# Arithmetic Properties of the Derived Category for Calabi-Yau Varieties

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**Abstract**

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This thesis develops a theory of arithmetic Fourier-Mukai transforms in order to obtain results about equivalences between the derived category of Calabi-Yau varieties over non-algebraically closed fields. We obtain answers to classical questions from number theory and arithmetic geometry using these results.

The main results of this thesis come in three types. The first concerns classifying moduli of vector bundles on genus one curves. Fourier-Mukai equivalences of genus one curves allow us to produce examples of non-isomorphic moduli spaces when a genus one curve has large period. The next result extends the result of Lieblich-Olsson which says that derived equivalent K3 surfaces are moduli spaces of sheaves over algebraically closed fields. We get the same result over arbitrary fields of characteristic $p \neq 2$.

The last class of results are about finding properties that are preserved under derived equivalence for Calabi-Yau threefolds. Examples of such arithmetic invariants include local Zeta functions, modularity, L-series, and the $a$-number. We then prove a Serre-Tate theory for liftable, ordinary Calabi-Yau threefolds in positive characteristic in order to show that the derived equivalence induces an isomorphism of their deformation functors that sends the canonical lift of one to the canonical lift of the other.

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# ACKNOWLEDGMENTS

I would like to thank my advisor, Max Lieblich, for all of his help. I often came into his office with grand, impossible ideas. He managed to reformulate them into questions that could actually be solved. The results of these conversations can be found in the following pages.

I would also like to thank the University of Washington math department for being such a supportive place to learn. The graduate students and faculty were always willing to listen and share ideas. In particular, the student run algebraic geometry club has been a continual source of useful conversations.

Lastly, I want to thank my family and friends for always believing I would finish.

# DEDICATION

*To my parents,*

*who instilled a love of learning*

*in me from an early age.*

# NOTATIONAL CONVENTIONS

There are certain categories and conventions that are made in this thesis which may require a bit of backtracking to find. To ease the burden on the reader these are collected here. The letter $k$ will always denote a field. Unless otherwise stated, it is safe to assume it is a perfect, but not necessarily algebraically closed field. The letter $W$ denotes the ring of $p$-typical Witt vectors of $k$ defined in Section 1.1.

In the spirit of Schlessinger's work, we often deal with functors of Artin rings. The category $\mathrm{Art}_k$ is the category of local Artin $k$-algebras with fixed choice of augmentation to $k$. The category $\mathrm{Art}_{W,k}$ is the category of local Artin $W$-algebras with fixed choice of augmentation to $k$. The reader unfamiliar with the subtleties in defining this category should consult [Sch68].

The symbols $\Phi_X$ and $\Phi$ denote the Artin-Mazur formal group of $X$, see Definition 1.3.2. The symbols $\Psi_X$ and $\Psi$ denote the enlarged Artin-Mazur formal group of $X$, see Definition 4.5.6.

We use $D(X)$ to denote the bounded derived category of coherent sheaves on $X$. It is not standard to omit the extra ornamentation, because it is sometimes useful to consider the unbounded version or take caution with the difference between coherent cohomology and complexes of coherent sheaves. Since this thesis will never need to consider these variants, we opt for this simpler notation. We will also refer to this as "the derived category."

We use standard Galois cohomology conventions and write $H^j(k, M)$ to mean $H^j(Gal(k^s/k), M)$. Galois groups will have the profinite topology. Galois modules will have the discrete topology, and actions will be continuous.

# HISTORY AND MOTIVATION

The general topic of this thesis is the derived category of Calabi-Yau varieties. In the past thirty years, the derived category has had fantastic success as a tool in algebraic geometry. From the point of view of a grand classification of smooth varieties over algebraically closed fields, one could think of understanding varieties up to derived equivalence as a coarser relation (and hence more manageable) than birational equivalence or isomorphism. In the geometric situation of working over an algebraically closed field, curves have equivalent derived categories if and only if they are isomorphic.

A more general result of Bondal and Orlov states

**Theorem** ([BO01])**.** *If $X$ and $Y$ are varieties with ample or anti-ample canonical bundles, then $D(X) \simeq D(Y)$ if and only if $X \simeq Y$.*

Calabi-Yau varieties have trivial canonical bundle, so they do not satisfy the hypotheses of the Bondal-Orlov result. It turns out there is a rich theory developed to understand when Calabi-Yau varieties have the same derived category. Many open problems remain. The most progress has been made on K3 surfaces for which the Torelli theorem allows a full classification. It is a classical result of Mukai:

**Theorem** ([Muk02])**.** *Over $\mathbf{C}$, if $X$ and $Y$ are K3 surfaces, then $D(X) \simeq D(Y)$ if and only if $X$ is a moduli space of sheaves on $Y$.*

The classification in this case has a very simple numerical form as a triple of integers now known as the Mukai vector.

Many other classical cases have been examined such as the case of abelian varieties.

**Theorem** ([Orl02]). *If $A$ and $B$ are abelian varieties such that $D(A) \simeq D(B)$, then there is an isometric isomorphism $A \times \widehat{A} \to B \times \widehat{B}$.*

More recently, parts of the Minimal Model Program have been completed for certain classes of higher dimensional varieties using techniques from derived categories [BM12]. They exploit the idea that constructing nice moduli spaces is intimately tied to ideas that can be defined purely on the derived category.

Another main topic of this thesis is the arithmetic theory of Calabi-Yau varieties. A one-dimensional Calabi-Yau variety is a genus 1 curve. Over algebraically closed fields these can always be given the structure of an elliptic curve. It goes without saying that elliptic curves are ubiquitous across all of number theory. See [Sil09] for an entire book that has been devoted to this topic.

Some recent results show us that higher dimensional Calabi-Yau varieties enjoy many of the same arithmetic properties that make elliptic curves so important to study. Gouvêa and Yui show

**Theorem** ([GY11]). *All rigid Calabi-Yau threefolds over $\mathbf{Q}$ are modular.*

This was the famous Taniyama-Shimura conjecture in the case of elliptic curves that led to the proof of Fermat's Last Theorem. It was only settled in full in the one-dimensional case in 2003 by Breuil, Conrad, Diamond, and Taylor in [BCDT01]. It seems that higher dimensional Calabi-Yau varieties are probably the most in-reach class of varieties to prove such generalized modularity conjectures. A related theorem of Elkies and Schütt is

**Theorem** ([ES13]). *Every weight 3 newform with integral Fourier coefficients comes from a K3 surface over $\mathbf{Q}$.*

The arithmetic theory of Calabi-Yau varieties has served as a fertile ground for many of the classical problems in number theory. Several papers have discussed interesting problems related to the Hasse principle and weak approximation for K3

surfaces: [HVAV11], [HT08], and [Sch09]. The number fields over which the Néron-Severi group of singular K3 surfaces are fully generated are related to certain class groups of fields. The above discussion provides many examples of how K3 surfaces have become of increasing interest in number theory.

Higher dimensional Calabi-Yau varieties have arithmetic interest as well. All elliptic curves and K3 surfaces projectively lift from positive characteristic to characteristic 0, but Hirokado [Hir99] and Schröer [Sch04] have constructed Calabi-Yau threefolds that do not lift. Such pathologies seem to be related to the fact that these threefolds are "supersingular" or have infinite height. This follows the philosophy that a height 1 variety should have a canonical lift, finite height varieties should have quasi-canonical lifts, and infinite height varieties exhibit the most pathological liftability behavior.

### *Thesis Results*

At this point we have seen that derived categories in the geometric situation of algebraically closed fields of characteristic 0 have had great success with such classical problems as classification and moduli space theory. Calabi-Yau varieties have been a rich source of examples and theory for arithmetic situations where the base field is not algebraically (or separably) closed. The literature on derived categories is full of assumptions of working over algebraically closed fields. It is quite a new idea to attempt to translate these techniques to arithmetic situations to shed some light on the above problems.

For example, in [KL08], Krashen and Lieblich obtain period-index results using twisted Fourier-Mukai transforms. In [LO11], Lieblich and Olsson work out realizations of the action of the kernel of a Fourier-Mukai equivalence on various cohomology theories to extend the classical theory to work over arbitrary perfect fields. As an application, they obtain the result that every Fourier-Mukai partner of a K3 surface over an algebraically closed field is a moduli space of sheaves.

This thesis continues work in this direction. It is a (consequence of a) classical

result of Atiyah [Ati57] that every moduli space of stable vector bundles on an elliptic curve is fine and represented by the curve itself. It seems to be a long-standing open problem to determine whether or not this remains true for arbitrary genus one curves. Pumplün [Pum04] extends Atiyah's techniques to work in some partial generalizations but does not fully settle the question. From her work, it appears the answer might be true.

In the second chapter we show this conjecture to almost always be false when the Weil-Châtelet group of the Jacobian of the curve is sufficiently large. We use arithmetic Fourier-Mukai techniques to produce the following result:

**Theorem.** *For any $N > 0$, there exists a genus $1$ curve, $C$, that admits at least $N$ distinct moduli spaces of stable vector bundles, $M_C(r, d)$, mutually non-isomorphic as curves.*

In chapter 3, we extend Lieblich and Olsson's result to work over arbitrary fields of characteristic $p \neq 2$. More specifically we obtain

**Theorem.** *Every Fourier-Mukai partner of a K3 surface over any field of characteristic $p \neq 2$ is a moduli space of sheaves.*

We make use of the generality of this result to prove that every relative Fourier-Mukai partner of Schröer's non-liftable Calabi-Yau threefold is isomorphic to the original one. This shows that one cannot use relative Fourier-Mukai partners to produce new examples of non-liftable Calabi-Yau threefolds. The last main chapter in this thesis examines a large number of arithmetic invariants of a Calabi-Yau threefold that are preserved under derived equivalence. These include, but are not limited to: the height of the Artin-Mazur formal group, the Zeta function, the $L$-series, modularity, the $a$-number, and the $b$-number.

In order to prove such results a general setup is proved for how the Fourier-Mukai transform acts on various $\ell$-adic and $p$-adic cohomology theories. The idea is

to convert the information of the invariant into cohomological information and then check that this is preserved when there is a Fourier-Mukai equivalence.

We then go on to prove a Serre-Tate theory for ordinary, liftable Calabi-Yau threefolds. This involves a close study of the enlarged Artin-Mazur formal group $\Psi$ defined in [AM77].

**Theorem.** *If $X/k$ is an ordinary, liftable Calabi-Yau threefold over a perfect field of characteristic $p > 3$ with associated formal group $\Psi$, then the canonical map between the deformation functor of $X$ and the deformation functor of $\Psi$ is an isomorphism.*

This theory of canonical coordinates is then used to prove a corollary:

**Corollary.** *If $X$ and $Y$ are ordinary, liftable Calabi-Yau threefolds over a perfect field of characteristic $p > 3$ such that their derived categories are equivalent, then the equivalence induces an isomorphism $Def_X \to Def_Y$ that sends the canonical lift of $X$ to the canonical lift of $Y$.*

This thesis opens the door for a much more in-depth study of how derived category techniques can shed light on some arithmetic questions of Calabi-Yau varieties.

# Chapter 1

# INTRODUCTION AND BACKGROUND MATERIAL

The purpose of this chapter is to provide references to and explanations of the main tools that will be used in later chapters. None of the results in this chapter are new, but proofs of many of these theorems seem to be folklore or very difficult to track down. In these cases we provide full proofs for completeness.

## 1.1 Preliminaries on Mixed Characteristic Algebra

Let $k$ be a perfect field of characteristic $p > 0$. Theorem 3 of [Ser79] is the existence of a complete, absolutely unramified DVR with residue field $k$ and fraction field of characteristic 0. We will call this the ring of Witt vectors and denote it $W(k)$ or even $W$ when no confusion will arise. In the literature this is sometimes called the ring of $p$-typical Witt vectors to distinguish from a related ring of "big" Witt vectors.

