Applying Scaling Laws in Process Engineering

Chris Burk, P.E.
Burk Engineering

Capital cost estimation techniques in the chemical process industries (CPI) rely largely on scaling relationships. Understanding how scaling relationships relate to equipment characteristics enables you to use them more effectively, even when specific data is not available.

Put aside your graphing calculator and imagine for a minute that you are Ultraman — Japanese superhero, 40 m tall, battling oversized aliens to the delight of Tokyo's schoolchildren. Your thighs measure a monstrous 6 m in diameter, but somehow your legs feel wobbly. Something is not right. You better get back to that calculator and run some numbers.

Ultraman is 20 times your height, but otherwise normally proportioned: His arms are 20 times the length of your arms, his bald spot is 20 times the diameter of yours, and so on. The strength (S) of a pillar is proportional to its cross-sectional area (A), which is equal to $\pi d^2/4$, where d is the diameter. Based on this, you estimate that his legs are about 400 times as strong as yours:

$$\frac{\text{Strength}_2}{\text{Strength}_1} \propto \frac{A_2}{A_1} = \frac{\pi d_2^2/4}{\pi d_1^2/4} = \left(\frac{d_2}{d_1}\right)^2 = 20^2 = 400 \qquad (1)$$

Ultraman's mass (M) should be proportional to his volume (V). Just as area scales with the diameter squared, volume scales with the diameter cubed:

$$\frac{M_2}{M_1} \propto \frac{V_2}{V_1} = \frac{Const \times d_2^3}{Const \times d_1^3} = \left(\frac{d_2}{d_1}\right)^3 = 20^3 = 8{,}000 \qquad (2)$$

So, Ultraman is 8,000 times as heavy as you are, but only 400 times as strong. No wonder his legs seemed wobbly: Ultraman's strength-to-weight ratio is only 400:8,000 = 5% of yours.

This inconvenient discovery is one implication of the mathematical principle Galileo called the square-cube law, which states that when an object grows, its volume grows faster than its surface area: if its length increases by a factor of x, its surface area will increase by a factor of x^2 and its volume will increase by a factor of x^3. This principle also explains why larger animals have a harder time cooling themselves than smaller ones, why you can drop an ant from almost any height without injuring it, and why larger cyclists tend to be faster in time trials than smaller ones.

This is expressed as:

$$\frac{l_2}{l_1} = \left(\frac{A_2}{A_1}\right)^{1/2} = \left(\frac{V_2}{V_1}\right)^{1/3} \qquad (3)$$

or:

$$A_2 = A_1\left(\frac{V_2}{V_1}\right)^{2/3} \qquad (4)$$

where l is a length. This equation is now in the form of a power scaling law. It is stated as "area scales with volume to the power of two-thirds," and two-thirds is referred to as the scaling exponent. You could also say that "area is proportional to volume to the two-thirds:"

$$A \propto V^{2/3} \quad (5)$$

Since we live in three dimensions, most systems scale with exponents that are multiples of one-third.

Scaling for capital cost estimation

In the chemical process industries (CPI), scaling relationships are the basis for many capital cost estimation correlations. When you know the cost of a system or piece of equipment at one scale (*i.e.,* from the literature, records, or a quote), you can estimate its cost at another scale using:

$$C_2 = C_1\left(\frac{K_2}{K_1}\right)^b \quad (6)$$

where C is the cost, b is the scaling exponent, and K is a capacity parameter. A capacity parameter quantifies some essential characteristic of the system or piece of equipment, such as the volume of a tank, the heat-transfer area of a heat exchanger, or the production rate of a plant.

Scaling exponents are available in the literature for a wide variety of equipment and plants *(1, 2).* Table 1 gives some examples of scaling exponents ranging from 0.33 to 1.2 derived from empirical data. Understanding the relationships between scaling exponents and equipment characteristics allows you to use them effectively and apply them to cases where specific data is not available.

Breaking down a scaling exponent

Empirical scaling exponents reflect the influence of many factors. Each factor can be thought of as being its own scaling exponent and contributing a certain percentage to the overall scaling exponent. Table 2 shows the breakdown for a hypothetical process vessel. In practice, the fractional contribution of each component could come from a known base case or from the literature.

