1. **Complex Analysis**

We started with the professors trying to decide whether or not Alon should ask the first question since this was his first general exam. Alon said “I can try! I think I can do it.” He started with a reasonable question.

**Alon**: Tell us Liouville’s theorem.

Me: A bounded entire function is constant.

**Alon**: How do you prove it?

Me: Choose a point \( z \in \mathbb{C} \), and use Cauchy’s integral formula to write

\[
f'(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.
\]

A basic estimate shows that this is \( O(1/R) \), so let \( R \to \infty \) and conclude \( f' \) vanishes identically.

**Sarnak**: So suppose I have a bounded harmonic function in the entire complex plane. What can you say about it?

Me: It’s constant.

**Sarnak**: Proof?

Me: The harmonic function is the real part of an entire function \( f \). Then \( |e^{f(z)}| = e^{\Re(f(z))} \) is bounded, so \( e^{f(z)} \) is constant. This forces \( f \) (and hence also its real part) to be constant.

**Sarnak**: What if I’m in three dimensions?
Me: So the proof is basically that you use the Mean Value Property for harmonic functions along with the fact that a sequence of balls centered at the origin in $\mathbb{R}^n$ with radii tending to infinity forms a Følner sequence.

Sarnak: Woah! Just be careful. You opened that can of worms.

Me: Okay, I won’t open any more cans of worms. Anyway, suppose our bounded harmonic function (in $\mathbb{R}^n$) is $u$, and pick a point $x_0 \in \mathbb{R}^n$. Use the Mean Value Property for harmonic functions to say that $u(x_0)$ is equal to the average of $u$ on $B_R(x_0)$, the ball of radius $R$ centered at $x_0$. Also, $u(0)$ is the average value of $u$ on $B_R(0)$. If $|u(x)| \leq M$ for all $x \in \mathbb{R}^n$, then a simple estimate shows that

$$|u(x_0) - u(0)| \leq M \mu(B_R(x_0) \Delta B_R(0))/\mu(B_R(0)),$$

where $\mu$ is Lebesgue measure. This tends to 0 as $R \to \infty$.

Sarnak: Okay good. What’s a subharmonic function?

Me: So a subharmonic function is an upper semicontinuous function that—

Sarnak: (Laughing) Woah you’re going to have to remind me what that means. What does that mean?

Me: A function $f$ is upper semicontinuous at a point $x_0$ if $\limsup_{x \to x_0} f(x) \leq f(x_0)$. Then a function defined on a region is upper semicontinuous if it is upper semicontinuous at every point in the region. Then a function $u$ defined on a region is called subharmonic if it is upper semicontinuous and has the property that whenever you have a ball $B$ contained in the region and you have a harmonic function $\varphi$ defined on a region containing $\overline{B}$ such that $u(z) \leq \varphi(z)$ for all $z \in \partial B$, it follows that $u(z) \leq \varphi(z)$ for all $z \in B$.

Sarnak: Okay, so if I have a subharmonic function in the plane, and it’s bounded above by 3, what can I say about it?

Me: It’s constant.

Sarnak: And can you prove that?

Me: Yes, so let’s say our function is $u$. Then $u(z) \leq 3$ for all $z \in \mathbb{C}$. Let $M = \max_{z \in \mathbb{D}} u(z)$ ($\mathbb{D}$ is the unit disc). By the maximum principal for subharmonic functions, $M$ is the maximum of $u$ over the boundary of the unit disc, also known as the unit circle. So $M = \max_{z \in \partial \mathbb{D}} u(z)$.

Sarnak: I’m impressed! We used the put this on the quals at Stanford, and people usually wouldn’t get it! (He’s also asked it in previous general exams, which is how I knew how to solve it ....)

Me: Now let $v(z) = u(z) - \varepsilon \log |z|$.

Sarnak: He knows how to construct a barrier! Now I’m really impressed!
Again, he’s asked this before ....

Me: Now log |z| is harmonic away from the origin, so v(z) is subharmonic away from the origin. Notice that v(z) → −∞ as |z| → ∞ because u is bounded above. Also, M = \max_{z \in \partial D} v(z) because u and v agree on the unit circle. By the maximum principal for subharmonic functions, v(z) ≤ M for all z ∈ C with |z| ≥ 1. Let ε → 0 to see that u(z) ≤ M for all z ∈ C with |z| ≥ 1. This means that u(z) ≤ M for all z. However, u attains this maximum value M on the unit circle, which contradicts the maximum principal for subharmonic functions.

Alon: Do you know the Picard theorem?

Me: Well there’s the Little Picard theorem, which states that if f is entire and there are two distinct complex numbers that are not in the image if f, then f is constant. Also, there’s the Big Picard theorem, which says that if a function has an essential singularity, then on any punctured neighborhood of that essential singularity, the function attains every complex value, with the exception of one value, infinitely often.

Sarnak: Draw a semicircle.

I drew a semicircle. I knew what was coming, so I also marked a point in the interior of the semicircle and drew the angle subtended at that point by the diameter.

Me: Should I also draw this?

Sarnak: What? Do I ask this every time?

Me: Yeah.

Sarnak: (Laughing) All right well he already knows what I’m going to ask! I better ask something else.

Alon was curious to see what the question was, so Sarnak asked the question.

Sarnak: I have a holomorphic function that is bounded by 2 on the circular part of the semicircle and is bounded by 1 on the diameter. Give an upper bound for the function in the interior of the circle.

Me: You want to find a harmonic function u that has value log 2 on the circular part and has value 0 on the diameter. If f is the holomorphic function in question, then log |f(z)| is subharmonic and is bounded above by u(z) on the semicircular boundary. Hence, log |f(z)| ≤ u(z) for all z in the interior. You can find u by taking v(z) to be the angle subtended at z by the diameter. This function is harmonic, has value π/2 on the circular part of the semicircle, and has value π on the diameter. Then u(z) = \frac{2\log 2}{π}(π − v(z)).

Sarnak: So you know what harmonic measure is?

Me: Yes, it’s a measure, dependent on a point in a domain, which is defined on the boundary of a domain. It has the property that if you integrate boundary values against the harmonic
measure, then you obtain the value at the specified point of the harmonic function in the
domain that has those given boundary values.

Sarnak: All right, you seem to know everything.

I smiled and thought “No, I just know the things you’ve asked in previous exams.”

Sarnak: Draw a triply connected region!

In my head I was thinking “Why is he starting with triply connected? Why not simply
connected and doubly connected?” I had learned five proofs that annuli with different radial
ratios are not conformally equivalent because I knew he would ask about that and I wanted
to give multiple proofs. Now he was skipping over annuli and going for triply connected
regions. I drew a blob with two circular holes missing.

Sarnak: Draw another triply connected region.

I drew a blob with a circular hole missing and a slit missing (just for fun).