To give the reader a feel for some of the properties of this ring, we will now sketch the construction. An element of $W(k)$ is an infinite sequence of elements of $k$ which we denote $(a_0, a_1, \ldots)$. The addition and multiplication rules of the $n$-th component are given by an integral polynomial involving only the first $n$ components. This means we can form other rings of truncated Witt vectors $W_n(k)$. We give a few terms to get a feel for the rules:

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (a_0 + b_0, a_1^p + b_1^p - \frac{1}{p} \sum_{j=1}^{p-1} \binom{p}{j} a_0^{p-j} b_0^j, \ldots)$$

$$(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots) = (a_0 b_0, a_0^p b_1 + a_1 b_0^p - p a_1 b_1, \ldots)$$

The addition (and multiplication) law may look strange, because in our situation the characteristic is $p$, so the second term can be more compactly written as $a_1^p b_1^p - a_0^{p-1} b_0 - a_0 b_0^{p-1}$. This is because $W$ is actually a functor on all rings, some of which may not be of characteristic $p$.

For some examples, if $k = \mathbf{F}_p$, then $W(k) = \mathbf{Z}_p$, the $p$-adic integers. If $k = \mathbf{F}_{p^a}$, then $W(k)$ is the ring of integers in the unramified extension of $\mathbf{Q}_p$ of degree $a$. We can also form the truncated Witt vectors, $W_1(k) = k$, $W_2(k) = \{(a_0, a_1) : a_0, a_1 \in k\}$ with the above addition and multiplication. Note that $W(k)$ is a DVR with maximal ideal generated by $p = (0, 1, 0, 0, \ldots)$.

We have a shift operator

$$V : W_n(k) \to W_{n+1}(k)$$

that is given on elements by

$$(a_0, \ldots, a_{n-1}) \mapsto (0, a_0, \ldots, a_{n-1}).$$

Note that this map is additive, but is not a ring map. We also have the restriction map

$$R : W_{n+1}(k) \to W_n(k)$$

given by

$$(a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_{n-1}).$$

Lastly, we have the Frobenius endomorphism

$$F : W_n(k) \to W_n(k)$$

given by

$$(a_0, \ldots, a_{n-1}) \mapsto (a_0^p, \ldots, a_{n-1}^p).$$

This is also a morphism of rings, but only because of our assumption that $k$ is of characteristic $p$. By brute force checking on elements we see a few relations between

these operations:

$$V(x)y = V(xF(R(y)))$$

and

$$RVF = FRV = RFV = p,$$

the multiplication by $p$ map.

The restriction maps give us an inverse system of rings $\cdots \to W_3(k) \to W_2(k) \to k$. The inverse limit is quickly checked to be $W(k)$, but on the other hand this inverse system is exactly the one formed by the filtration determined by the maximal ideal, so we see that $W(k)$ is a complete DVR. Before moving on, it should be mentioned that in a similar fashion to $W(k)$ being absolutely unramified, there is a complete DVR with residue field $k$ and of characteristic 0 with absolute ramification index $e$ for all positive integers $e$.

Here we record some basic facts about $p^{-1}$-linear algebra which will be needed in Chapter 3. We suppose now that $k$ is an algebraically closed field of characteristic $p > 0$. Recall that a $p^{-1}$-linear map between $W$-modules $\phi : M \to M'$ is an operator that satisfies the relations:

1. $\phi(0) = 0$

2. $\phi(v + w) = \phi(v) + \phi(w)$

3. $\phi(a \cdot v) = a^{1/p} \cdot \phi(v)$ for all $a \in W$.

**Lemma 1.1.1.** *Any $p^{-1}$-linear endomorphism $\phi : M \to M$ of finitely generated torsion $W$-modules has a Jordan decomposition into two $\phi$-stable components: $M = M_s \oplus M_n$ where $\phi$ acts bijectively on $M_s$ and nilpotently on $M_n$.*

*Proof.* The idea of the proof is simple. Consider, $\phi^j$, the iterates of our map. We get a descending chain of submodules $\phi^j(M) \supset \phi^{j+1}(M)$, and since $M$ is Artinian it

stabilizes somewhere. Note that even though $\phi$ is not a linear map the image is still a submodule of $M$. Let $r$ be the smallest integer such that $\phi^r(M) = \phi^{r+1}(M)$. This means that $\ker \phi^r$ stabilizes at $r$ as well.

Now we take as our definition $M_s = \phi^r(M)$ and $M_n = \ker \phi^r$. By construction, the above properties follow immediately. It is the kernel/image decomposition and hence a direct sum. By the minimality of the choice of $r$ we get that $\phi$ maps $M_s$ to $M_s$ and $M_n$ to $M_n$. Also, $\phi|_{M_s}$ is bijective by construction. Lastly, if $m \in M_n$, then $\phi^j(m) = 0$ for some $0 \leq j \leq r$ and hence $\phi$ is nilpotent on $M_n$. $\qquad\square$

**Proposition 1.1.2.** *If $\phi : V \to V$ is a bijective $p^{-1}$-linear map between finite dimensional $k$-vector spaces, then there is a basis for $V$ consisting of fixed points of $\phi$.*

*Proof.* The proposition as stated is what we will need later, but it suffices to prove this for a $p$-linear map by duality. The proofs are exactly the same, but working with a power of $1/p$ everywhere adds unnecessary clutter to an already tedious calculation.

We induct on the dimension of $V$. If we can find a single $v_1$ fixed by $\phi$, then we would be done for the following reason. We quotient by the span of $v_1$, then by the inductive hypothesis we can find $v_2, \ldots, v_n$ a fixed basis for the quotient. Together these make a fixed basis for all of $V$.

Now we need to find a single fixed $v_1$. We produce it by hand. Consider any nonzero $w \in V$. We start taking iterates under $\phi$. Eventually they will become linearly dependent, so we consider $w, \phi(w), \ldots, \phi^m(w)$ for the minimal $m$ such that this is a linearly dependent set. This means we can find some coefficients that are not all $0$ for which $\sum_{j=0}^{m} a_j \phi^j(w) = 0$.

Suppose $v_1 \in \mathrm{span}_k \langle w, \phi(w), \ldots, \phi^m(w) \rangle$ such that $\phi(v_1) = v_1$. We examine some necessary and sufficient conditions for such an element to exist. First, $v_1 = \sum_{j=0}^{m} b_j \phi^j(w)$ is fixed if and only if $v_1 = \phi(v_1) = \sum_{j=0}^{m} b_j^p \phi^{j+1}(w)$.

By assumption of the minimality of $m$ before, the coefficient $a_m \neq 0$, so we rewrite the top power

$$\phi^m(w) = -\sum_{j=0}^{m-1}(a_j/a_m)\phi^j(w).$$

The other equation is

$$
\begin{aligned}
\sum_{j=0}^{m-1} b_j \phi^j(w) &= \sum_{j=0}^{m-1} b_j^p \phi^{j+1}(w) \\
&= \sum_{j=1}^{m-1} b_{j-1}^p \phi^j(w) - b_{m-1}^p \left( \sum_{j=0}^{m-1}(a_j/a_m)\phi^j(w) \right) \\
&= -b_{m-1}^p(a_0/a_m)w + \sum_{j=1}^{m-1}(b_{j-1}^p - b_{m-1}^p(a_j/a_m))\phi^j(w).
\end{aligned}
$$

We will have produced our desired fixed vector if we can find a set of $b_j \in k$ that satisfy the conditions that we get by comparing coefficients. The first equation is $b_0 = -(a_0/a_2)b_{m-1}^p$. The second equation is $b_1 = b_0^p - b_{m-1}^p(a_1/a_m)$. We can use the first to eliminate $b_0$ in the second.

In general, the $j$-th relation will give $b_j$ in terms of $b_{m-1}$ and $b_0, \ldots, b_{j-1}$. By repeatedly forward substituting at each step as above, we can get a single variable polynomial equation in $b_{m-1}$ over $k$, since the $a_j$ are known scalars coming from the linear dependence relation. Since $k$ is algebraically closed we can solve to find such a $b_{m-1}$. Then since we wrote all of our other coefficients in terms of $b_{m-1}$ during the forward substitution process we produce a fixed $v_1$. $\square$

**Corollary 1.1.3.** *If $v_1, \ldots, v_n$ is a basis of fixed vectors for $\phi$, then $V^\phi = \mathrm{span}_{\mathbf{F}_p}\langle v_1, \ldots, v_n\rangle$.*

*Proof.* First, the $\mathbf{F}_p$-span is contained in the fixed points, because the prime subfield of $k$ consists of exactly the fixed elements of $x \mapsto x^{1/p}$. On the other hand, if $c = \sum a_j v_j$ is fixed, then

$$c = \phi(c) = \sum a_j^{1/p}\phi(v_j) = \sum a_j^{1/p}v_j.$$

This shows that all the coefficients must be fixed by Frobenius and hence in $\mathbf{F}_p$. $\square$

**Corollary 1.1.4.** *If $\phi : V \to V$ is a bijective $p^{-1}$-linear endomorphism, then $\phi - Id_V$ is surjective.*

*Proof.* Let $w \in V$. We want to find some $z$ such that $\phi(z) = w$. Let $\{v_1, \ldots, v_n\}$ be a basis for $V$ consisting of fixed points of $\phi$. We construct the coefficients in the fixed basis by hand. We know $w = \sum c_i v_i$ for some $c_i \in k$. An element $z = \sum a_i v_i$ satisfies $\phi(z) - z = w$ if and only if we can find $a_i$ that satisfy $a_i^p - a_i = c_i$, which comes from comparing coefficients. These are polynomial equations, so we can solve this over our algebraically closed field to get our coefficients. $\square$

## 1.2 Tools from Deformation Theory

The reader familiar with standard deformation theory of smooth projective varieties can safely skip this section. We merely collect some theorems that will be used later. All theorems and proofs can be found in all the standard sources. See, for example, [Ill05], [Har10], or [Ser06].

Suppose $X_0$ is a smooth projective variety over $k$. A lift (or deformation) to $W_2$ is a flat morphism $X_2 \to \operatorname{Spec} W_2$ with the property that the special fiber is isomorphic to $X_0$ (this isomorphism is part of the data of the lift). This is a first-order infinitesimal deformation of $X_0$.

We can repeat this process and look for another first order deformation of $X_2$ to $X_3 \to \operatorname{Spec} W_3$. Continuing this process we get via the isomorphism an inductive system $X_0 \to X_2 \to X_3 \to \cdots$. Taking the limit we get a formal scheme $\mathfrak{X} \to \operatorname{Spf} W$. This will be called a formal lift of $X_0$ over $W$.

Conversely, given any scheme $X \to \operatorname{Spec} W$ we get a formal scheme by considering each of the base changes to $W_n$. This is called the $\mathcal{I}$-adic completion of $X$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion of the special fiber $X_0 \hookrightarrow X$. We will denote this by $\widehat{X}$. We will call a formal scheme $\mathfrak{X} \to \operatorname{Spf} W$ algebraizable if there exist some actual scheme $X \to \operatorname{Spec} W$ for which $\widehat{X} = \mathfrak{X}$. This tells us that producing a formal

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lift using these infinitesimal deformations gives an actual lift if and only if the formal lift is algebraizable.

This brings us to the Grothendieck Existence Theorem. The general form given in [Ill05] by Illusie is the following theorem:

**Theorem 1.2.1.** *Let $X$ be a noetherian scheme, separated and of finite type over $Y$. Let $\widehat{X}$ be its mathcalI-adic completion. The functor $F \mapsto \widehat{F}$ from the category of coherent sheaves on $X$ with proper support over $Y$ to the category of coherent sheaves on $\widehat{X}$ with proper support over $\widehat{Y}$ is an equivalence of categories.*

The version that is directly useful to our lifting problem is:

**Theorem 1.2.2.** *If $\mathfrak{X}$ is a formal scheme $\varinjlim X_i$ over $\operatorname{Spf} R$ and $X_0$ is projective with an ample line bundle $\mathcal{L}_0$ that extends to a line bundle on all of $\mathfrak{X}$, then there exists $X \to \operatorname{Spec} R$ such that $\widehat{X} = \mathfrak{X}$.*

Our lifting problem has been converted to two deformation problems. First, given $X_0/k$ we have to do an inductive sequence of deformations $X_n \to X_{n+1}$ over $\operatorname{Spec} W_n \to \operatorname{Spec} W_{n+1}$. In order to see that this algebraizes we must then do an inductive sequence of deformations of an ample line bundle on $X_0$. Note that this is not the only method for proving a lift exists, but it has had such widespread success that one can use it to prove that every smooth projective curve lifts, every K3 surface lifts, and every abelian variety lifts.

