of Bologna, the Scuola Normale Superiore di Pisa offered Ricci what he needed most in 1873—a thriving school of mathematics connected to the University of Pisa. Ten students competed for one of the school's coveted spots in the physical and mathematical section in 1873—the examiners rejected five of them.

In his first year as a *Normalista*, Ricci juggled four required math and science classes at the university with four classes at the Scuola Normale. His professors included Ulisse Dini and Enrico Betti, both of whom had few peers in the classroom. Dini introduced Ricci to mathematical analysis and higher geometry. Of Dini's lectures, it was said that he "stripped the principles of mathematical analysis of all superfluity and so gave its teaching a perfect rigor." [3] Dini was to leave an indelible mark on Ricci's approach to mathematics.

During his second year at the Scuola Normale, Ricci attended Betti's lectures on celestial mechanics and mathematical physics while working on his doctorate under Dini's supervision. In June 1875 he received his doctor's degree in mathematical physics with full honors, for his thesis on linear differential equations. He spent another year in Pisa doing post-graduate work in higher analysis under Dini's watchful eye, and in July 1876, he completed a second dissertation, which qualified him to teach mathematics in high school. Soon after, he satisfied the one remaining requirement, to present a lecture to the mathematics faculty. In his diary, he recorded the event as follows: "A formality that lasted about ten minutes, after which came the vote on my thesis," [4] which he obtained with praise.

Almost immediately, Ricci discovered that a freshly minted license to teach mathematics did not pay the rent or automatically open any classroom doors. However, when the chair of rational mechanics at Bologna became vacant that year, he entered the national competition for that. Had he consulted professors Betti or Dini beforehand, they might have advised him to lower his expectations and look elsewhere. Still, Betti wrote the strongest letter of recommendation he could, given that Ricci had yet to publish a single paper. Nothing came of these efforts. In Ricci's personnel file there is a cursory, undated statement, scribbled in red pencil, that reads simply: "He was not declared eligible." [5] Ricci tried again, in 1878, for teaching positions at Milan's Polytechnic and again at Bologna. By now he had publications to his credit, and his dossier was sufficiently impressive to place him second for the position at Milan and fourth for the chair of algebra and analytical geometry at Bologna.

With no immediate Italian prospects in sight, Ricci widened his horizons. Armed with a postgraduate fellowship for study abroad, he spent the 1878-1879 academic year in Munich, doing advanced work with Felix Klein, whose youthful contributions to non-Euclidean geometry and the links between geometry and group theory had established his reputation and made him a celebrated mathematician. Ricci followed Klein's courses on the theory of algebraic equations and on the theory of elliptic modular functions. He participated in Klein's mathematical seminar and read widely in the literature, including Riemann's paper on abelian functions. In many ways, Ricci came of age as a mathematician during his year in Munich.

After his return to Italy, Ricci served as Dini's assistant in his calculus course while filing applications for vacant chairs of mathematical physics at Rome, Bologna, Padua, and Palermo. This time, he placed first in the national competitions for Rome and Padua, and in 1880 he accepted the position of professor extraordinario (associate professor would be our equivalent) of mathematical physics at Padua. Soon afterward, Ricci sent Betti an urgent message concerning his first classroom lecture. He wrote: "I have learned they are in the truly regrettable habit of having the new [associate] professor inaugurate their courses with an opening address", which everyone at the university was free to attend. He continued, "Many people have told me to talk about something of no importance, but I need to take this seriously, although a mathematics course especially presents difficulties."[6] Only an undated draft of Betti's reply has survived. It consists of a few hastily scribbled lines on a scrap of paper tucked among his correspondence files in Pisa. "*Caro* Ricci," he began, "It seems to me that a subject for an introductory lecture could be the influence of mathematical physics on the progress of mathematical analysis."[7] This was followed by a scramble of incomplete, sometimes, illegible, words and phrases.

The mathematician Vito Volterra, who overlapped with Ricci during his last year at Pisa, described Betti as someone who lived and breathed theoretical physics and displayed "the greatest enthusiasm and desire to use it in a practical way."[8] It was his usual habit, Volterra added, "to reconcile analytical concepts with natural phenomena." It would have been unthinkable for Betti to encourage Ricci to talk down to his audience or to speak on a topic about which Ricci knew very little. Although he later dramatically altered the arc of his research, Ricci's initial publications at Padua suggest he took the advice Betti had offered him to heart. Over the course of more than four decades, Ricci added other topics to his teaching repertoire, including infinitesimal analysis and higher geometry. But abandoning his first love, mathematical physics, was not something he ever seriously considered. Even when he finally secured his promotion to full professor in 1890 (at the cost of relinquishing his own chair for the chair of algebra), Ricci continued to teach his original course in mathematical physics.

Accounts of Ricci's classroom performance are scarce. Although he taught at Padua for nearly half a century, no school of mathematics grew up around him.