Sarnak: Ahh, I see he knows the answer!

I’m not sure why he said this. This time I didn’t know what he was planning on asking, so
I actually didn’t know the answer yet.

Sarnak: Can I map the first region to the second biholomorphically?

Me: Not necessarily, since these are just arbitrary triply connected regions. There are
restrictions for when you can.

Sarnak: Okay, so can I map these regions to some standard types of regions?

Me: Oh yeah! You could map them to annuli with circular slits removed.

Sarnak: Draw that.

I drew an annulus with one slit removed.

Sarnak: Okay, so how do I map the first region to this new region?

I didn’t know what he meant by the word “how.” I knew you could do this theoretically
by solving the Dirichlet problem on the first region, but I thought at the time that he was
asking me for an explicit map. Looking back on it now, it should have been clear to me
what he wanted.

Me: I’m not sure.

Sarnak: Okay, suppose I just have a doubly connected region and I want to map it to an
annulus. How can I do that?
In my head: “Yes! My plan is back on track!”

Me: I can use the Uniformization theorem to—

Sarnak: What is the Uniformization theorem?

Me: Every simply connected Riemann surface is conformally equivalent to the Riemann sphere, the complex plane, or the upper half-plane.

Sarnak: Okay good, so go on.

Me: I can use the Uniformization theorem to say that my simply connected region is a quotient of its universal cover by some group of Deck transformations, where the universal cover is the Riemann sphere, the complex plane, or the upper half-plane. The group of Deck transformations is isomorphic to $\mathbb{Z}$ since that’s the fundamental group of my region. Now, any nontrivial Deck transformation has no fixed points. Since every automorphism of the Riemann sphere has a fixed point, the universal cover can’t be the Riemann sphere. So let’s suppose it’s the complex plane. In this case, you can use the fact that the Deck transformation group acts properly discontinuously on $\mathbb{C}$, along with the fact that nontrivial Deck transformations have no fixed points, to show that the only possible Deck transformation groups for $\mathbb{C}$ are $\{1\}$, $\mathbb{Z} \cdot \omega$ (with $\omega \in \mathbb{C} \setminus \{0\}$), and $\mathbb{Z} \cdot \omega_1 \times \mathbb{Z} \cdot \omega_2$ (with $\omega_1, \omega_2$ linearly independent over $\mathbb{R}$). Again, our group is isomorphic to $\mathbb{Z}$, so it must be $\mathbb{Z} \cdot \omega$. Now the map $z \mapsto e^{2\pi i z/\omega}$ gives a biholomorphic map from $\mathbb{C} / (\mathbb{Z} \cdot \omega)$ to $\mathbb{C} \setminus \{0\}$, showing that in this case our region is conformally equivalent to the punctured plane. Next, suppose the universal cover is the upper half-plane $\mathbb{H}$. The automorphism group of $\mathbb{H}$ is $\text{PSL}_2(\mathbb{R})$. Take a generator for the Deck transformation group. This generator is not elliptic since elliptic elements have fixed points in $\mathbb{H}$, so it is either parabolic or hyperbolic. If it is parabolic, the Deck group is conjugate to the group generated by the map $T$ given by $T(z) = z + 1$. In this case, the map $z \mapsto e^{2\pi i z}$ gives an explicit biholomorphic map from $\mathbb{H} / \langle T \rangle$ to $\mathbb{D} \setminus \{0\}$. Finally, if the generator of the Deck group is hyperbolic, then it is conjugate to the map $V_\lambda$ given by $V_\lambda(z) = \lambda^2 z$, where $\lambda > 1$ is some real number. In this case, the map $z \mapsto e^{-\pi i \log z / \log \lambda}$ is a biholomorphic map from $\mathbb{H} / \langle V_\lambda \rangle$ to $A(1, e^{\pi^2 / \log \lambda}) = \{z \in \mathbb{C} : 1 < |z| < e^{\pi^2 / \log \lambda}\}$.

Sarnak: All right, do you know how to prove the Riemann mapping theorem?

Me: Yes.

Sarnak: Is there a relation between the Riemann mapping theorem and the Uniformization theorem?

Me: Yes, the Riemann mapping theorem is basically the Uniformization theorem for simply connected regions in the complex plane that are not the whole complex plane.

Sarnak: Who gave the first correct proof?

I said I had heard different things. I said that I had heard it was Carathéodory (this is what is written on Wikipedia and in a note in the book by Einsiedler and Ward that I used to study ergodic theory) and that I had also heard it was Koebe (this is what is written
in virtually every general exam transcript written by students who had Sarnak on their committee because he always tells them it was Koebe).

Sarnak: It was Koebe.

Me: Okay, Koebe.

Sarnak: No, Koebe (correcting my pronunciation by pronouncing it in what seemed to be the exact same way). He actually proved that every multiply connected region can by mapped to a circular region (this historical remark can also be found in some other general exam transcripts).

Sarnak: How did Riemann try to prove the Riemann mapping theorem?

Me: Well let’s say we start with a simply connected domain $\Omega$ and a biholomorphic map $f: \Omega \to \mathbb{D}$ such that $f(z_0) = 0$. Then we can write $f(z) = (z - z_0)e^{g(z)}$ for some function $g$ which is holomorphic on $\Omega$. If $f$ is supposed to map $\partial \Omega$ to the unit circle, then $1 = |f(z)| = |z - z_0|e^{u(z)}$ for all $z \in \partial \Omega$, where $u(z) = \Re(g(z))$. This means $u(z) = -\log|z - z_0|$ for $z \in \partial \Omega$. So Riemann said that if he could solve the Dirichlet problem on $\Omega$, then he could find the function $u$, find a harmonic conjugate to reconstruct $g$, and then obtain the function $f$. He then had to show that $f$ is bijective.

Sarnak: Do you know how to solve the Dirichlet problem without appealing to the Riemann mapping theorem?

Me: I know a few methods, but don’t know any too deeply. I know you can use Green’s functions, Perron’s method, or the Dirichlet principle.

Sarnak: What do you know about Perron’s method?

Me: If your region is $D$ and your boundary values are given by a function $\phi$, then you can define the Perron function $H_D\phi$ by $H_D\phi(z) = \sup_{u \in \mathcal{U}} u(z)$, where $\mathcal{U}$ is the set of all subharmonic functions $u$ on $D$ that satisfy $\limsup_{\zeta \to \zeta} u(z) \leq \phi(\zeta)$ for all $\zeta \in \partial D$. If the Dirichlet problem has a solution in this situation, it is given by $H_D\phi$.

Sarnak: Okay good. Should we move on to real?

Me: Before we do ...

Sarnak: Oh! I’ve never had a guy not want to move on!

Me: Well the thing is ... I was talking to Levent, and he said I have to show you this proof that annuli with different radial ratios are not conformally equivalent.