The two theorems that tell us when such deformations are possible are as follows:

**Theorem 1.2.3.** *Suppose $X/k$ is a nonsingular variety over a field. Let $A$ be an Artin $W$-algebra with augmentation to $k$, $0 \to I \to A \to k \to 0$. The obstruction to deforming $X$ over $A$ is a class in $H^2(X, \mathcal{T} \otimes I)$ and if this class vanishes, then the set of deformations is a torsor under $H^1(X, \mathcal{T})$.*

**Theorem 1.2.4.** *Suppose $X/k$ is a nonsingular variety over a field. Let $A$ be an Artin $W$-algebra such that $0 \to I \to A \to k \to 0$. The obstruction to deforming a line*

bundle, $\mathcal{L}$, over $A$ is a class in $H^2(X, \mathcal{O}_X \otimes I)$ and if this class vanishes, then the set of deformations is a torsor under $H^1(X, \mathcal{O}_X)$.

Since this thesis is about Calabi-Yau varieties, it is interesting to note that the lifting problem in dimensions 2 and 3 are in some sense opposite to each other. A K3 surface has $H^2(X, \mathcal{T}) = 0$, so the formal lift always exists, but $H^2(X, \mathcal{O}_X) \simeq k$, so it may not algebraize. A Calabi-Yau threefold has $H^2(X, \mathcal{T})$ as an unknown (but is often quite large), so it may have no formal lift, but once a formal lift is known it automatically algebraizes because $H^2(X, \mathcal{O}_X) = 0$.

The modern point-of-view is to package this together in terms of a deformation functor. Consider $\text{Art}_{W,k}$, the category of local Artin $W$-algebras with fixed choice of augmentation $A \to k$ (morphisms make all diagrams commute). Given a smooth, projective variety $X_0/k$, there is a functor $\text{Def}_{X_0} : \text{Art}_{W,k} \to \text{Set}$. The set $\text{Def}_{X_0}(A)$ consists of deformations of $X/A$ together with a choice of isomorphism to the base change $X_0 \xrightarrow{\sim} X_k$ up to equivalence.

It is a consequence of the above theorem that this functor has more structure. It is what Schröer calls a semihomogeneous cofibered groupoid. If there exists a complete, Noetherian $W$-algebra, $R$, such that $\text{Def}_{X_0}$ is naturally isomorphic to the functor $A \mapsto \text{Hom}_W(R, A)$, then we call $\text{Def}_{X_0}$ prorepresentable by $R$.

In later chapters we will examine non-liftable Calabi-Yau threefolds more closely, but for now we will record a theorem that will be needed when they do lift. The following is a special case of [Sch03] Theorem 3.4. We fix a perfect field $k$ of characteristic $p \geq 3$.

**Theorem 1.2.5.** *If $X/k$ is a Calabi-Yau threefold that lifts to characteristic $0$ and has torsion-free crystalline cohomology, then $\text{Def}_X$ is prorepresentable by a formal power series ring over $W$ in $\dim H^1(X, \Omega^2)$ variables.*

In particular, this tells us that under mild hypotheses, deformations of liftable Calabi-Yau threefolds are completely unobstructed despite the possibility of having

large obstruction spaces. The structure of the deformation functor will be of vital importance in Chapter 4 for Serre-Tate theory.

## 1.3 The Height of a Variety

In the conjectures and theorems that come in later chapters we will need a thorough understanding of an invariant of a variety in positive characteristic invented by Artin and Mazur called the height. In order to define and work with this we will need some of the theory of $p$-divisible groups. That is where we start this section.

### p-Divisible Groups

A more detailed account of all facts found in this section can be found in [Sha86] and [Dem72]. In order to avoid technicalities that are unnecessary for the purposes of this thesis we will work over a perfect field $k$ (possibly of characteristic 0). Fix a natural number $h$. A $p$-divisible group of height $h$ over $k$ is a directed system $(G_\nu, i_\nu)_{\nu \in \mathbb{N}}$ where each $G_\nu$ is a finite, flat, commutative group scheme over $k$ of order $p^{\nu h}$ that also satisfies the property that $0 \to G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{p^\nu} G_{\nu+1}$ is exact. In other words, the maps of the system identify $G_\nu$ with the kernel of multiplication by $p^\nu$ in $G_{\nu+1}$.

Some authors refer to the $p$-divisible group as the inductive limit of the system $\varinjlim G_\nu$. Note that if everything is affine, $G_\nu = \mathrm{Spec}(A_\nu)$ and the limit $\varinjlim G_\nu = \mathrm{Spec}(\varprojlim A_\nu) = \mathrm{Spf}(A)$. It can be checked that a $p$-divisible group over $k$ is a $p$-torsion commutative formal group $G$ for which $p \colon G \to G$ is an isogeny.

The motivating example is the Barsotti–Tate group of an abelian variety. Let $A$ be an abelian variety of dimension $g$ over an algebraically closed field $k$. Multiplication by $p^\nu$ has kernel $A[p^\nu]$. This is always a finite group scheme over $k$ of order $p^{2g\nu}$ (even if it is purely infinitesimal and hence topologically consists of a singleton). The natural inclusions satisfy the conditions for the limit to be a $p$-divisible group of height $2g$. We denote this $p$-divisible group by $A[p^\infty]$.

Similarly, the kernel of raising to the $p^\nu$ power on $\mathbf{G}_m$ is a group scheme $\boldsymbol{\mu}_{p^\nu}$. The

limit $\varinjlim \boldsymbol{\mu}_{p^\nu} = \boldsymbol{\mu}_{p^\infty}$ is a connected $p$-divisible group of height 1. To reiterate, the height can be read off from knowing the orders of $G_\nu$, but it is often useful to have a more theoretical description.

Given a $p$-divisible group $G$, each individual $G_\nu$ has a Cartier dual $G_\nu^D$, since they are all group schemes. There are also maps $j_\nu$ that make the composite $G_{\nu+1} \overset{j_\nu}{\to} G_\nu \overset{i_\nu}{\to} G_{\nu+1}$ multiplication by $p$ on $G_{\nu+1}$. After taking duals, the composite is still the multiplication by $p$ map on $G_{\nu+1}^D$, so it is easily checked that $(G_\nu^D, j_\nu^D)$ forms a $p$-divisible group called the Cartier dual.

One of the important properties of the Cartier dual is that one can determine the height of a $p$-divisible group (often a hard task when in the abstract) using the information of the dimension of the formal group and its dual. For any $p$-divisible group, $G$, we have the formula

$$\text{ht}(G) = \text{ht}(G^D) = \dim G + \dim G^D.$$

To continue with our examples from before we have $\boldsymbol{\mu}_{p^\infty}^D \simeq \mathbf{Q}_p/\mathbf{Z}_p$ which is the étale $p$-divisible group $\varinjlim \mathbf{Z}/p^\nu$. For an abelian variety $A$, the dual is $A[p^\infty]^D = A^t[p^\infty]$ where $A^t$ denotes the dual abelian variety. Another proof that $A[p^\infty]$ has height $2g$ is to note that $A$ and $A^t$ have the same dimension $g$, so using our formula for height we get $\text{ht}(A[p^\infty]) = 2g$.

*The Dieudonné Module*

Recall that we have an inductive system formed by shifting $V : W_n(k) \to W_{n+1}(k)$ given by $V(a_0, \ldots, a_{n-1}) = (0, a_0, \ldots, a_{n-1})$. We normally treat $W$ as a $W$-module by taking $\lambda \in W$ and then using Witt multiplication. When we do this, $V$ is a $p^{-1}$-linear map. It has the property that $V(\lambda a) = \lambda^{1/p} V(a)$. It is not a map of $W$-modules, and hence the inductive system is not a system of $W$-modules.

We can alter the $W$-structure to make $V$ a linear map. For notation, we will say that $\overline{\lambda}$ is the image of $\lambda$ via $W(k) \to W_n(k) = W(k)/V^n(W(k)) \simeq W_n(k)$. Let $W_n$ be

a $W$-module by the altered action $\lambda \star x = \overline{\lambda}^{p^{1-n}} x$, where two elements next to each other means Witt multiplication.

We check that this makes $V$ into a $W$-linear map:

$$
\begin{aligned}
V(\lambda \star a) &= V(\overline{\lambda}^{p^{1-n}} a) \\
&= \overline{\lambda}^{p^{-n}} V(a) \\
&= \lambda \star V(a)
\end{aligned}
$$

Thus we have an inductive system of $W$-modules. If $G$ is a finite, commutative, unipotent, group scheme, then we use this inductive system to define $D(G)$, the Dieudonné module of $G$ by taking $D(G) = \varprojlim \operatorname{Hom}(G, W_n(k))$. Modifications give a construction for other finite, flat group schemes. A similar construction in the Ind-category exists, so we have a Dieudonné module for formal group schemes and $p$-divisible groups. For a more rigorous definition along with its various properties see either Sections 5 and 6 of [Dem72] or Section 3 of [Oda69].

Since all the $V$ operators are monomorphisms, we get that

$$
\operatorname{Hom}(G, W_n(k)) \to \operatorname{Hom}(G, W_{n+1}(k))
$$

are all injective and hence we can identify $\operatorname{Hom}(G, W_n(k))$ with a submodule of $D(G)$. More explicitly, various properties of $G$ can be extracted from $V$. For example, if $G$ is unipotent, we know that

$$
\operatorname{Hom}(G, W_n(k)) = \{m \in D(G) : V^n(m) = 0\}.
$$

Thus every element of $D(G)$ is killed by a power of $V$. Thus $V$ acts nilpotently on $D(G)$.

The main importance of this for us is that if $G$ is a $p$-divisible group of height $h$ then $D(G)$ is a free $W$-module of rank $h$. There is a natural action of Frobenius on $D(G)$ coming from the Frobenius on $W$ (it is not a $W$-linear endomorphism). Let CartMod be the category of Cartier modules. The objects are pairs $(M, \phi)$ where

$M$ is a free $W$-module, and the morphisms $\phi : M \to M$ are $F$-semilinear injective endomorphisms of $M$.

One can find a basis which diagonalizes $\phi$, and the eigenvalues have well-defined valuation (coming from $W$) called the slopes of $(M, \phi)$. We arrange these in increasing order. A theorem of Dieudonné-Manin [Man63] tells us CartMod is semi-simple via this slope decomposition. Thus $M \simeq \bigoplus_{\lambda \in \mathbb{Q}} M_\lambda$ where each $M_\lambda$ is the part corresponding to slope $\lambda$. Note that $(D(G), F)$ has the property that $FV = VF = p$ and hence all slopes for $D(G)$ are in $[0, 1]$, or equivalently $F(D(G)) \supset pD(G)$.

**Theorem 1.3.1.** *The functor $D : \mathrm{pDivGps} \to \mathrm{CartMod}$ is a fully-faithful embedding of categories whose essential image is the subcategory of Cartier modules with slopes contained in $[0, 1]$.*

For a more detailed analysis of the theorem see Chapter IV of [Dem72] on the classification of $p$-divisible groups. This theorem tells us that for any free $W$-module of rank $h$ with an $F$-semilinear injective endomorphism there is a unique $p$-divisible group $G$ with $D(G) = M$ and $F = \phi$ up to isomorphism. Some sources use the term $F$-crystal for Cartier module. In practice, it is sometimes easier to work with the finite dimensional vector space $M \otimes_W K$ called the $F$-isocrystal. The Dieudonné functor to the category of $F$-isocrystals uniquely determines the $p$-divisible group up to isogeny. The ambiguity comes from possibly differing by a power of $p$ which is invertible over $K$ but not over $W$.