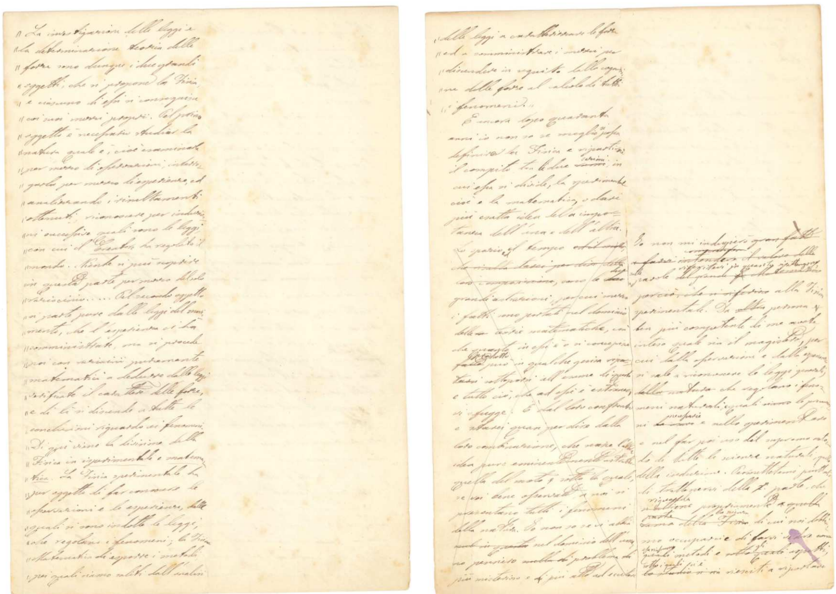

Fig. 2. Pages from the manuscript for Ricci's 1881 opening lecture in Padua. © Liceo Sc. St. "G. Ricci Curbastro", Lugo.

Numerous students came and went, among them Levi-Civita, who entered the university as a freshman in 1890, and once proudly remarked on his "good fortune and honor to be Ricci's favorite disciple." According to him, Ricci was not a charismatic speaker, but his lessons "were admirable for their precision and their extreme fluidity of form: whoever wrote them down would never have had to revise them." [9]

The Italian analyst Angelo Tonolo, who took a degree in mathematics at Padua in 1908 and went on to spend his entire career there, also considered Ricci his mentor. In a talk given on the occasion of Ricci's centenary in 1954, Tonolo recalled his professor's lectures as "a model of precision, never any hesitation, delivered by a tall, dark-haired and well-mannered scholar in a gentle, mild voice." Ricci, he added, was always "the dignified figure in compassionate contemplation, the sober gesture, and the carefully weighed and calm word, which gave him a diffident and distracted air." [10] Tonolo would later become the driving force behind the publication of Ricci's collected works in the 1950s.

For a time, Ricci taught higher geometry in Padua's teacher-training program. The lectures were scheduled for Saturday afternoons, and the earliest classes gave Ricci's students a foretaste of their professor's "diffident and distracted" aspect. As one of them later recalled, "After a couple of weeks, waiting in vain for the professor to appear, my classmates and I were leaving ... when he arrived in a carriage, completely out of breath (evidently he had [walked] part of the way), ran through the door, and went into the lecture hall, followed by us. Seating himself in

Fig. 3. Tullio Levi-Civita (1873–1941). © Institute Mittag-Leffler.

front of the blackboard, with his eyes lowered like a guilty schoolboy, he confessed, 'I forgot. This will not happen again.' "[11]

In the winter of 1884, shortly before his thirty-first birthday, Ricci began courting a young woman by the name of Bianca Bianchi Azzarani, who lived in nearby Imola, a small town dating back to the time of Cicero. Shy and reserved, and perhaps also fearing the disapproval of his father, who had many harsh words to say about the woman his older brother, Domenico, had fallen in love with, Ricci had enlisted the help of his parish priest in selecting a suitable mate. After identifying an appropriate candidate and securing the approval of the prospective bride and respective parents, the cleric arranged for Gregorio, who typically returned to Lugo from Padua during the university's winter recess, to meet Bianca at her family home in Imola that December. The first meeting must have gone well because, almost immediately after arriving back in Lugo, Ricci wrote to Bianca, expressing his own feelings about marriage and inviting Bianca to share hers. She did so in a series of sometimes thoughtful, sometimes saucy letters, suitably peppered with expressions of awe at his professional accomplishments. Barely two weeks after first setting eyes on each other, Bianca and Gregorio were ready to announce their engagement. They were married on August 25, 1884.