Sarnak: You mean not the standard proof?

Me: Well I learned five proofs hoping to give them here, but there’s one that Levent said I have to show you.
Sarnak: Okay, go ahead.

Me: Suppose I have a biholomorphic map \( f : A(1,r) \rightarrow A(1,R) \), where \( r, R > 1 \). By inverting if necessary, I can assume \( f \) maps the inner circle to the inner circle and maps the outer circle to the outer circle. Expand \( f \) as a Laurent series, say \( f(z) = \sum_{n \in \mathbb{Z}} c_n z^n \). It’s an easy consequence of Green’s theorem that the area bounded by a Jordan curve \( \gamma \) is

\[
\text{Area}(\gamma) = \frac{1}{2i} \int_{\gamma} z \, dz.
\]

Let \( C(t) = \{ z \in \mathbb{C} : |z| = t \} \). We have

\[
\text{Area}(f(C(t))) = \frac{1}{2i} \int_{f(C(t))} z \, dz = \frac{1}{2i} \int_0^{2\pi} f(te^{i\theta})f'(te^{i\theta})tie^{i\theta} d\theta.
\]

You can then just plug in the Laurent series and its derivative for \( f \) and \( f' \). After you rearrange terms, you get

\[
\text{Area}(f(C(t))) = \pi \sum_{n \in \mathbb{Z}} n|c_n|^2 t^{2n}.
\]

This holds for \( 1 < t < r \), but by continuity, it also holds for \( t = 1 \) and \( t = r \). If we set \( t = 1 \), we get

\[
\pi = \pi \sum_{n \in \mathbb{Z}} n|c_n|^2.
\]

If we set \( t = r \), we get \( \pi R^2 = \pi \sum_{n \in \mathbb{Z}} n|c_n|^2 r^{2n} \). We can combine these to find that

\[
\pi R^2 - \pi r^2 = \pi r^2 \sum_{n \in \mathbb{Z}} n|c_n|^2 (r^{2n-2} - 1).
\]

Now the neat thing is that, whether \( n \) is positive, negative, or zero, each of the terms \( n|c_n|^2 (r^{2n-2} - 1) \) is nonnegative! This shows that \( R \geq r \). Applying the same argument with \( f \) replaced by \( f^{-1} \) shows that \( r \geq R \), so \( r = R \).

Sarnak said it was a cute proof that he hadn’t seen before.

Sarnak: Now go back and tell Levent I tested you on triply connected regions instead!

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**2. Real Analysis**

**Sarnak:** Define the Fourier transform of an \( L^1 \) function on the real line.

I wrote down \( \hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy} \, dx \).

**Sarnak:** Okay, so what is \( L^1 \)?

Me: It’s the set of functions whose absolute values have finite integrals.

Sarnak: Any functions?

Me: Measurable functions!
Sarnak: Can you give a nonmeasurable function?

For some reason I started thinking about trying to do something with the Cantor-Lebesgue function. After ten or twenty seconds, I realized that I could just take the characteristic function of a nonmeasurable set.

Sarnak: Okay, give a nonmeasurable set.

Me: Take a bounded set $A$ of positive measure, say $A \subseteq [-M,M]$. Put an equivalence relation on $A$ by saying $a \sim b$ if and only if $a - b \in \mathbb{Q}$. Use the axiom of choice to find a set $V$ of representatives of the equivalence classes. Then

$$A \subseteq \bigcup_{q \in \mathbb{Q} \cap [-2M,2M]} (V + q).$$

If $V$ is measurable, then this is a countably infinite union of disjoint sets that all have the same measure as $V$. Since the union contains $A$, $V$ must have positive measure. But this means that the union has infinite measure, which is impossible because it is a bounded set.

Sarnak: So what can you say about the Fourier transform of an $L^1$ function?

Me: It’s in $C_0(\mathbb{R})$.

Sarnak: Is the image all of $C_0(\mathbb{R})$?

Me: No.

Sarnak: Why not?

Me: The Fourier transform is an injective bounded linear map from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$. If it were surjective, I could use the Banach Isomorphism theorem to say that it has a bounded inverse. However, if you define $f_n(x) = \frac{\sin(2\pi x)\sin(2\pi nx)}{\pi^2 x^2}$ and $h_n = 1_{[-1,1]} * 1_{[-n,n]}$, then you can show that $\hat{f_n} = h_n$. You can then show that $\|f_n\|_1 \to \infty$ while $\|h_n\|_\infty$ remains bounded.

Alon: Can you give an example of a continuous nowhere-differentiable function?

Me: I can write one down.

Sarnak: Are you sure?! Many people have tried and failed!

At this point Sarnak and Alon started talking to each other about something. I wasn’t really paying attention to what they were saying. While they were talking, I turned to the board and wrote

$$\sum_{n \geq 1} a^n \cos(b^n \pi x), \quad 0 < a < 1, \quad b \text{ odd}, \quad ab > 1 + \frac{3}{2}\pi.$$ 

I said I thought this was what Weierstrass originally did.
Sarnak: Uhh okay I don’t know how to check that .... Do you know how to give a conceptual proof?

I said something about taking a sequence of functions that have lots of triangular spikes that get really small so you get a uniform limit of continuous functions. Then the limit is continuous while the spikes keep it from being differentiable anywhere. They agreed that the $a^n$ in the expression I wrote down was giving the uniform convergence while the $b^n$ was giving the oscillation.

Sarnak: What’s the dual of $L^1$?

Me: $L^\infty$.

Sarnak: What’s the dual of $L^\infty$?

Me: Something that properly contains $L^1$. I can use the Hahn-Banach theorem to construct an element of the dual of $L^\infty$ that isn’t in $L^1$.

Sarnak: How about the dual of $L^p$?

Me: $L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Sarnak: So is the dual of the dual the space?

Me: It is when for $L^p$ when $1 < p < \infty$.

Alon: Can you take a square, partition it into finitely many pieces, then rearrange them to form a circle?

Sarnak: And then can you do it measurably?

Alon: Right, so the next question is if you can do it measurably.

Me: Well, the answer to the first question is “yes” because otherwise you wouldn’t have asked the second question.

Alon: Right, so very good. What about the second?

Me: I don’t know.

Alon said that it is actually possible to do this measurably, and Sarnak mentioned that this was a recent result featured in the New York Times. I guess I should read the news more often.

Sarnak: Okay write down $f(x) = \sum_{n=N}^{2N} e^{2\pi i n^2 x}$. What can you say about its $L^2$ norm on the circle?
Me: It’s $\sqrt{N + 1}$.

**Sarnak:** What can you say about the $L^1$ norm?

Me: Well the $L^1$ norm is at most the $L^2$ norm (maybe there’s a constant factor depending how you normalize Lebesgue measure on the circle, but I didn’t bother with this).