This ability to turn the classification of $p$-divisible groups into a semilinear algebra problem will be crucial in the next section and Chapter 4. Already, we can see the utility because one can read off all possible $p$-divisible groups of small height over an algebraically closed field. For example, there are 2 of height 1; 4 of height 2; and 6 of height 3. It is an enlightening exercise to work out why a semilinear injective map $\phi : W \oplus W \to W \oplus W$ can only have 4 possible slope decompositions if all slopes are assumed to be in $[0, 1]$ (in fact, the only possible slopes are 0, 1/2, and 1).

*The Artin-Mazur Formal Group*

A more thorough treatment of this section can be found in [AM77]. Let $X$ be a smooth, proper $n$-dimensional variety over an algebraically closed field $k$ of positive characteristic $p$.

**Definition 1.3.2.** *Define the functor* $\Phi : \mathrm{Art}_k \to \mathrm{Grp}$ *by*

$$\Phi(S) = \ker(H^n_{et}(X \otimes S, \mathbf{G}_m) \to H^n_{et}(X, \mathbf{G}_m)).$$

The foundational result of Artin-Mazur is that under these hypotheses the functor is prorepresentable by a $d$-dimensional formal group where $d = \dim_k H^n(X, \mathcal{O}_X)$. We define this to be the Artin-Mazur formal group.

For a curve, this group is often called the formal Picard group, $\widehat{\mathrm{Pic}}$, since the groups involved are $H^1(-, \mathbf{G}_m) \simeq Pic(-)$. Geometrically, these are the line bundles that extend the trivial line bundle over an infinitesimal thickening. Similarly, for a surface, this group is called the formal Brauer group $\widehat{Br}$.

This thesis concerns itself with Calabi-Yau varieties in which case $\dim_k H^n(X, \mathcal{O}_X) = 1$. This means that for any $X$ we now have an associated connected, one-dimensional formal group $\Phi_X$. It is a well-known theorem that these are completely determined up to isomorphism by their height. If $\Phi_X \simeq \widehat{\mathbf{G}}_a$, then the formal group is said to have infinite height. Otherwise $\Phi_X$ is a $p$-divisible group and the height is as described in the previous section.

For an elliptic curve, the height is either 1 in which case it is ordinary or 2 in which case it is supersingular. Along these same lines, we define a Calabi-Yau variety to be *ordinary* if it has height 1. For K3 surfaces the height is less than or equal to 10 or infinite, but for all higher dimensional Calabi-Yau varieties the finite heights have no known bound. We define infinite height Calabi-Yau varieties to be *supersingular*.

*Other Characterizations of Height*

The above definition of height is very cumbersome to work with in practice. This section will describe very briefly two alternate ways to see the height of a variety. No proofs will be given. In the next section more details will be given in the particular cases we care about.

An early attempt at constructing a Weil cohomology theory by Serre involved making a sheaf version of the ring of Witt vectors. If $X$ is a variety over a perfect field of positive characteristic, then one can make $\mathcal{W}_n$, a sheaf on $X$ whose sections are $\mathcal{W}_n(U) = W_n(\Gamma(U, \mathcal{O}_X))$. This is an abelian sheaf, so we can take sheaf cohomology. The Witt cohomology is defined to be $H^i(X, \mathcal{W}) := \varprojlim H^i(X, \mathcal{W}_n)$.

In [AM77] it is shown that $D(\Phi_X) \simeq H^n(X, \mathcal{W})$ where $n$ is the dimension of $X$. In other words, if we set $K = \mathrm{Frac}(W)$, then the height of $X$ is $\dim_K H^n(X, \mathcal{W}) \otimes_W K$. The Witt cohomology is slightly obscure and hard to calculate, so we will look at one other cohomology theory that gives a way to find the height.

Suppose that $X$ is a smooth variety, then the torsion-free part of the crystalline cohomology $H^n_{crys}(X/W)$ is a Cartier module under the action of Frobenius. We can consider the part with slopes less than 1, i.e. $H^n_{crys}(X/W) \otimes_W K_{[0,1)}$. Under appropriate hypotheses which will be formally stated later, the dimension of this vector space is the height of $X$, since it is isomorphic to $D(\Phi_X) \otimes_W K$.

<div style="text-align: center">

Chapter 2

# PRELIMINARY RESULTS ON CALABI-YAU VARIETIES

</div>

This chapter is similar to the previous one. Its main purpose is to provide background on tools that will be used later in the thesis. The reason for splitting this into its own chapter is that many of these results, though preliminary to the main results, are new. Some of them were known, but hopefully the presentation here sheds a new light on how they fit together.

The flow of the rest of the thesis is roughly organized as follows. The end of this chapter gives a complete description of derived equivalent 1-dimensional Calabi-Yau varieties. It serves as a motivating example for just how dissimilar this arithmetic setting can be from the typical geometric situation. We then move on to Calabi-Yau surfaces, better known as K3 surfaces, in Chapter 3. Finally, we finish with Calabi-Yau threefolds in Chapter 4. Some of the concepts may fit better together by presenting all dimensions at once, but the progression is designed to gradually increase in dimension with each chapter.

## 2.1 Ordinary Calabi-Yau Threefolds

In this section we will prove that several previously studied properties are equivalent in the case of a Calabi-Yau threefold.

**Definition 2.1.1.** *A* Calabi-Yau variety $X/k$ *of dimension $n$ is a smooth, projective variety that satisfies $\omega_X \simeq \mathcal{O}_X$ and $H^j(X, \mathcal{O}_X) = 0$ for all $0 < j < n$.*

Fix $X/k$ a Calabi-Yau threefold over a perfect field of characteristic $p > 3$. Define $\mathcal{W} := W(\mathcal{O}_X)$ to be Serre's sheaf of Witt vectors on $X$. A few of the following equivalences have been proved in other sources such as [JR03] and [KvdG03]. Others

follow the same proof as for K3 surfaces. We collect them together in one place for convenience and provide most of the proofs.

**Theorem 2.1.2.** *The following are equivalent:*

1. *$X$ is ordinary.*

2. *The Frobenius map $F : H^3(X, \mathcal{O}_X) \to H^3(X, \mathcal{O}_X)$ is bijective.*

3. *$H^i(X, B^j) = 0$ for all $i$, $j$ except possibly $(i, j) = (1, 2)$ or $(2, 2)$.*

The sheaf $B^j$ is the sheaf of exact $j$-forms. This can be realized as the sheaf-theoretic image of the operator $d : \Omega^{j-1} \to \Omega^j$ and is sometimes written $d\Omega^{j-1}$. In postive characteristic, this sheaf plays an important role in many exact sequences due to the Cartier operator.

The last cohomological criterion will be called *Kato's criterion* for ordinarity, because it is a modified form of the definition found in Bloch and Kato's paper [BK86]. It would be nice if the height 1 definition of ordinary corresponded to Bloch and Kato's definition, because we will need to assume their stronger form in Chapter 4. Unfortunately, we will present evidence that this is probably not the case (unlike for elliptic curves or K3 surfaces in which the definitions coincide).

**Lemma 2.1.3.** *For a Calabi-Yau threefold $\dim_k[H^3(X, \mathcal{W})/V H^3(X, \mathcal{W})] = 1$.*

*Proof.* Consider the short exact sequence induced by the shift operator $V$ and the restriction $R$. These satisfy

$$0 \to W_{n-1}\mathcal{O}_X \xrightarrow{V} W_n\mathcal{O}_X \xrightarrow{R^{n-1}} \mathcal{O}_X \to 0.$$

Taking cohomology, and using the Calabi-Yau condition we get

$$0 \to H^3(X, W_{n-1}\mathcal{O}_X) \xrightarrow{V} H^3(X, W_n\mathcal{O}_X) \to H^3(X, \mathcal{O}_X) \to 0.$$

Since $H^3(X, W_n\mathcal{O}_X)$ satisfies the Mittag-Leffler condition, we can take the limit while preserving exactness. This gives us

$$0 \to H^3(X, \mathcal{W}) \xrightarrow{V} H^3(X, \mathcal{W}) \to H^3(X, \mathcal{O}_X) \to 0.$$

Thus

$$H^3(X, \mathcal{W})/VH^3(X, \mathcal{W}) \simeq H^3(X, \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X)^\vee$$

and we get that the quotient is 1-dimensional. $\qquad\square$

**Lemma 2.1.4.** *Given any finite height p-divisible group $\Phi$ over $k$, the Dieudonné module, $M$, satisfies* $\mathrm{rk}_W(M) = \dim_k(M/VM) + \dim_k(M/FM)$.

*Proof.* It was pointed out in Section 1.3 that for a $p$-divisible group we always have

$$\mathrm{ht}(\Phi) = \dim(\Phi) + \dim(\Phi^D).$$

By standard Dieudonné theory we have that $\mathrm{ht}(\Phi) = \mathrm{rk}_W(M)$. But also $\dim(\Phi) = \dim_k(M/VM)$. Since the Cartier dual has the same height and $F$ and $V$ are interchanged we see that $\dim(\Phi^D) = \dim_k(M/FM)$. Putting these together gives the lemma. $\qquad\square$

**Proposition 2.1.5.** *The height of a Calabi-Yau threefold is given by*

$$h = \min\{i \geq 1 : [F : H^3(W_i\mathcal{O}_X) \to H^3(W_i\mathcal{O}_X)] \neq 0\}.$$

Note that this is Theorem 2.1 in [KvdG03]. The proof is omitted there as a direct generalization of the proof for K3 surfaces. This is a replica of Katsura and van der Geer's proof adapted from K3 surfaces to threefolds.

*Proof.* We will prove that

$$h \geq i + 1 \text{ if and only if } F : H^3(W_i\mathcal{O}_X) \to H^3(W_i\mathcal{O}_X) \text{ is zero.}$$

For the backward direction there are two cases. If $h = \infty$, then there is nothing to show, so suppose that $h$ is finite and that $F$ is the zero map for a fixed $i$. Proposition 2.13 of [AM77] shows that the Dieudonné module of the formal group, $M = D(\Phi_X)$, is isomorphic to $H^3(X, \mathcal{W})$. We checked in Lemma 2.1.3 that

$$\dim_k \left( H^3(X, \mathcal{W})/VH^3(X, \mathcal{W}) \right) = 1.$$

By Lemma 2.1.4, $h = \mathrm{rk}_W(M) = \dim_k(M/FM) + \dim_k(M/VM)$, so we get

$$\dim_k(H^3(X, \mathcal{W})/FH^3(X, \mathcal{W})) = h - 1.$$

Since we have a surjection $H^3(X, \mathcal{W}) \twoheadrightarrow H^3(X, W_i\mathcal{O}_X)$ we get a surjection

$$
\begin{aligned}
H^3(X, \mathcal{W})/FH^3(X, \mathcal{W}) &\twoheadrightarrow H^3(X, W_i\mathcal{O}_X)/FH^3(X, W_i\mathcal{O}_X) \\
&\simeq H^3(X, W_i\mathcal{O}_X)
\end{aligned}
$$

where the last isomorphism comes from our assumption that $F$ is zero on $H^3(X, W_i\mathcal{O}_X)$. But now we note that $\dim_k H^3(X, W_i\mathcal{O}_X) = i$, so in order to have a surjection, we must have the inequality $h - 1 \geq i$ or $h \geq i + 1$. This proves one direction.

Now we prove the forward direction. If $h = \infty$, then the formal group is $\widehat{\mathbf{G}}_a$, so $F$ is equivalently 0 on $H^3(X, \mathcal{W}) \simeq D(\widehat{\mathbf{G}}_a)$. Since $F$ commutes with $R$, $F$ acts as 0 on all $H^3(X, W_i\mathcal{O}_X)$. This proves the infinite height case. Suppose $h$ is finite and $h \geq i + 1$. Consider

$$0 \subset V^{h-2}H^3(\mathcal{O}_X) \subset \cdots \subset VH^3(W_{h-2}\mathcal{O}_X) \subset H^3(X, W_{h-1}\mathcal{O}_X).$$

By the $VR$-sequence in the proof of Lemma 2.1.3 we see each successive quotient is one-dimensional, so all the inclusions are proper.