The shell material accounts for the largest fraction of the vessel cost. The amount of metal required for the vessel shell is proportional to its surface area, and A scales with $V^{2/3}$ as described in Eq. 4. Therefore, cost of the shell material also scales with $V^{2/3}$:

$$C_{shell} \propto M_{shell} \propto A_{surface} \propto V^{2/3} \quad (7)$$

The welding labor is proportional to the length of welding on the vessel, which is a linear dimension. In Eq. 3, l scales with $V^{1/3}$, so:

$$C_{weld} \propto l_{weld} \propto V^{1/3} \quad (8)$$

Instrument-port material has a scaling exponent of 0. This implies that the number and size of instrument ports are independent of vessel volume. On the other end of the spectrum, support material has a scaling exponent of 1, which indicates that it scales directly with volume. The amount of material used for vessel supports needs to scale

Table 1. Empirical scaling exponents have been tabulated for a variety of equipment and plants *(1)*.

Equipment	Range	Exponent
Blender, Double Cone Rotary, Carbon Steel	1.4–7.1 m^3	0.49
Blower, Centrifugal	0.5–4.7 m^3	0.59
Centrifuge, Solid Bowl, Carbon Steel	7.5–75 kW	0.67
Crystallizer, Vacuum Batch, Carbon Steel	15–200 m^3	0.37
Compressor, Reciprocating, Two-Stage, with 1,035-kPa Discharge	0.005–0.19 m^3/s	0.69
Fan, Centrifugal	10–35 m^3/s	1.17
Heat Exchanger, Shell-and-Tube with a Floating Head, Carbon Steel	10–40 m^2	0.6
Motor, Squirrel Cage, Induction, 440 V, Explosionproof	15–150 kW	0.99
Pump, Centrifugal, Horizontal, Cast Steel	4–40 m^3/s-kPa	0.33
Reactor, Stainless Steel, 2,070 kPa	0.4–4 m^3	0.56
Tank, Flat Head, Carbon Steel	0.4–40 m^3	0.57
Tray, Bubble Cap, Carbon Steel	1–3 m dia.	1.20
Plant	**Range**	**Exponent**
Acetic Acid	3,300–30,000 ton/yr	0.68
Ammonia	33,000–300,000 ton/yr	0.53
Chlorine	17,000–150,000 ton/yr	0.45
Polyethylene	1,700–15,000 ton/yr	0.65

Nomenclature

- A = area
- b = scaling exponent
- C = cost
- d = diameter
- l = length
- M = mass
- S = strength
- V = volume

Table 2. The scaling exponent for a process vessel reflects the influence of a variety of factors.					
Cost Component	**Cost Scales With**	**Scaling Exponent Relative to Volume**	**0.1× Scale Contribution to Cost, %**	**1× Scale Contribution to Cost, %**	**10× Scale Contribution to Cost, %**
Shell Material	Surface Area	2/3	18	35	34
Instrument Port Material	N/A	0	24	10	2
Process Port Material	Volume	1	2	8	17
Support Material	Mass	1	2	8	17
Design Labor	N/A	0	24	10	2
Welding Labor	Length	1/3	16	14	6
Other Labor	Volume	1	2	10	21
Code Stamp	N/A	0	12	5	1

with the vessel mass to maintain an adequate strength-to-weight ratio (and avoid the scenario we encountered with Ultraman's legs):

$$C_{support} \propto \text{Support Material} \propto M_{vessel} \propto V^1 \qquad (9)$$

The blended scaling exponent

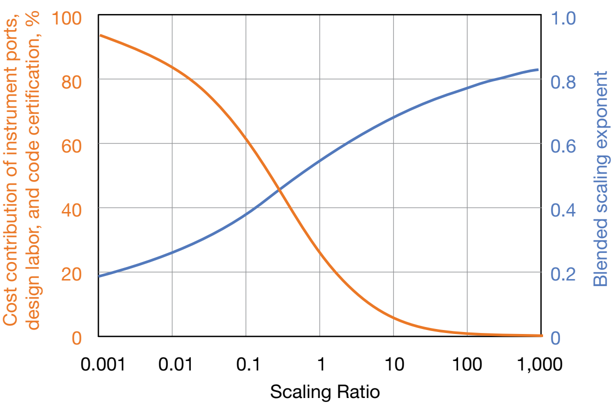
The blended scaling exponent for the vessel described in Table 2 is 0.54, which is the weighted average of the component exponents. However, this value will not remain constant after scaling — the component costs scale at different rates, so their individual contributions change.