**Sarnak:** Okay, let’s try to estimate the $L^4$ norm.

I wrote down $||f||_4^4 = ||f^2||_2$.

Sarnak: All right, let’s square $f$.

I got a little confused here because of the strange limits of summation used to define $f$. Sarnak said to just make the sum go from 1 to $N$ instead. I said that in this case the coefficient of $x^m$ in $f(x)^2$ is the number of ways of writing an integer as a sum of two squares, which you can write down explicitly in terms of divisor functions. Sarnak then said we could estimate the divisor function and find that $||f||_4 \ll N^{1/2+\varepsilon}$.

**Sarnak:** Do you know a relation between different $L^p$ norms?

Me: I know that if $\frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$, then $||f||_r \leq ||f||_p^\lambda ||f||_q^{1-\lambda}$.

Sarnak: Great! So this shows that the $L^1$ norm of $f$ is at least $N^{1/2-\varepsilon}$.

Sarnak probably said something else here, but I don’t think he asked any more questions in real analysis. We switched to algebra.

### 3. Algebra

Sarnak: Chris, you’ve been awfully quiet, why don’t you start?

Chris: All right. What’s your fav—

Sarnak: You know, we used to put this on the quals at Stanford, and most people wouldn’t get it. (He was referring to previous question from real analysis.)

Alon: Yes, but that’s a written test, right?

Sarnak: Right. There’s a trade-off because you have more time, but you don’t someone guiding you to the answer.

I think Alon had a point here. I doubt I would have gotten this right on a written test if the question just told me to find a lower bound for the $L^1$ norm.

At this point, I was hoping that Skinner would ask for my favorite proof of the Fundamental Theorem of Algebra when he got to finish his sentence (I like the Galois-theoretic proof),
but this didn’t happen. Now that I think about it, that probably wouldn’t have been a good algebra question. I once had a professor who liked to say that the Fundamental Theorem of Algebra is fundamentally not a theorem about algebra.

**Skinner**: How would you classify conjugacy classes of \( \text{GL}_2(\mathbb{F}_p) \)?

**Me**: Rational canonical form. Should I write down representatives?

**Skinner**: Sure.

**Me**: So the possible invariant factor decompositions are \( x - \alpha, x - \alpha \) for \( \alpha \neq 0 \) and \( x^2 + ax + b \), where \( b \neq 0 \). These give the matrices

\[
\begin{pmatrix}
\alpha & 0 \\
0 & \alpha
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & -b \\
1 & -a
\end{pmatrix}.
\]

**Skinner**: What information does the number of conjugacy classes give you?

**Me**: It’s the number of irreducible representations of \( \text{GL}_2(\mathbb{F}_p) \).

**Sarnak**: Over what?

**Me**: Over, say, the complexes.

**Skinner**: Do you know anything about representations over finite fields?

**Me**: I know where some things can go wrong ....

**Sarnak, Skinner**: What can go wrong?

**Me**: Well, for example, Maschke’s theorem is false if we work over a field whose characteristic divides the order of the group.

**Sarnak**: What does Maschke’s theorem say?

**Me**: If \( F \) is a field whose characteristic does not divide the order of a finite group \( G \), and if \( V \) is an \( FG \)-module with a submodule \( U \), then there exists a submodule \( W \) of \( V \) such that \( V = U \oplus W \).

**Sarnak**: How do you prove this?

**Me**: You can find a vector subspace \( W_0 \) of \( V \) such that \( V = U \oplus W_0 \). Then let \( \pi_0 \) be the projection from \( V \) onto \( U \) along \( W \). If we put

\[
\pi = \frac{1}{|G|} \sum_{g \in G} g\pi_0 g^{-1},
\]

then we can show that \( \pi \) is a projection onto \( U \) which is also an \( FG \)-module homomorphism. Then we can show that \( W = \ker(\pi) \) gives us what we want.
Sarnak: Can you do something like this to make the matrices in a representation friendlier?

Me: Like unitary?

Sarnak: Yes.

Me: Sure. You just stick an arbitrary inner product on the vector space. Then average over the group to get a new inner product. Then your matrices will be unitary with respect to this inner product.

Sarnak: Can you make this argument work for a compact group?

Me: Yeah, you just integrate instead of summing.

Sarnak: Do you need the fact that the group is compact? Can you get this to work for $\text{SL}_2(\mathbb{R})$, for example?

Me: No (so yes, you do need the compactness assumption).

Sarnak: You had rational canonical form. Do you know another canonical form if we are working over an algebraically closed field?

I mentioned Jordan Normal Form. I think he asked me to say what it is, so I just described the shape of a matrix that is in Jordan Normal Form.

Sarnak: Write down $A, B, C, D$. These are complex $n \times n$ matrices. When can you find a matrix that simultaneously conjugates $A$ into $C$ and conjugates $B$ into $D$? I only ask because I just learned of this recently. The solution is very beautiful, but it’s highly nontrivial.

Before I had a chance to say anything, Skinner saved me from trying to fumble around with that “highly nontrivial” problem.

Skinner: How do you prove Jordan Normal Form?

Me: You use the Fundamental Theorem of Finitely Generated Modules over a PID.

I had gotten a lot of practice reciting the name of this theorem while going through past general exams (it comes up extremely often), so I said it so quickly that Sarnak burst out laughing.

Skinner: What are the module and the PID?

Me: Well the matrix acts via a linear transformation on some vector space $V$. Say we’re working over $\mathbb{C}$. Then $V$ becomes a $\mathbb{C}[x]$-module, where $x$ acts via the linear transformation given by the matrix. So $V$ is the module, and $\mathbb{C}[x]$ is the PID.

Skinner: What’s your favorite Dedekind domain that isn’t a PID?
It seemed as though this might have been the sentence Skinner had been unable to finish earlier. I still didn’t give an answer because Sarnak suggested we stay away from algebraic number theory for the moment.

**Alon:** Can you state Sylow’s theorems?

Me: Let $G$ be a group of order $p^\alpha m$, where $p$ is a prime, $\alpha$ and $m$ are positive integers, and $p \nmid m$. A Sylow $p$-subgroup of $G$ is a subgroup of order $p^\alpha$. Sylow’s first theorem says that $G$ has a Sylow $p$-subgroup. Sylow’s second theorem says that any two Sylow $p$-subgroups of $G$ are conjugate. The third theorem says that the number of Sylow $p$-subgroups is congruent to 1 modulo $p$ and divides $m$.

**Skinner:** What can you say about groups of order $pq$, where $p$ and $q$ are distinct primes?

Me: If $p \nmid q - 1$, the only group of order $pq$ is the cyclic group of order $pq$. If $p \mid q - 1$, then there’s the cyclic group of order $pq$ and also a nonabelian semidirect product.