By Dieudonné theory, we have that $H^3(X, \mathcal{W})$ is a free $W$-module of rank $h$. We now claim that $V^{h-1}H^3(X, \mathcal{W}) = FH^3(X, \mathcal{W})$. As noted $D(\phi) \simeq W[F, V]/(W[F, V](F - V^{h-1}))$. Using the law $FV = p = VF$ we can note that

$$
\begin{aligned}
F(\sum a_{ij} F^i V^j) &= \sum a_{ij}^\sigma F F^i V^j \\
&= \sum (a_{ij}^\sigma F^i V^j) F \\
&= \sum (a_{ij}^\sigma F^i V^j) V^{h-1} \\
&= V^{h-1} (\sum a_{ij}^{\sigma^h} F^i V^j).
\end{aligned}
$$

This gives us a rule for how to compute $F$ acting on the left in terms of $V^{h-1}$. But this shows that $FH^3(X, W_{h-1}\mathcal{O}_X) = 0$, since after adjusting the element we act on appropriately we shift by $h-1$ which gives 0's in every entry. Thus $F$ acts as 0 on $H^3(X, W_i\mathcal{O}_X)$ for $i \leq h-1$. $\qquad\square$

**Proposition 2.1.6.** *If $X$ is a Calabi-Yau threefold, then the following are equivalent:*

1. *$X$ has height 1.*

2. *$X$ is $F$-split.*

3. *The Frobenius induces an isomorphism $H^3(X, \mathcal{O}_X) \to H^3(X, \mathcal{O}_X)$.*

Note that $H^3(X, \mathcal{O}_X)$ is one-dimensional and $k$ is perfect, so we need only check that the Frobenius is non-trivial in order for it to be bijective. The natural Frobenius map on cohomology comes from the induced absolute Frobenius on the structure sheaf $\mathcal{O}_X \to F_*\mathcal{O}_X$. The map in the proposition uses that $F$ is a finite morphism to give

$$
F : H^3(X, \mathcal{O}_X) \to H^3(X, F_*\mathcal{O}_X) \xrightarrow{\sim} H^3(X, \mathcal{O}_X).
$$

The first map of the composition is non-trivial if and only if the whole composition is, so we check whichever is most convenient. The term $F$-split has become standard terminology for "Frobenius-split." This means that map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits.

*Proof.* Proposition 2.1.5 gives us a formulation of the height in terms of the action of Frobenius on $H^3(X, W_i\mathcal{O}_X)$. Taking $i = 1$ gives the equivalence between 1 and 3. To prove the equivalence with being $F$-split, we will reformulate the $F$-split condition as being equivalent to Frobenius acting injectively on $H^3(X, \mathcal{O}_X)$ and the theorem follows from the above comments.

Consider the sequence in question

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B^1 \to 0. \tag{2.1}$$

The splitting of this sequence is the definition of Frobenius splitting. The derivation of $B^1$ appearing uses Grothendieck duality, and the fact that the Cartier operator is the trace map for $F$. This is tangential to the proof, but the curious reader can consult Section 2 of [JR03].

This sequence is split if and only if the dual sequence $0 \to (B^1)^\vee \to (F_*\mathcal{O}_X)^\vee \to \mathcal{O}_X \to 0$ is split (recall our smoothness hypothesis which guarantees these sheaves are all locally free). This happens if and only if we can lift the identity $\mathcal{O}_X \to \mathcal{O}_X$ through $(F_*\mathcal{O}_X)^\vee$. This happens if and only if $H^0(X, (F_*\mathcal{O}_X)^\vee) \to H^0(X, \mathcal{O}_X)$ is non-zero.

Now by Serre duality and triviality of the canonical bundle, we have this last map is dual to $H^3(X, \mathcal{O}_X) \to H^3(X, F_*\mathcal{O}_X)$. Thus being $F$-split is equivalent to Frobenius acting non-trivially. $\square$

**Corollary 2.1.7.** *If $X/k$ is an ordinary Calabi-Yau threefold and $p > 3$, then $X$ lifts to $W_2$ and the Hodge-de Rham spectral sequence degenerates at $E_1^{i,j}$.*

*Proof.* This is the proof of Theorem 9.1 in [Jos07]. We reproduce it here, because it does not seem to be well-known. The class $\zeta \in \mathrm{Ext}^1(\Omega^1, B^1)$ that corresponds to the sequence $0 \to B^1 \to Z^1 \to \Omega^1 \to 0$ is the obstruction to lifting the pair $(X, F)$ to a pair $(X^{(2)}, F^{(2)})$. Here $X^{(2)}$ is a lift of the variety to $W_2(k)$, and $F^{(2)}$ is a lift of Frobenius.

If we look at the sequence $0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B^1 \to 0$ and take $\mathrm{Hom}(\Omega^1, -)$ we get a connecting homomorphism in the long exact sequence

$$\mathrm{Ext}^1(\Omega^1, B^1) \xrightarrow{\delta} \mathrm{Ext}^2(\Omega^1, \mathcal{O}_X).$$

The obstruction to lifting lies in $\mathrm{Ext}^2(\Omega^1, \mathcal{O}_X) \simeq H^2(X, \mathcal{T}_X)$, and the obstruction class in this case is exactly the image of $\zeta$ under $\delta$. Note that $\delta$ acts as a forgetful map for the obstruction to lifting the pair to the obstruction to lifting the variety without lifting Frobenius.

Frobenius splitting gives us

$$\mathrm{Ext}^1(\Omega^1, F_*\mathcal{O}_X) \twoheadrightarrow \mathrm{Ext}^1(\Omega^1, B^1) \xrightarrow{\delta} \mathrm{Ext}^2(\Omega^1, \mathcal{O}_X),$$

so in fact $\delta = 0$. The obstruction to lifting $X$ is the image of $\zeta$ under $\delta$ which is 0. $\quad\square$

We return to the proof of Theorem 2.1.2.

*Proof.* In light of Proposition 2.1.6, all that is left to show is that Kato's criterion which says that $H^i(X, B^j) = 0$ for all $i$ and $j$ except possibly $(i, j) = (1, 2)$ or $(2, 2)$ is equivalent to one of the listed forms of being ordinary.

Consider again the short exact sequence (2.1). Taking cohomology we see that 3 implies 2, since if $H^j(X, B^1) = 0$ we have exactly

$$F : H^n(X, \mathcal{O}_X) \xrightarrow{\sim} H^n(X, \mathcal{O}_X)$$

for all $n$. In particular, take $n = 3$.

For the other direction, there are ten cohomology groups that must be checked to be 0. Proposition 3.1 in [KvdG03] tells us that $H^i(X, B^1) = 0$ for all $i$ for a height 1 Calabi-Yau variety. Then the duality induced by the Cartier operator tells us that $H^i(X, B^1) \simeq H^{3-i}(X, B^3)^\vee$, and hence all of these are zero as well.

We can prove all the necessary $H^i(X, B^2)$ are trivial as follows. We consider the standard defining sequence

$$0 \to Z^1 \to \Omega^1 \xrightarrow{d} B^2 \to 0.$$

The end of the long exact sequence gives

$$\to H^3(X, Z^1) \to H^3(X, \Omega^1) \to H^3(X, B^2) \to 0.$$

But the Calabi-Yau condition implies that $H^3(X, \Omega^1) = 0$, so $H^3(X, B^2) = 0$. Note the duality from the Cartier operator gives us $(B^2)^\vee \simeq B^2$. An application of Serre duality gives $H^0(X, B^2) \simeq H^3(X, B^2) = 0$. This proves Kato's criterion and hence all of Theorem 2.1.2. □

*A Candidate Counterexample*

To get a feel for exactly why the Bloch-Kato definition of ordinary is strictly stronger than height 1 we could note that their definition requires $H^1(X, B^2) = 0$. Unfortunately, this does not give a useful distinction. For the purpose of the construction of a counterexample, we will put some extra hypotheses on our Calabi-Yau threefold. Assume now that $X/k$ lifts to characteristic 0, and that the crystalline cohomology is torsion-free.

Under these hypotheses, [BK86] Proposition 7.3 says that being Bloch-Kato ordinary is equivalent to the Newton polygon of the $F$-crystal $H^3_{crys}(X/W)$ being exactly the same as the Hodge polygon defined by the numbers $h^{3-i}(X, \Omega^i)$. Using this idea we can examine why height 1 forces this condition in low dimensional examples. For an elliptic curve $C$, the $F$-crystal $H^1_{crys}(C/W)$ is 2-dimensional, and height 1 means that there is exactly 1 slope in the range $[0, 1)$ (and that slope is 0). By the Hard Lefschetz Theorem the other slope is 1. Thus the Newton and Hodge polygons coincide.

For a K3 surface, $X$, the Hodge polygon is completely determined independently of its height: there is 1 of slope 0, 20 of slope 1, and 1 of slope 2. If $X$ has height 1, then we again are forcing $H^2_{crys}(X/W)$ to have exactly one slope in $[0, 1)$ (and that

slope is 0). The Hard Lefschetz theorem tells us that the only slope in the range $(1, 2]$ is 2. Thus all other 20 slopes must be 1. This is interesting because height 1 is merely a condition about one of the slopes, but there is so much symmetry in small dimensional examples that this forces the shape of the entire Newton polygon.

Now we see why such a trick cannot be used for Calabi-Yau threefolds. Consider the case where $b_3(X) = 4$. If $X$ has height 1, then we can account for 2 of the slopes: one is 0, and the other is 3. We know the other two for the Hodge polygon which are a single slope 1 and a single slope 2. This is one possibility for the Newton polygon and corresponds to being ordinary in the Bloch-Kato sense. The other possiblity is that the Newton polygon has two slopes of the form $3/2$. This would be a height 1 Calabi-Yau threefold that is not Bloch-Kato ordinary.

## 2.2 Arithmetic Theory of Derived Categories

*Mukai Motives for Odd Dimensional Varieties*

Suppose that $X/k$ is a smooth projective variety of odd dimension $d$. The *derived category $D(X)$* is defined to be $D^b(Coh(X))$, the bounded derived category of coherent sheaves on $X$. We omit the superscript and shorten the inside, because we will not need any of the variants for this thesis.

In [LO11], all the various realizations of the even Mukai motives for even dimensional varieties are described in detail. Rather than recount every one of these constructions, we will point out the modifications needed for the crystalline realization when the variety is odd and then state the other realizations. All the proofs follow verbatim.

Suppose $X$ and $Y$ are two smooth projective varieties over $k$. We call $X$ and $Y$ *derived equivalent* if there is a triangulated, $k$-linear equivalence between the derived categories. We write this as $D(X) \simeq D(Y)$. Now let $\mathcal{P} \in D(X \times Y)$. By using the two natural projections $p : X \times Y \to Y$ and $q : X \times Y \to X$ we can construct a

functor between the two derived categories $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ defined by $\Phi_{\mathcal{P}}(\mathcal{E}) = p_*(q^*\mathcal{E} \otimes \mathcal{P})$. To avoid cluttered notation, all functors are derived in this section. The functor $\Phi_{\mathcal{P}}$ is called a *Fourier-Mukai transform* with kernel $\mathcal{P}$. If $\Phi_{\mathcal{P}}$ is an equivalence of categories, then we call $X$ and $Y$ *Fourier-Mukai partners*.

A major question that was solved by Orlov is whether or not every derived equivalent pair of smooth projective varieties are also Fourier-Mukai partners.