Whether we increase or decrease the size of the vessel, the costs for higher-exponent components will change faster than those for lower-exponent components. The cost of any component with an exponent of zero will remain constant regardless of scale. This explains why the cost for instrumentation ports, design, and code certification are proportionally high for small vessels (Figure 1). This also accounts for the hockey-stick-shaped curve seen in Figure 2, which is common in log-log plots of cost vs. capacity. Since scaling exponents can vary, it is important that they only be used within their applicable ranges.

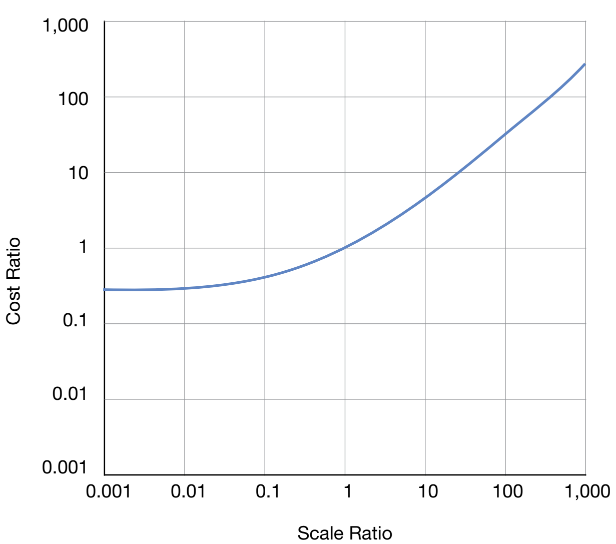
The importance of similarity

Accurate scaling relies on similarity. In cost estimation, this means similarity in configuration, materials of construction, and process conditions. For example, do not scale from a centrifugal pump to a positive-displacement pump, from a carbon steel reactor to a stainless steel reactor, or from a low-pressure column to a high-pressure column.

It is difficult to extrapolate costs to different equipment configurations. However, factors can be applied to estimate the cost of a new system that has the same configuration but uses different materials and/or pressures.



▲ **Figure 1.** Costs for small vessels are dominated by components that do not scale with volume, resulting in a lower blended scaling exponent. This graph is for the hypothetical case described in Table 2.



▲ **Figure 2.** Graphs of cost versus scale tend to be shaped like hockey sticks. This one is for the hypothetical case described in Table 2.

Closing thoughts

Proper scaling is critical to the success of any new development in the CPI. The principles and techniques described in this article can be applied to any type of equipment or plant, even when empirical scaling exponents are not available. First, break down the total cost into its major components and estimate their relative contributions. Then, choose a representative capacity parameter and determine how you would expect each component to scale. The blended scaling exponent will be the contribution-weighted average of the component exponents. Check your results against empirical exponents for similar cases.

This kind of analysis is valuable even with approximation and guesswork. It provides a framework to probe and understand the fundamental relationships connecting scale, form, function, and cost — for process plants and Ultraman's strength. CEP

CHRIS BURK, P.E., is a chemical engineer and an independent consultant (Email: chris@BurkEngineeringLLC.com; Website: www.BurkEngineeringLLC.com). He works with companies that are developing or investing in new chemical and bioprocess technologies, helping them use techno-economic modeling to better understand their economics at a commercial scale. His clients have included venture capital firms, universities, national laboratories, independent startups, and startup incubators. He is also a member of Lee Enterprises Consulting's team of renewable fuel and chemical experts. Prior to starting his consulting practice, he spent 12 years working in industry, where he gained diverse experience with research and development (R&D), startups, and pilot plant engineering, procurement, and construction. He is a licensed professional engineer and he holds BS and MEng degrees in chemical engineering from Cornell Univ.

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Additional Reading

West, G. B., "Scale: The Universal Laws of Growth, Innovation, Sustainability, and the Pace of Life in Organisms, Cities, Economies, and Companies," Penguin Press, London, U.K. (2017).