**Sarnak:** What is a semidirect product?

Me: Well, if you have groups $N$ and $H$, you can define a semidirect product $N \rtimes \varphi H$, where $\varphi : H \to \text{Aut}(N)$ is a homomorphism. The elements are pairs of the form $(n, h)$ with $n \in N$ and $h \in H$. Then you can use the map $\varphi$ to determine how to conjugate elements of $N$ by elements of $H$ (viewing $N$ and $H$ as subgroups of $N \rtimes \varphi H$ in the obvious way).

**Sarnak:** Okay, so write down a formula for the multiplication.

Looking back now, I should have been able to translate what I had just said into a formula for the multiplication, but I froze up and couldn’t figure out what to write down. I knew to write $(n_1, h_1)(n_2, h_2) = (\ , h_1 h_2)$, but couldn’t see what I should put in the first position. After I fumbled for about a minute, they decided to have me focus on a specific example of a semidirect product. They told me to write down the group of $2 \times 2$ upper triangular matrices, and had me decompose this group as the semidirect product $N \rtimes D$, where $N$ is the set of upper triangular matrices with 1’s on the diagonal and $D$ is the set of diagonal matrices. They had me conjugate a general element of $N$ by a general element of $D$. At this point I was basically just blindly multiplying matrices when they told me to do so. I finally got to the end, and Sarnak suggested we take a five-minute break.

### 4. Algebraic Number Theory

**Sarnak:** Okay Chris, why don’t you start?

**Skinner:** Okay—

Me: The ring of integers in $\mathbb{Q}(\sqrt{-5})$ (finally answering his question from earlier).

**Skinner:** Okay, why is it not a PID?
Me: Because I can write $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. You can easily show that these are irreducibles that are not associated to each other.

Skinner: What’s the class number of $\mathbb{Q}(\sqrt{-5})$?

Me: 2.

Skinner: How would you prove that?

I said I would write down the Minkowski bound, find a set of generators for the class group, and try to find relations among them. I asked if I should actually go through this on the board. Sarnak interjected with something that I’ve forgotten. We eventually got to talking about quadratic forms.

Skinner: What’s the relation between quadratic forms and class groups?

Me: Well if we have a quadratic number field, then there is a bijection between the narrow class group of the ring of integers and the set of $\text{SL}_2(\mathbb{Z})$ equivalence classes of integral binary quadratic forms with discriminant equal to the discriminant of the number field.

Skinner: Can you write down the bijection?

Me: Let’s say the quadratic field is $K = \mathbb{Q}(\sqrt{d})$. Given an ideal $A$ of $\mathcal{O}_K$, we can find a $\mathbb{Z}$-basis and write $A = a_1 \mathbb{Z} + a_2 \mathbb{Z}$. We then get a quadratic form $Q_{a_1,a_2}(X,Y) = \frac{1}{N(A)}N_{K/\mathbb{Q}}(a_1X + a_2Y)$. You can then show that the map $A \mapsto Q_{a_1,a_2}(X,Y)$ induces the bijection you want.

Sarnak: Do you know how to show that there are only finitely many positive squarefree integers $d$ such that $\mathbb{Q}(\sqrt{-d})$ has class number 1?

I started to explain that you could use the analytic class number formula along with Siegel’s theorem, but Sarnak cut me off.

Sarnak: Write down the class number formula.

I wrote down

$$\text{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} \text{Reg}_K h_K}{w_K \sqrt{|\Delta_K|}}.$$

Sarnak: Define everything in that formula.

Me: Okay so 2 is 2, $2\pi$ is $2\pi$, $r_1$ is the number of real embeddings of $K$, $r_2$ is the number of complex conjugate pairs of nonreal embeddings of $K$, $\text{Reg}_K$ is the regulator of $K$, $h_K$ is the class number of $K$, $w_K$ is the number of roots of unity in $K$, and $\Delta_K$ is the discriminant of $K$. If $\ell: \mathcal{O}_K^\times \to \mathbb{R}^{r_1+r_2}$ is the standard logarithmic map, then $\text{Reg}_K$ is the covolume of the lattice $\ell(\mathcal{O}_K^\times)$ in the trace-zero subspace of $\mathbb{R}^{r_1+r_2}$. If $\beta_1,\ldots,\beta_n$ is a $\mathbb{Z}$-basis for $\mathcal{O}_K$, then $\Delta_K$ is the determinant of the matrix whose $i,j$ entry is $\text{Tr}(\beta_i \beta_j)$. 
Sarnak: What does this give you when $K$ is the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$?

I was allowed to assume $d$ was greater than 3 so that $w_K = 2$. I wrote down
\[
\frac{\pi h_K}{\sqrt{|\Delta_K|}}.
\]

Skinner: That’s the right-hand side. What’s the left-hand side?

I wrote down $L(1, \chi)$.

Sarnak: What is $\chi$?

Me: It’s the character you get from the Legendre symbol.

Sarnak: What kind of character?

Me: A character modulo $|\Delta_K|$.

Sarnak: Well there are lots of those.

Me: A primitive character modulo $|\Delta_K|$.

Skinner: What’s its order? What are its values?

Me: Oh! It’s a quadratic character!

At this point, Sarnak asked me something that I don’t fully remember. I thought he was asking me about Siegel’s theorem, so I wrote down $L(1, \chi) \gg d^{-\varepsilon}$, but then he said we would talk about $L(1, \chi)$ in a minute. He asked what $\Delta_K$ is. I said it’s $d$ if $d \equiv 1 \pmod{4}$ and it’s $4d$ otherwise. He said to ignore the constants and just say that the expression I had written was roughly $h_K/\sqrt{d}$. He then told me to write down $L(1, \chi)$ as a product, so I wrote
\[
L(1, \chi) = \prod_p \frac{1}{1 - \chi(p)p^{-1}}.
\]

Sarnak: Now suppose I want the class number to be 1. Then $1/\sqrt{d}$ is small, so what signs should you choose for the terms $\chi(p)$?

I said I’d need them to be negative for lots of small primes $p$. Sarnak then told me to observe that this meant lots of small primes had to be inert. He then started talking about Siegel’s theorem and its ineffectivity.

Sarnak: So do you know how to prove Sieg— Wait .... What are we testing him on?

Skinner: We’re doing algebraic number theory ....

Sarnak: Right, so maybe we should go back to that.
This led to one of my favorite parts of the exam.

Skinner: So I feel obliged to ask you about class field theory.

Alon: Do you know the Cauchy-Davenport theorem?

Me: (???) Uhh yeah ... it says that if $A, B \subseteq \mathbb{F}_q$, then either $A + B = \mathbb{F}_q$ or $|A + B| \geq |A| + |B| - 1$.

Alon: Do you know how to prove it?

I smiled and said that you can use the Combinatorial Nullstellensatz. I think Skinner and Sarnak got a kick out of that.