**Theorem 2.2.1** (Orlov [Orl97]). *Let $X$ and $Y$ be smooth projective varieties over $k$, and let $F : D(X) \to D(Y)$ be a fully faithful exact functor. If $F$ admits a right and left adjoint, then there is some $\mathcal{P} \in D(X \times Y)$ together with a natural isomorphism $F \simeq \Phi_{\mathcal{P}}$.*

We see that any equivalence, $F$, satisfies the hypothesis of the theorem, so in particular there is a unique (up to isomorphism) kernel $\mathcal{P}$ and a natural isomorphism $F \simeq \Phi_{\mathcal{P}}$.

The importance of equivalences being given by a Fourier-Mukai transform is that we can now use the kernel to induce maps on various cohomology theories. We first describe how this works for crystalline cohomology. Suppose $k$ is a perfect field of positive characteristic $p$ and $W$ is the ring of $p$-typical Witt vectors. Let $K$ be the fraction field of $W$. Recall that the semilinear action of Frobenius on the crystalline cohomology

$$H^i_{crys}(X/K) := H^i_{crys}(X/W) \otimes_W K$$

turns this vector space into an $F$-isocrystal.

There is a crystalline Chern character [GM87] satisfying the standard relations $ch_{crys} : K(X) \to H^{2*}_{crys}(X/K)$. As shown in [LO11], the Frobenius action interacts with the $2i$-th component via $\varphi_X(ch^i_{crys}(E)) = p^i ch^i_{crys}(E)$. Thus cup product with $ch^i_{crys}(E)$ as a map of $F$-isocrystals must take into account a negative twist $H^j_{crys}(X/K) \to H^{j+2i}_{crys}(X/K)(-i)$.

Now we define the odd Mukai isocrystal to be

$$\tilde{H}_o(X/K) := \bigoplus_{i=1}^{d} H_{crys}^{2(d-i)+1}(X/K)\left(\frac{d-2i+1}{2}\right). \tag{2.2}$$

This is the exact analogue for odd dimensional varieties as the standard Mukai weighting known for K3 surfaces. It aligns the weight to the middle degree, $d$. We define the even Mukai isocrystal to be

$$\tilde{H}_e(X/K) := \bigoplus_{i=0}^{d} H_{crys}^{2i}(X/K)(i). \tag{2.3}$$

This weights everything in degree 0, because there is no middle degree.

We now relate this to the Fourier-Mukai transform. Since we are concerned with derived equivalences, we only consider the case where both $X$ and $Y$ have the same odd dimension $d$. Suppose $\mathcal{P} \in K(X \times Y)$. In order to be compatible with Grothendieck-Riemann-Roch we define $\Phi_{\mathcal{P}}$ to be the correspondence on the sum of the Mukai isocrystals given by the Mukai vector $v(\mathcal{P}) := ch(\mathcal{P}).\sqrt{Td(X \times Y)}$.

Generically, on the odd Mukai isocrystal the map is given by

$$\begin{array}{ccc}
H_{crys}^{2(d-i)+1}(X/K)(\frac{d-2i+1}{2}) & \xrightarrow{pr_1^*} & H_{crys}^{2(d-i)+1}(X \times Y/K)(\frac{d-2i+1}{2}) \\
& & \downarrow{ch^{i-j+d}} \\
H_{crys}^{2(d-j)+1}(Y/K)(\frac{d-2j+1}{2}) & \xleftarrow{pr_{2*}} & H_{crys}^{4d-2j+1}(X \times Y/K)\left(\frac{3d-2j+1}{2}\right)
\end{array}$$

and the map on the even part is given as a sum of

$$\begin{array}{ccc}
H_{crys}^{2i}(X/K)(i) & \xrightarrow{pr_1^*} & H_{crys}^{2i}(X \times Y/K)(i) \\
& & \downarrow{ch^{j-i+d}} \\
H_{crys}^{2j}(Y/K)(j) & \xleftarrow{pr_{2*}} & H_{crys}^{2(j+d)}(X \times Y/K)(j+d).
\end{array}$$

Following the usual theory, when the Fourier-Mukai transform on the derived category $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ is an equivalence, this descends to an isomorphism on

cohomology. As shown in the previous two diagrams, the isomorphism preserves the even and odd parts of the isocrystal, and hence will give isomorphisms of F-isocrystals $\tilde{H}_o(X/K) \overset{\sim}{\to} \tilde{H}_o(Y/K)$ and $\tilde{H}_e(X/K) \overset{\sim}{\to} \tilde{H}_e(Y/K)$. In particular, both isomorphisms are Frobenius equivariant and hence the slopes of Frobenius acting on both sides are the same.

The étale realization follows in exactly the same way. Fix a separable closure $k \hookrightarrow \overline{k}$, and let $\ell$ be a prime different from $p$. We form the even and odd $\ell$-adic realizations of the Mukai motives by

$$\tilde{H}_o(X, \mathbf{Z}_\ell) \quad := \quad \bigoplus_{i=1}^{d} H_{\acute{e}t}^{2(d-i)+1}(X_{\overline{k}}, \mathbf{Z}_\ell)\left(\frac{d-2i+1}{2}\right) \tag{2.4}$$

and

$$\tilde{H}_e(X, \mathbf{Z}_\ell) \quad := \quad \bigoplus_{i=0}^{d} H_{\acute{e}t}^{2i}(X_{\overline{k}}, \mathbf{Z}_\ell)(i). \tag{2.5}$$

In general, this integral $\ell$-adic cohomology will not be preserved under a derived equivalence because the Chern character involves dividing by numbers. The object that is preserved is the cohomology with $\mathbf{Q}_\ell$ coefficients, which is the standard $\ell$-adic cohomology. We denote those realizations by $\tilde{H}_o(X, \mathbf{Q}_\ell)$ and $\tilde{H}_e(X, \mathbf{Q}_\ell)$. This will be dealt with more carefully as it comes up later.

On each realization, we can define the Mukai pairing which is preserved by the Fourier-Mukai isomorphism. A generic element of the direct sum decomposition of the odd realization of the motive looks like $(a_1, a_3, \ldots, a_{2d-1})$. We define the Mukai pairing on the $\ell$-adic realization

$$\langle \cdot, \cdot \rangle : \tilde{H}_o(X, \mathbf{Q}_\ell) \times \tilde{H}_o(X, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell(-d)$$

given by

$$\langle (a_1, \ldots, a_{2d-1}), (b_1, \ldots, b_{2d-1}) \rangle = \sum_{i=1}^{d} (-1)^{i-1} a_{2i-1} b_{2d-2i+1}.$$

The pairing on the even $\ell$-adic realization

$$\langle \cdot, \cdot \rangle : \tilde{H}_e(X, \mathbf{Q}_\ell) \times \tilde{H}_e(X, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell$$

is similarly given by

$$\langle (a_0, \ldots, a_{2d}), (b_0, \ldots, b_{2d}) \rangle = \sum_{i=0}^{d} (-1)^i a_{2i} b_{2d-2i}.$$

The last definition we need about Fourier-Mukai equivalences will be a crucial component of the main result in Chapter 3. We call a Fourier-Mukai equivalence *filtered* if the induced map on the Chow groups

$$\Phi_{\mathcal{P}}^{CH} : CH(X) \to CH(Y)$$

preserves the codimension filtration given by $F^s := \bigoplus_{i \geq s} CH^i(X)$. For a surface, this is equivalent to $\Phi_{\mathcal{P}}^{CH}(F_X^2) = F_Y^2$. This is a very strong condition, because for a K3 surface this descends to cohomology to give a Hodge isometry as in the Torelli theorem. It is Theorem 5.1 in [LO11] that if $X$ and $Y$ are K3 surfaces with a filtered equivalence between them, then they are isomorphic.

## 2.3   Derived Equivalence of Curves

In this section we prove some facts and provide some examples in the one-dimensional case. It will serve as a motivating tool for the ideas in the later sections.

*Definitions and Background*

Fix an arbitrary field $k$. An elliptic curve $E$ is defined to be a smooth, geometrically connected, genus 1 curve over $k$ with a distinguished rational point $e : \operatorname{Spec} k \to E$.

Suppose $E$ and $F$ are elliptic curves and $\Phi : D(E) \to D(F)$ defines an equivalence. As before, there is an object $\mathcal{P} \in D(E \times F)$ such that $\Phi$ is naturally isomorphic to the Fourier-Mukai transform with kernel $\mathcal{P}$. We will use $\Phi$ to denote this map with no indication of the kernel or spaces. Define $\Phi^i(\mathcal{K}) := \mathcal{H}^i(\Phi(\mathcal{K}))$ to be the cohomology sheaf of the corresponding complex.

For any closed point $x \in E(k)$, we notate the skyscraper sheaf at $x$ by $\mathcal{O}_x$. A complex $\mathcal{K}$ is called $WIT_i$ if there is a quasi-isomorphism $\Phi(\mathcal{K}) \simeq \mathcal{G}[-i]$ where $\mathcal{G}$ is a coherent sheaf.

Define $M_E^P : \mathrm{Sch}_k^{op} \to \mathrm{Set}$ to be the functor

$$M_E^P(T) = \left\{ \mathcal{E} \in Coh(E \times T) : \begin{array}{c} \mathcal{E} \text{ flat over } T \\ \mathcal{E}_t \text{ has Hilbert polynomial } P \end{array} \right\} \Big/ \simeq$$

where two sheaves are equivalent if they differ by the pullback of a line bundle from $T$. Define the subfunctor $M_E^{P,i}$ to also require that $\mathcal{E}_t$ is $WIT_i$ for all fibers.

**Proposition 2.3.1.** *If $M \subset M_E^{P,i}$ is a fine subfunctor, then $\Phi$ induces an isomorphism of schemes $M \to \Phi(M)$ where $\Phi(M)$ is the moduli space of sheaves over $F$ of the form $\Phi^i(\mathcal{E})$ for $\mathcal{E}$ in $M$.*

*Proof.* First, given any $i$, the Fourier-Mukai equivalence gives a map of functors $\Phi : M_E^{P,i} \to M_F^{P'}$ for some Hilbert polynomial $P'$. This is an abuse of notation, since $\Phi(T) : M_E^{P,i}(T) \to M_F^{P'}(T)$ via the Fourier-Mukai transform with kernel $\Delta_T \boxtimes \mathcal{P}$. Now Corollary 1.9 in [BBHR09] tells us that $\mathcal{E}_t$ is $WIT_i$ for all $t$ if and only if $\mathcal{E}$ is $WIT_i$ for the Fourier-Mukai transform on the product and $\Phi^i(\mathcal{E})$ is flat. We have assumed that $\mathcal{E} \in M_E^{P,i}(T)$, and hence $\mathcal{E}_t$ is $WIT_i$ for all $t$. Thus, $\Phi^i(\mathcal{E})$ is flat.

This tells us that the Hilbert polynomial is well-defined and the same for all objects in the image of $\Phi$. Let $M$ be any fine subfunctor of $M_E^{P,i}$. Since $\Phi$ is an equivalence, the induced map $M \to \Phi(M)$ is still fully faithful. By definition, $\Phi(M)$ is the essential image. Thus $\Phi$ induces an isomorphism of schemes $M \to \Phi(M)$. One sees that $\Phi(M)$

is the moduli space of objects of the form $\Phi^i(\mathcal{E})$ over $F$ and hence this moduli space is isomorphic to $M$. $\qquad\square$

**Lemma 2.3.2.** *There is an isomorphism*

$$\text{Hom}^i_{D(E)}(\mathcal{M},\mathcal{N}) \xrightarrow{\sim} \text{Hom}^i_{D(F)}(\Phi(\mathcal{M}),\Phi(\mathcal{N})).$$

*Proof.* This is standard. See for example Theorem 2.6 in [BBHR09]. $\qquad\square$

**Theorem 2.3.3.** *If $E,F$ are elliptic curves over a field $k$ such that $D(E) \simeq D(F)$, then $E \simeq F$.*

*Proof.* Let $x \in E(k)$ be a closed point. First we show that there is a unique value of $i$ such that $\Phi^i(\mathcal{O}_x) \neq 0$. Consider the spectral sequence associated to a double complex

$$E_2^{p,q} = \bigoplus_i \text{Hom}^p_{D(F)}(\Phi^i(\mathcal{O}_x),\Phi^{i+q}(\mathcal{O}_x)) \Rightarrow \text{Hom}^{p+q}_{D(F)}(\Phi(\mathcal{O}_x),\Phi(\mathcal{O}_x)).$$

The sequence of low degree terms gives an injection

$$\bigoplus_i \text{Hom}^1_{D(F)}(\Phi^i(\mathcal{O}_x),\Phi^i(\mathcal{O}_x)) \hookrightarrow \text{Hom}^1_{D(F)}(\Phi(\mathcal{O}_x),\Phi(\mathcal{O}_x)).$$

By Lemma 2.3.2, this latter group is isomorphic to $\text{Ext}^1_E(\mathcal{O}_x,\mathcal{O}_x) \simeq k$. Now the $\Phi^i(\mathcal{O}_x)$ are all simple, so by Serre duality and a dimension count there can only be one $i$ for which $\Phi^i(\mathcal{O}_x)$ is non-zero.