It was in the happy, hectic months leading up to and immediately following his wedding that Ricci began the construction of his absolute differential calculus, which he methodically elaborated in a series of papers published over the course of the next decade. These ranged from ten-page notes in the *Annali di Matematica pura e applicata* to the first systematic presentation of his calculus in a publication issued in connection with the 800th anniversary of the University of Bologna. Ricci's first paper on this topic, completed in Padua in February 1884, dealt with the theory of quadratic differential forms and their transformation properties (these are now called second rank tensors). The research problems he had earlier worked on under Betti and Dini's guidance had been consigned to a desk drawer and replaced by investigations into a challenging new branch of mathematics involving analysis and differential geometry. Building on the fundamental work of Gauss and Riemann on the foundations of geometry in the first half of the nineteenth-century, Ricci's first paper marked his entrance into this new mathematical arena where the rules of non-Euclidean geometry held sway.

In 1827, Gauss had published an important treatise covering his investigations of curved surfaces in two dimensions. He had also arrived at a theory of a non-Euclidean geometry after failing to prove Euclid's parallel axiom. However, fearing the controversy that would greet such a new and perplexing geometry, he did not publish it. The Russian mathematician Lobachevsky and his Hungarian counterpart Bolyai independently constructed and published a non-Euclidean geometry soon after. Their discoveries brought them neither fame nor followers. Most mathematicians of that era simply continued to work on projective and algebraic geometries.

Two decades later, however, Bernhard Riemann, a mathematician whom Felix Klein once described "as the only proper pupil of Gauss" [12] was quick to grasp this work's importance. In 1854, in his probationary lecture, Riemann offered a penetrating discussion dealing with the hypotheses underlying geometry's foundation—the lecture contained no calculations and was only published in 1868, two years after Riemann's death. In this lecture, Riemann advanced an all-encompassing view of geometry, in which space, which he now referred to as a manifold, consisted of any number of dimensions in any kind of space. The distance between two points that are infinitesimally close together on a curved surface in such a manifold, now known as the Riemannian metric, was defined by way of a positive quadratic differential form. Six years later, in 1861, in a memoir submitted to the French Academy of Sciences, Riemann remarked, in passing, "that some analytic expression could be illustrated as the curvature components of a metric manifold." [13] The significance of this work, in which Riemann "developed the whole apparatus of quadratic differential forms, now used in the theory of relativity," [14] apparently escaped the Academy's notice. His memoir was only published in 1876 in his collected works.

After the posthumous publication of these two lectures, Riemannian geometry came into its own as a vibrant branch of mathematics. By the time Ricci took up this subject in earnest in 1884, he had mastered the key papers in the field. As Ricci had discovered while devouring this literature, the mathematicians who took Riemann's writings on geometry and quadratic differential forms as their own starting point had carved out different approaches to the problem. As a mathematician who prized clarity, precision, and rigor above all else, Ricci felt sure that this branch of mathematics was ripe for improvement. Theory was the first order of business for Ricci, before he would allow himself any discussion of geometrical, mechanical, or physical applications.

In 1893, after nearly a decade of publishing papers on the subject, Ricci formally christened the mathematical methodology he had developed and refined as an "absolute differential calculus"—the same methodology that was to rivet Einstein's attention early in the next century. Years later, Levi-Civita would retrace, in largely non-technical language, the long and intricate mathematical road that Ricci followed on his way to the absolute calculus. He began by recalling how his professor "gradually seized upon and perfected the calculations" [15] in his first writings on differential forms, differential parameters, and partial differential equations, and then "modified the usual procedures employed in the differential calculus in such a way that the formulas and results always remained in the same form, no matter what system of coordinates was being used." He continued,

> This explains why we have . . . a system of functions [now called tensors] that behave in the same way when the coordinates are changed, independent of the choice of these coordinates. In addition, certain operations are introduced that are equally independent of the coordinates chosen, i.e. they are absolute, giving the name to the calculus.

Turning next to the advantages offered by Ricci's methods, Levi-Civita listed some of "the most useful applications [that] arise when the nature of the material under consideration requires a quadratic differential form," as in the case of general relativity, where the very small "interval between two events in space-time" is expressed using a differential form. At that point, explained Levi-Civita, Ricci took "this differential form as given—that is, as absolute—and this is where the essential element of the new calculus arises: in the notion of covariant differentiation, which has the essential characteristics of ordinary differentiation, but also respects the invariant behavior (i.e., is independent of the choice of coordinates) of the system to which it is applied." [16]

Ricci's absolute calculus attracted little notice in the Italian mathematical community at the time, including the monograph, "Methods of the absolute differential calculus and their applications," which he and Levi-Civita jointly published at the request of Felix Klein in *Mathematische Annalen*, in 1900. It would be another twelve years before Marcel Grossmann, a good friend and colleague of Einstein's at Zurich, called Einstein's attention to this work, with results that are well known to the members of my audience.

As for Ricci, he continued to develop his calculus; indeed, he worked almost exclusively on this subject for the rest of his life. "Nobody who really grasped it [the general theory of relativity]," Einstein wrote in 1915, "can escape from its charm, because it signifies a real triumph of the general differential calculus as founded by Gauss, Riemann, Christoffel, Ricci, and Levi-Civita."[17]

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