Alon: So do you know why it is called the Cauchy-Davenport theorem? They couldn’t have had a joint paper.

Me: I don’t know.

Alon: Well Davenport proved it several years after Cauchy died, but then he later found out that Cauchy had proved it. So you see, it’s never too late to prove a theorem.

We laughed a bit.

Alon: So does this count as algebraic number theory?

Sarnak: Not in this world!

Skinner decided to get us back on track.

Skinner: So I feel obliged to ask you about class field theory. State your favorite version of the main theorems of class field theory.

I struggled with choosing a favorite (mainly because I was trying and failing to anticipate what would come afterward), but eventually decided to go with the statements of global class field theory in terms of ideals.

Me: Should I define the Artin map?

Skinner: Yes.

I defined the Artin map and stated the Reciprocity Law in terms of ideals. I think Sarnak made some comment about some famous mathematician, but I don’t remember what it was. Eventually, he turned the floor over to Skinner again.

Skinner: Do you know the Kronecker-Weber theorem?

Me: Every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension.
Skinner: Can you prove it from what you've written?

I think I said that we just needed to show that the ray class fields of moduli of the form \((m)\infty\) were cyclotomic extensions. I tried to indicate that I actually needed the Existence theorem, which I hadn’t stated yet, but they cut me off (I never got back to stating the Existence theorem or Classification theorem).

Sarnak: Can you deduce quadratic reciprocity from what you’ve written? (This might have been asked by Skinner, but I’m not sure. It seemed like they were often trying to ask things at the same time.)

Me: Let \(p\) and \(q\) be distinct odd primes. Let \(L = \mathbb{Q} (\zeta_p)\). The map \((\mathbb{Z}/p\mathbb{Z})^\times \to \text{Gal}(L/\mathbb{Q})\) given by \(a \mapsto \sigma_a\) is an isomorphism, where \(\sigma_a(\zeta_p) = \zeta_p^a\). Let \(H\) be the image of \(((\mathbb{Z}/p\mathbb{Z})^\times)^2\) under this isomorphism. Then \(H\) is the unique subgroup of \(\text{Gal}(L/\mathbb{Q})\) of index 2, so \(L^H\) is the unique quadratic extension of \(\mathbb{Q}\) contained in \(L\). This extension is \(\mathbb{Q}(\sqrt{p^*})\), where

\[
p^* = \begin{cases} 
p, & \text{if } p \equiv 1 \pmod{4}; \\
-p, & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

Now, \(\left(\frac{q}{p}\right) = 1\) if and only if \(\sigma_q \in H\), and this occurs if and only if \(\sigma_q\) fixes \(\mathbb{Q}(\sqrt{p^*})\). It follows from the Reciprocity Law that \(\sigma_q\) is the Frobenius element of \(q\) in \(\text{Gal}(L/\mathbb{Q})\). Choose a prime \(Q\) in \(L\) lying over the ideal \((q)\), and form the decomposition group \(D(Q \mid (q))\) (the group doesn’t actually depend on the choice of the prime \(Q\) since the extension is abelian). The Frobenius element \(\sigma_q\) generates \(D(Q \mid (q))\), so we find that \(\sigma_q\) fixes \(\mathbb{Q}(\sqrt{p^*})\) if and only if \(\mathbb{Q}(\sqrt{p^*}) \subseteq L^{D(Q \mid (q))}\). This fixed field is the maximal subfield of \(L\) in which the prime ideal \((q)\) splits completely, so \(\mathbb{Q}(\sqrt{p^*}) \subseteq L^{D(Q \mid (q))}\) if and only if \(q\) splits in \(\mathbb{Q}(\sqrt{p^*})\). This happens if and only if \(\left(\frac{p^*}{q}\right) = 1\), so \(\left(\frac{q}{p}\right) = \left(\frac{p^*}{q}\right)\).

The night before the exam, I had studied the history of the development of Artin \(L\)-functions (Sarnak seems to like asking historical questions). I was almost sure this subject would arise since Sarnak and Skinner were on my committee. Strangely enough, it never did. I tried to get them to ask me something about this, but they decided we should move on to Ergodic Number Theory.

## 5. Ergodic Number Theory

We started with Sarnak making fun of me for making up the phrase “ergodic number theory.” I didn’t really know what else to call it. I could have called it “applications of ergodic theory in number theory,” but that doesn’t really roll off the tongue as nicely as “ergodic number theory.” Sarnak asked what I had read. I said that I had read a book of Einsiedler and Ward and that I had read Furstenberg’s monograph. I didn’t mention this at the time, but I had also reviewed a very recent paper of Frantzikinakis and Host concerning Sarnak’s Möbius Disjointness conjecture. This turned out to be useful when Sarnak asked me about the theorem of Host, Kra, and Ziegler.
Sarnak: What’s in Einsiedler and Ward?

Me: Some standard ergodic theory (such as the basic ergodic theorems), Weyl’s polynomial equidistribution theorem (proven ergodically), Furstenberg’s proof of Szemerédi’s theorem, and some homogeneous dynamics (it also covers some diophantine approximation and the theory of continued fractions, but I chose not to mention that because I hadn’t reviewed it).

Sarnak: How did Furstenberg proved Szemerédi’s theorem (Alon was also involved in deciding to ask this question)?

I essentially prepared for this question by pretending I was going to give a seminar talk about Szemerédi’s theorem. Below is the outline of the proof in the order that I had prepared. I think this is the most natural order in which to organize the proof. During the actual exam, I had to readjust the order because Sarnak told me to start in the middle, go back to the beginning, and then go to the end.

Szemerédi’s theorem states that every set of integers with positive upper Banach density contains arbitrarily long arithmetic progressions. In order to prove this, Furstenberg first proves the following.

Multiple Recurrence Theorem: If $T_1, \ldots, T_\ell$ are commuting measure-preserving transformations of a measure space $(X, \mathcal{B}, \mu)$ and $A \in \mathcal{B}$ is a set with $\mu(A) > 0$, then there exists an integer $b \geq 1$ such that

$$\mu(T_1^{-b}(A) \cap \cdots \cap T_\ell^{-b}(A)) > 0.$$  

From the Multiple Recurrence Theorem, we can actually prove the multiple-dimensional analogue of Szemerédi’s theorem quite easily.

Multiple-Dimensional Szemerédi’s Theorem: If $S \subseteq \mathbb{Z}^r$ is a set of positive upper Banach density and $F = \{u_1, \ldots, u_\ell\} \subseteq \mathbb{Z}^r$, then there exist $a \in \mathbb{Z}^r$ and $b \geq 1$ such that $a + bF \subseteq S$.