The $i$ is independent of choice of $x$, because by [BBHR09] Proposition 6.5 for any fixed $j$ the set of $x$ such that $\mathcal{O}_x$ is $WIT_j$ is open in $E$. Since all $\mathcal{O}_x$ are $WIT_i$ for some $i$, none can specialize to a complex that is not concentrated in a single degree. Since $E$ is connected, this $i$ is the same for all $x$. Note the moduli functor, $M$, of skyscraper sheaves is a fine subfunctor of $M_E^{P,i}$ representable by $E$. Thus by Proposition 2.3.1, $E$ is isomorphic to the moduli space of simple sheaves of the form $\Phi^i(\mathcal{O}_x)$ on $F$. We denote this by $M$.

If the $\Phi^i(\mathcal{O}_x)$ are torsion-free, then they are stable. Define $r = \text{rk}(\Phi^i(\mathcal{O}_x))$ and $d = \deg(\Phi^i(\mathcal{O}_x))$. In this notation, the equivalence realizes $E$ as the moduli space $M_F(r,d)$ of vector bundles of rank $r$ and degree $d$.

Since $\Phi^i(\mathcal{O}_x)$ is stable it has the property that $\gcd(r, d) = 1$. Thus $E$ is isomorphic to the moduli space of indecomposable vector bundles of rank $r$ and degree $d$ over $F$. Taking the determinant gives an isomorphism of this moduli space with the moduli space of degree $d$ line bundles over $F$, i.e. $Pic^d(F) \simeq F$. Thus $E \simeq F$. For details on this isomorphism see [Ati57].

Our other case is if $\Phi^i(\mathcal{O}_x)$ has torsion. In this case they are skyscraper sheaves, and hence $E \simeq F$ via the transform itself. $\qquad\square$

*Genus 1 Curves*

We can use the previous result to show that the primes of bad reduction are preserved under derived equivalence for any genus 1 curve.

**Proposition 2.3.4.** *If $C, C'/k$ are derived equivalent genus one curves, then they have the same set of primes of bad reduction.*

*Proof.* Since being derived equivalent is stable under base change to the completion of a field at any place it suffices to prove that $C$ has good reduction if and only if $C'$ has good reduction where the curves are defined over some local field $k$. Suppose $C$ has good reduction. Then its reduction is a smooth genus 1 curve over a finite field. It is geometrically an elliptic curve, so by Lang's theorem there is a rational point. By Hensel's lemma, this point lifts, and hence $C(k) \neq \emptyset$. This shows that $C$ is an elliptic curve. By the previous theorem this implies that $C \simeq C'$. Thus $C'$ also has good reduction. By symmetry we repeat the argument to show that if $C'$ has good reduction, then $C$ also has good reduction. $\qquad\square$

Now we come to the main theorem of this section. This is an analogue of the main result of the paper [AKW]. This fully settles when two curves are derived equivalent over any base field. Before stating the theorem we can simplify the statement using the following lemma.

**Lemma 2.3.5.** *If $C, C'$ are genus 1 curves over a field $k$ and $D(C) \simeq D(C')$, then $C$ and $C'$ are homogeneous spaces under the same elliptic curve, $Jac(C)$.*

*Proof.* First note that $C' \simeq M_C(r, d)$. We may assume $r \geq 1$ or else they are skyscraper sheaves and hence $C \simeq C'$. We may also assume $(r, d) = 1$, because $r$ and $d$ are independent of base change, but over $k^s$ the curves become isomorphic elliptic curves. Here we know that the Fourier-Mukai partner must be a moduli of simple sheaves which are exactly the stable sheaves ([Bri98] Theorem 6.4.3).

There is a natural action of $Jac(C) \simeq M_C(1, 0)$ on $C'$. We check this action is simply transitive. Consider

$$M_C(1, 0) \times M_C(r, d) \to M_C(r, d)$$

given by $(L, V) \mapsto L \otimes V$. This is an action defined over $k$ and hence we may check it is simply transitive after base change to $k^s$. Over $k^s$ we apply the corollary to Theorem 7 in [Ati57] to see that the kernel of the action is contained in $Jac(C)[r]$, and hence the quotient $Jac(C)/Jac(C)[r] \simeq Jac(C)$ acts simply transitively over $k^s$ which is what we sought to show. $\qquad\square$

Now we fix $E/k$, the elliptic curve satisfying $Jac(C) \simeq E$, and we may assume that both $C$ and $C'$ are torsors under $E$.

**Theorem 2.3.6.** *If $C, C'$ are genus 1 curves over a field $k$, then $D(C) \simeq D(C')$ if and only if there exists a separable $\phi \in End_k(E)$ such that $[C] = \phi^*[C'] \in H^1(k, E)$ where $(\deg(\phi), \text{per}([C])) = 1$.*

Before beginning the proof, we recall some fairly well-known lemmas about torsor classes.

**Lemma 2.3.7.** *If $[C], [C'] \in H^1(k, E)$ are classes of genus 1 curves thought of as torsors under $E$, then the genus 1 curves are isomorphic as $k$-curves if and only if they are in the same orbit under the natural action of $Aut(E)$ on the cohomology.*

Note that for any elliptic curve over any field the group $Aut(E)$ is finite and of order less than or equal to 24.

*Proof.* Suppose $(C, \mu)$ and $(C, \mu')$ are principal homogeneous spaces for $E$. We wish to find some $\phi \in Aut(E)$ such that $(C, \mu') = (C, \mu \circ (id \times \phi))$. Both $\mu$ and $\mu'$ define a "subtraction" $C \times C \to E$. If we fix a point $p_0 \in C$, then we get a bijection $C \to E$ given by $p \mapsto p_0 - p$. This allows us to fill in a set-theoretic bijection $\phi : E \to E$ making the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{p_0 - \mu} & E \\
{\scriptstyle id}\big\downarrow & & \big\uparrow{\scriptstyle \phi} \\
C & \xrightarrow[p_0 - \mu']{} & E
\end{array}
$$

Lemma X.3.1 from [Sil09] tells us $\phi$ is a group homomorphism, and Proposition 3.2 tells us it is an algebraic morphism defined over $k$. Thus $\phi \in Aut(E)$ and is constructed to satisfy the appropriate condition. $\square$

We now prove one direction of Theorem 2.3.6. Fix $[C], [C'] \in H^1(k, E)$.

**Proposition 2.3.8.** *If there exists $\phi \in End_k(E)$ that has coprime degree to the order of $[C]$ such that $[C'] = \phi^*[C]$, then we have $D(C) \simeq D(C')$ over $k$.*

*Proof.* First we consider the various elements in the $End(E)$-orbit of $[C]$. Fix $p_0 \in C$ (note that $k(p_0)$ is necessarily a non-trivial extension of $k$ if $C$ is non-split). The class is represented by the twisted homomorphism $\rho : G_k \to E$ given by $\rho(\sigma) = p_0^\sigma - p_0$ (where subtraction is defined using the torsor structure).

Adding, we see $n[\rho] = [n]^*[\rho]$. In other words, the subgroup generated by some class is the same thing as considering the $\mathbf{Z}$-orbit given by the $\mathbf{Z} \subset End(E)$-action on cohomology via pullback.

Suppose $[C'] = \phi^*[C]$. Let $d = \deg \phi$. By assumption we have $(d, \mathrm{per}(C)) = 1$. As before, $C'$ is a moduli of stable vector bundles on $C$, so by taking the determinant we can assume that $[C'] \simeq M_C(1, s)$.

The idea now is to show that the obstruction to the existence of the universal sheaf on $M_C(1, s)$ vanishes so that we can use the universal sheaf as a Fourier-Mukai kernel to produce an explicit equivalence $D(C) \simeq D(M_C(1, s)) = D(C')$.

The obstruction to the existence of the universal sheaf is a (torsion) Brauer class $\alpha \in Br(C')$. We directly check that this obstruction vanishes. Let $\eta$ be the generic point of $C'$ and $K = k(C')$ be the function field.

The Leray spectral sequence shows that we have an injection given by specialization at the generic point: $Br(C') \hookrightarrow Br(K)$. Thus it is enough to check that $\alpha|_\eta = 0$. The inclusion of fields $k \hookrightarrow K$ gives us a base change map $C'_K \to C'$ and we get a pullback map on Brauer groups: $Br(C') \to Br(C'_K)$.

The specialization map factors $Br(C') \to Br(C'_K) \to Br(K)$. Thus it suffices to check that after base change to the function field the class $\alpha$ vanishes. But after base change, $C'_K$ is split as a torsor, because it has a $K$-rational point, $\eta$.

Our assumption that $C' = \phi^* C$ implies that under the natural action of $\ker \phi$ on $C$ we have $C' \simeq C/\ker \phi$, or in other words $C$ is a degree $d$ cover of $C'$. Since $(d, \mathrm{per}(C)) = 1$, the pullback of $\eta$ under this map gives $C_K$ a rational point, and hence $C$ is split after base change to $K$ as well.

Thus we have that $M_{C_K}(1, s)$ is unobstructed which shows $\alpha|_\eta = 0$. This shows that the moduli problem has a universal sheaf (over $k$). Taking this sheaf to be the kernel of a Fourier-Mukai transform, we get that $D(C) \simeq D(C')$. $\qquad \square$

We now finish the proof of Theorem 2.3.6.

*Proof.* Proposition 2.3.8 proves one direction. Now we assume that $D(C) \simeq D(C')$ over $k$. We must find a separable isogeny $\phi \in End_k(E)$ such that $[C'] = \phi^*[C]$ and $(\deg \phi, \mathrm{per}([C])) = 1$. As before, we may assume $C' \simeq M_C(1, d)$ for some $d$. We can

now make a cocycle computation to show that $[d] : E \to E$ is a separable isogeny with the property that $[d]^*[C] = [C']$. If $p_0 \in C$, then $dp_0 \in C'$. Thus

$$
\begin{aligned}
[C'] &= (\sigma \mapsto (dp_0)^\sigma - (dp_0)) \\
&= d(p_0^\sigma - p_0) \\
&= d[C] \\
&= [d]^*[C].
\end{aligned}
$$

This gives us our candidate element of $End_k(E)$. We note that the moduli problem $M_C(1, d)$ has an unobstructed universal sheaf, because the kernel of the Fourier-Mukai transform is (quasi-isomorphic to) such an object. To finish the proof, it suffices to check that $(d, \mathrm{per}([C])) \neq 1$ implies that the obstruction class is not trivial. This is Theorem 4.4 in [AKW]. $\qquad\square$

*Examples and Applications*

In this section we show that in some typical situations it is still the case that genus one curves are derived equivalent if and only if they are isomorphic. We also show that when this is not the case, it provides counterexamples to a long standing open problem about moduli of stable vector bundles on genus 1 curves.