To deduce this last theorem from the Multiple Recurrence theorem, start by putting $X = \{0,1\}^\mathbb{Z}^r$. This has the structure of a compact metric space, so we can let $\mathcal{B}$ denote the Borel $\sigma$-algebra. For $u \in \mathbb{Z}^r$, let $T_u : X \to X$ be the “shift by $u$” map. Because $S$ has positive upper Banach density, we can find a sequence $(B_n)$ of blocks with widths tending to infinity such that $\frac{|B_n \cap S|}{|B_n|} > \eta$ for all $n$, where $\eta > 0$ is some fixed constant. Let

$$\mu_n = \frac{1}{|B_n|} \sum_{u \in B_n} \delta_{T_u(1_S)},$$

where $\delta_x$ denotes the Dirac measure at a point $x$ and $1_S$ is the indicator function of $S$, which we can view as an element of $X$. By the Banach–Alaoglu theorem, the sequence $(\mu_n)$ has a weak* subsequential limit $\mu$. If we let $A = \{\omega \in X : \omega(0) = 1\}$, then it follows from our definition of $\mu_n$ that $\mu_n(A) > \eta$. Thus, $\mu(A) \geq \eta > 0$. This is why we need $S$ to have positive upper Banach density. If we now apply the Multiple Recurrence theorem with the commuting transformations $T_{u_1}, \ldots, T_{u_\ell}$, then we find that there exists $b \geq 1$ such that

$$\mu(T_{u_1}^{-b}(A) \cap \cdots \cap T_{u_\ell}^{-b}(A)) > 0.$$
Note that $A$ is an open set and that the shift maps $T_{u_i}$ are all continuous. Thus, $T_{u_1}^{-b}(A) \cap \cdots \cap T_{u_\ell}^{-b}(A)$ is open. The measure $\mu$ is supported on the closure of the set of translates of $1_S$, so it follows that $T_a(1_S) \in T_{u_1}^{-b}(A) \cap \cdots \cap T_{u_\ell}^{-b}(A)$ for some $a \in \mathbb{Z}^r$. If we unwind the definitions, we find that this is saying precisely that $a + bF \subseteq S$.

Now how does Furstenberg go about proving the Multiple Recurrence theorem? He starts by making the following definition.

Definition: Say a system $(X, \mathcal{B}, \mu, \Gamma)$ has the SZ property if

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T_1^{-n}(A) \cap \cdots \cap T_\ell^{-n}(A)) > 0$$

for all $T_1, \ldots, T_\ell \in \Gamma$ and $A \in \mathcal{B}$ with $\mu(A) > 0$.

Here, $\Gamma$ is a free abelian group of finite rank acting via measure-preserving transformations on $(X, \mathcal{B}, \mu)$. Furstenberg actually proves that every measure-preserving system (with some very mild regularity conditions) has the SZ property. This is much stronger than the Multiple Recurrence theorem, but he decides to prove the stronger statement because he wants to use transfinite induction. In order to make the inductive argument work, he needs this stronger inductive hypothesis.

Morally speaking, why should we expect every system to have the SZ property? Well, there are two opposite extreme types of systems. There are systems that are very rigid and predictable. A canonical example of this would be a rotation on a compact abelian group. In this type of system, the sets $T_1^{-n}(A), \ldots, T_\ell^{-n}(A)$ move around as $n$ increases in a predictable fashion. They overlap significantly with each other at very predictable times. This leads to a significant positive contribution to the average at very regular time intervals, which leads to the positivity of the liminf. The other extreme type of system is a chaotic system that mixes everything together. In this case, the sets $T_1^{-n}(A), \ldots, T_\ell^{-n}(A)$ become almost independent, so $\mu(T_1^{-n}(A) \cap \cdots \cap T_\ell^{-n}(A))$ is roughly $\mu(A)^\ell$ for most positive integers $n$. This again leads to positivity of the liminf.

Now, Furstenberg’s idea is to show that every system can be built up from rigid parts and chaotic parts. These parts have the SZ property for different reasons, and together they form a system that still has the SZ property. More precisely, Furstenberg defines two types of extensions of systems. The first is a compact extension. This is an extension of systems in which the extended system is very rigid relative to the base system. The other type of extension is a weak-mixing extension. This is an extension in which the extended system is very chaotic relative to the base system. He also defines a primitive extension to be an extension that is, in a precise sense, formed by combining a compact extension with a weak-mixing extension (I offered to define these terms formally, but Sarnak said I didn’t need to do that). Furstenberg proves that weak-mixing extensions and compact extensions both preserve the SZ property, and he deduces that primitive extensions preserve the SZ property. He also defines how to take limits of systems and shows that a limit of systems with the SZ property has the SZ property. He is then able to show that every system (with very mild regularity conditions) can be obtained from the trivial system by a (possibly transfinite)
sequence of primitive extensions and limits of extensions. The trivial system certainly has
the SZ property, so he deduces that every system has the SZ property.

Sarnak: What does it mean for a system to be ergodic?

It’s not often that you define ergodicity after sketching an ergodic-theoretic proof of Szemeredi’s theorem. I gave the standard definition (for a system with a single measure-preserving transformation).

Sarnak said some things about how he thought Furstenberg’s method was so incredible. Somehow this led him to ask the following.

Sarnak: Do you know the theorem of Host, Kra, and Ziegler?

Me: Yes. Suppose \((X, \mathcal{B}, \mu, T)\) is an ergodic measure-preserving system. There is a factor \((Z_\infty, \mathcal{C}, \mu_\infty, T)\) of \((X, \mathcal{B}, \mu, T)\), called the infinite-step nilfactor, with the following properties. First, \((Z_\infty, \mathcal{C}, \mu_\infty, T)\) is isomorphic to an infinite-step nilsystem. Second, if we are given any \(f_1, \ldots, f_\ell \in L^\infty_\mu(X)\), then

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{j=1}^{\ell} f_j \circ T^{nj} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{j=1}^{\ell} E(f_j | Z_\infty) \circ T^{nj},
\]

where the equality is in \(L^2_\mu(X)\). Here, \(E(f_j | Z_\infty)\) denotes the conditional expectation.

There is actually an explicit description of the infinite-step nilfactor, but Sarnak didn’t expect me to go into that (which is good because I didn’t remember it).

Sarnak: Can you prove that the horocycle flow is uniquely ergodic? Let’s say we have a quotient of \(SL_2(\mathbb{R})\) by some discrete subgroup.

I asked if I could assume the quotient is compact. Then I realized that I had to assume the quotient is compact because the statement is false otherwise.

Sarnak: So we have a lattice \(\Gamma\) in \(SL_2(\mathbb{R})\), and the quotient is compact. Can you give an example?

Me: \(SL_2(\mathbb{Z})\)!

Sarnak: That’s not compact!

Me: Oh! Compact! Right. Well, you can take a hyperbolic quadrilateral on the upper half-plane in which the interior angles are all \(\pi/3\) and—

Sarnak: Oh! Then you take the reflection group?

Me: Yes.

Sarnak: All right then! Why don’t we use the Uniformization theorem?
I said “okay” and started trying to show how to construct a compact Riemann surface of genus 2. Sarnak quickly stopped me.

Sarnak: No, we can assume those exist!

Me: Oh okay. Then just take a compact Riemann surface of genus at least 2. By the Uniformization theorem, it will be a quotient of the upper half-plane by some group of Deck transformations. Then that group of Deck transformations is the cocompact subgroup we want.

Sarnak made some comment about moduli that I don’t remember. Then we got back to wanting to show proving the unique ergodicity. I first went to a separate board to write down all the notation I gave (although Sarnak cut me off just before the end because it was getting late). I’ll probably forget many of them within the next week. Below is the sketch of the proof I wrote the whole proof out on a blackboard four times. While preparing for the exam, I didn’t know how much detail I would need to know if Sarnak asked me this question. To compensate, I memorized more than I probably needed to. A couple nights before the exam, I wrote the whole proof out on a blackboard four times.

Let \( X = \Gamma \backslash \text{PSL}_2(\mathbb{R}) \), \( T = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}) \right\} \), \( a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} \), \( u^{-}(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \).

I also let \( m_X \) be the Haar measure on \( X \) (that is, the push-forward of the Haar measure on \( \text{PSL}_2(\mathbb{R}) \) under the quotient map). Finally, I defined \( R_g : X \to X \) by \( R_g(x) = xg^{-1} \) for each \( g \in \text{PSL}_2(\mathbb{R}) \).

Let \( B^T_\delta \) denote the ball in \( T \) of radius \( r \) centered at the identity. Choose \( \eta > 0 \) such that the map \( u^{-}([-0, \eta])B^T_\delta \to X \) given by \( g \mapsto yg \) is injective for every \( g \in X \) (this is possible because \( X \) is compact). The necessity of choosing this \( \eta \) is mostly a technicality that I won’t discuss. Choose \( f \in C(X) \) and \( x_0 \in X \). Fix \( \varepsilon > 0 \). Since \( f \) is uniformly continuous, there exists \( \delta \in (0, \eta) \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( d_X(x, y) < \delta \). Here, \( d_X \) is a metric on \( X \) induced by a left-invariant metric on \( \text{PSL}_2(\mathbb{R}) \). We will consider \( x_0 u^{-}([-0, \eta \varepsilon^t]) \), the stretch of the orbit of \( x_0 \) under the horocycle flow of length \( \eta \varepsilon^t \). We want to find the average of \( f \) along this stretch. Instead, we will form a thin “tube” along this stretch and use the uniform continuity of \( f \) to say that the average of \( f \) on that tube is close to the average of \( f \) on the stretch of the horocycle orbit.

Let \( Q_\delta = u^{-}([-0, \eta])B^T_\delta \). Let \( B_t = R^{-1}_{a_t}(R_{a_t}(x_0)Q_\delta) \). The set \( B_t \) is our tube. Indeed, one can show that \( B_t \subseteq x_0 u^{-}(-[0, \eta \varepsilon^t])B^T_\delta \). In other words, every element of \( B_t \) can be written in the form \( x_0 u^{-}(-s)h \), where \( s \in [0, \eta \varepsilon^t] \) and \( h \in B^T_\delta \). For such \( s \) and \( h \), we have

\[
X(x_0 u^{-}(-s)h, x_0 u^{-}(-s)) \leq d_{\text{PSL}_2(\mathbb{R})}(x_0 u^{-}(-s)h, x_0 u^{-}(-s)) = d_{\text{PSL}_2(\mathbb{R})}(h, I) < \delta.
\]

It follows from the choice of \( \delta \) that \( |f(x_0 u^{-}(-s)h) - f(x_0 u^{-}(-s))| < \varepsilon \).
There is a way (discussed in Eisiedler and Ward) to decompose the Haar measure on $\text{PSL}_2(\mathbb{R})$ into two “pieces.” One piece is a left Haar measure on $\{u^-(s) : s \in \mathbb{R}\}$, which is essentially the Lebesgue measure $ds$. The other piece is a right-invariant Haar-measure $m^r$ on $T$. Using this decomposition, we can write

$$\frac{1}{m_X(B_t)} \int_{B_t} f \, dm_X = \frac{1}{\eta e^t} \int_0^{\eta e^t} \frac{1}{m^r_T(a_t^{-1}B_\delta a_t)} \int_{a_t^{-1}B_\delta a_t} f(x_0u^-(-s)h) \, dm^r_T(h) \, ds$$

(imagine that we are decomposing the integral over the tube into an integral in the “$s$ direction” of an integral in the “$h$ direction”). Using our above estimate, we find that this last integral is within $\varepsilon$ of

$$\frac{1}{\eta e^t} \int_0^{\eta e^t} f(x_0u^-(-s)) \, ds.$$ 

Our whole goal here is to show that this last integral approximates $\frac{1}{m_X(X)} \int_X f \, dm_X$ when $t$ is large. To do this, we argue that $\frac{1}{m_X(B_t)} \int_{B_t} f \, dm_X$ approximates $\frac{1}{m_X(X)} \int_X f \, dm_X$ when $t$ is large. This is essentially because the set $B_t$ is defined as a preimage of a set under the geodesic flow map $R_{a_t}$ and because the action of the geodesic flow is mixing. In fact, this argument would complete the proof immediately if $B_t$ were the preimage of a fixed set under the geodesic flow (this is essentially what it means for an action to be mixing). However, the set $R_{a_t}(x_0)Q_\delta$ is not fixed; it is dependent on $t$. We can get around this by using the fact that $X$ is compact. Roughly speaking, the sets of the form $R_{a_t}(x_0)Q_\delta$ all have the same “shape”; it is really the position that varies with $t$. With a compactness argument, one can show that all of these sets can be approximated by finitely many fixed sets. We can then apply the mixing argument with each of these fixed sets and finish the proof with a final approximation argument.

Sarnak ended by saying some things about Ratner’s theorem, but I don’t remember anything he said.

6. Aftermath

They kicked me out. I went into the hall and jumped around because I was so happy to have this behind me. It felt like they kept me waiting for ten minutes, but maybe it wasn’t that long. They eventually opened the door and told me I passed.

The whole exam lasted about 2.5 hours. It was actually really fun. I don’t have too much advice that differs significantly from the advice given in the other past general exams. My strategy for studying for the standard topics was to go through every question that has ever been asked. I did the same for algebraic number theory. For ergodic number theory, I read Einsiedler and Ward along with Furstenberg’s monograph. I had to guess what I thought Sarnak would ask in this area. This wasn’t too hard since I knew roughly what he found interesting within ergodic theory (such as homogeneous dynamics).
Good luck to all the students who have yet to pass their generals. Don’t worry too much about it. Have fun with it!