**Example 2.3.9.** *If $C$ and $C'$ are derived equivalent genus $1$ curves over a finite field $k = \mathbf{F}_q$, then $C \simeq C'$.*

*Proof.* Lang's theorem tells us that in general that if $X/k$ is geometrically an abelian variety, then $X(k) \neq \varnothing$. Thus $C$ and $C'$ are elliptic curves over $k$. Now by Theorem 2.3.3 we have that $C \simeq C'$. $\qquad\square$

**Example 2.3.10.** *If $C$ and $C'$ are derived equivalent genus $1$ curves over $\mathbf{R}$, then $C \simeq C'$.*

*Proof.* Let $E = Jac(C)$. If $C$ and $C'$ are not isomorphic, then they must represent two distinct $Aut(E)$-orbits in $H^1(\mathbf{R}, E)$. Our goal is to show that $H^1(\mathbf{R}, E)$ is too small to allow non-isomorphic torsor classes. There are two cases corresponding to whether or not $E$ has full 2-torsion defined over $\mathbf{R}$. We merely check that $\left|H^1(\mathbf{R}, E)\right| \leq 2$.

Since $\mathrm{Gal}(\mathbf{C}/\mathbf{R}) \simeq \mathbf{Z}/2$, we know that $H^1(\mathbf{R}, E)$ is killed by 2. This tells us computing the whole group is equivalent to computing the 2-torsion part. In the case that $E$ has full 2-torsion defined over $\mathbf{R}$. Consider the Kummer sequence:

$$0 \to E[2] \to E \xrightarrow{2} E \to 0$$

This induces

$$0 \to E(\mathbf{R})/2E(\mathbf{R}) \to H^1(\mathbf{R}, E[2]) \to H^1(\mathbf{R}, E) \to 0.$$

The middle group consists of non-twisted homomorphisms $\mathrm{Hom}(\mathbf{Z}/2, \mathbf{Z}/2 \times \mathbf{Z}/2)$ because the 2-torsion is fully defined over $\mathbf{R}$, so the Galois action is trivial. Since the Weil pairing is non-degenerate and Galois invariant, we know that the full 4-torsion is not defined over $\mathbf{R}$ otherwise $\mathbf{R}^\times$ would contain four distinct roots of unity. This means that $[2] : E(\mathbf{R}) \to E(\mathbf{R})$ is not surjective and hence $E(\mathbf{R})/2E(\mathbf{R})$ is non-trivial. This proves the inequality.

The other case is that $E[2](\mathbf{R}) \simeq \mathbf{Z}/2$. In this case we can explicitly write down in terms of elements $E[2] = \{1, a, b, c\}$. Without loss of generality, we assume $a^\sigma = a$, $b^\sigma = c$ and $c^\sigma = b$ where $\sigma$ is the non-trivial element of $G_\mathbf{R}$. The condition on $\rho : G_R \to E[2]$ being a twisted homomorphism forces $\rho(1) = 1$ and $\rho(\sigma) = 1$ or $a$. Thus there are only two possible cocycles. The non-trivial one is actually a coboundary, since $\rho(\sigma) = b^\sigma - b$. Thus $H^1(\mathbf{R}, E[2]) = 0$ which forces $H^1(\mathbf{R}, E) = 0$. $\qquad\square$

**Example 2.3.11.** *There exists non-isomorphic derived equivalent genus 1 curves.*

*Proof.* Fix $E/\mathbf{Q}$, a non-CM elliptic curve with $j(E) \neq 0, 1728$. Consider a genus one curve $[C] \in H^1(\mathbf{Q}, E)$ with period 5 (or for arbitrarily large period see [Cla06]). The

cyclic subgroup generated by $[C]$ has order 5 and hence all four non-split classes are generators. Only one other of these generators can be isomorphic as a $\mathbf{Q}$-curve by the Lemma 2.3.7. But by Theorem 2.3.8 take a non-isomorphic generator to get a non-isomorphic derived equivalent curve. $\qquad\square$

**Example 2.3.12.** *For any $N > 0$, there exists a genus $1$ curve that admits at least $N$ distinct moduli spaces of stable vector bundles $M_C(r,d)$ on $C$ mutually non-isomorphic as curves.*

*Proof.* We can again fix $E/\mathbf{Q}$ a non-CM elliptic curve with $j(E) \neq 0, 1728$ and $N > 0$. Choose a prime $p > 3N$. There exists a cyclic subgroup of $H^1(\mathbf{Q}, E)$ of order $p$. Since $Aut(E) \simeq \mathbf{Z}/2$, there are more than $p/2 > N$ non-isomorphic generators. By the proof of Theorem 2.3.8 any two of these generators, $C$ and $C'$, are derived equivalent. Thus $C' \simeq M_C(r,d)$ for some $r$ and $d$ coprime. $\qquad\square$

This result is a rather surprising contrast to the results of Atiyah which says that fine moduli spaces of vector bundles on an elliptic curve are always isomorphic to the elliptic curve. More recently, Pumplün [Pum04] even extended some of these results to work in a more general genus 1 setting suggesting that these types of examples should not exist.

## Chapter 3

# FOURIER-MUKAI PARTNERS OF K3 SURFACES

## *3.1   Introduction*

In this chapter we prove that every Fourier-Mukai partner of a K3 surface is a moduli
space of sheaves. This was already known for algebraically closed fields of arbitrary
characteristic from [LO11]. The proof here will be broken into a few steps. The result
will follow as a formal consequence of a derived version of the Torelli theorem. Thus
the bulk of the work is in proving this Torelli-like theorem. To do this, we prove the
result in characteristic 0. Then we use a lifting argument to prove the result over any
perfect field. Lastly, we use a standard reduction to show that it is enough to prove
the result over perfect fields.

## *3.2   The Derived Torelli Theorem*

First, we recall the classical strong Torelli theorem as stated in [SS71]:

**Theorem 3.2.1.** *If $(X, \mathcal{L})$ and $(Y, \mathcal{L}')$ are polarized, complex K3 surfaces such that
there is a Hodge isometry $\phi : H^2(X, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ that takes the class of $\mathcal{L}$ to $\mathcal{L}'$,
then there is a unique isomorphism $f : Y \xrightarrow{\sim} X$ inducing the isometry.*

Let $k$ be a field. Let $(X, \mathcal{L})$ and $(Y, \mathcal{L}')$ be polarized K3 surfaces over $k$. Given a
Fourier-Mukai equivalence $\Phi_{\mathcal{P}}$ we will denote the induced map on $\ell$-adic cohomology
by $\Phi_{\mathcal{P}}^H$ (see 2.5). We first prove the following derived analogue of the strong Torelli
theorem:

**Theorem 3.2.2.** *If $\Phi_{\mathcal{P}} : D(Y) \to D(X)$ is a filtered equivalence whose associated
cohomological Fourier-Mukai transform satisfies $\Phi_{\mathcal{P}}^H(\mathcal{L}') = \mathcal{L}$, then there is an isomor-*

*phism* $Y \overset{\sim}{\to} X$ *defined over* $k$ *inducing* $\Phi_{\mathcal{P}}^H$ *on cohomology. Moreover, in characteristic* 0 *there is a unique such isomorphism.*

Before beginning the proof, we consider a few facts about the Isom scheme between two K3 surfaces.

**Lemma 3.2.3.** *The quasi-projective scheme* $\mathbf{Isom}_k(X, Y)$ *is étale over* $k$.

*Proof.* The functor $\mathbf{Isom}_k(X, Y)$ is defined by

$$\mathbf{Isom}_k(X, Y)(A) = \{X \otimes_k A \overset{\sim}{\to} Y \otimes_k A\}$$

where the isomorphisms are over $A$. Since this is an open subscheme of the Hilbert scheme, see [Nit05] Theorem 5.23 for details, it is quasi-projective. We define $\mathbf{Isom}$ to be the scheme representing this functor.

Since $\mathbf{Isom}$ is defined over a field, we check that the functor is formally unramified. Let $A$ be a $k$-algebra and $I$ a square-zero ideal. We must check that the natural map $\mathbf{Isom}(A) \to \mathbf{Isom}(A/I)$ given by restricting an isomorphism $X \otimes A \to Y \otimes A$ to $X \otimes (A/I) \to Y \otimes (A/I)$ is injective.

Suppose $\phi_1$ and $\phi_2$ are isomorphisms over $A$ that agree over $A/I$. Then $\phi_2^{-1} \circ \phi_1$ is an infinitesimal automorphism of $X$. It is well-known in deformation theory that infinitesimal automorphisms are parametrized by $H^0(X, \mathcal{T} \otimes I) \simeq H^0(X, \Omega^1) \otimes I = 0$ since $\Omega^1 \simeq \mathcal{T}$ for a K3 surface. Thus $\phi_2^{-1} \circ \phi_1$ is the identity isomorphism over all of $A$. In other words, $\phi_1 = \phi_2$.

This shows that $\mathbf{Isom}(A) \to \mathbf{Isom}(A/I)$ is injective and hence $\mathbf{Isom}$ is étale. $\quad\square$

We will need to make a careful analysis of how certain results over $\mathbf{C}$ translate to other fields. Implicit in the hypotheses of some of these lemmas is the assumption that $k$ is a finitely generated extension of $\mathbf{Q}$.

**Lemma 3.2.4.** *If we fix an embedding* $\overline{k} \hookrightarrow \mathbf{C}$, *then there is a canonical bijection* $\mathbf{Isom}(\overline{k}) \to \mathbf{Isom}(\mathbf{C})$ *given by base change.*

*Proof.* The injective part of the lemma is true by definition. For the surjective part we may reduce to a special case of field theory since **Isom** is étale. By localizing, we may suppose that **Isom** $\simeq \operatorname{Spec} L$ where $L/k$ is a finite, separable field extension. The lemma can now be reformulated: given $L \to \mathbf{C}$ over $k$, does there exist an embedding $L \to \overline{k}$ factoring our map $L \to \overline{k} \to \mathbf{C}$?

Such an embedding can always be constructed by taking the algebraic closure of $L$ in $\mathbf{C}$ which we denote $\overline{L}$. This is abstractly isomorphic to $\overline{k}$. The set theoretic images of the two maps to $\mathbf{C}$ are the same, and the map $\overline{k} \to \overline{L}$ is a field isomorphism. We can invert it to find $L \hookrightarrow \overline{k}$ making the diagram commute. $\qquad\square$

The reader familiar with étale schemes should not be alarmed by the strange argument above. If $X/k$ is a finite, étale scheme, then $X \simeq \operatorname{Spec} L_1 \bigoplus \cdots \bigoplus L_n$ where each $L_j/k$ is finite and separable. Thus $X_{\overline{k}}$ has $n$ points and $X_{\mathbf{C}}$ has $n$ points. Therefore all $\mathbf{C}$-points descend because they already must exist over $\overline{k}$. This type of reasoning would suffice if we were considering $X_{\overline{k}}$ as a $\overline{k}$-scheme and $X_{\mathbf{C}}$ as $\mathbf{C}$-scheme, but we are intentionally working over $k$ which means all automorphisms of $\mathbf{C}$ over $k$ give different points. In order to prove the stronger result which involves uniqueness of the map, we must keep track of this extra information.

The idea now is to convert the necessary part of the classical strong Torelli theorem into a theorem on $\ell$-adic cohomology. We can then prove the derived strong Torelli theorem by using the $\ell$-adic realization of the Mukai motive. Let

$$\eta : \mathbf{Isom}_{\overline{k}}(Y_{\overline{k}}, X_{\overline{k}}) \to \mathbf{Isom}(H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_\ell), H^2_{\acute{e}t}(Y_{\overline{k}}, \mathbf{Z}_\ell))$$

be the natural map given by pullback. The left side consists of polarized isomorphisms over $\overline{k}$, and the right side consists of isomorphisms that preserve the ample class.

The rest of this section will consist of proving Theorem 3.2.2. Most of the work is in proving it for an arbitrary field of characteristic 0. We then make a series of reductions to prove the result over any field. Let $k$ be a field of characteristic 0, and fix an embedding $\overline{k} \hookrightarrow \mathbf{C}$. We may assume such an embedding exists by the Lefschetz

principle (if $k$ is too large, then we work over the compositum of the fields of definition of $X$ and $Y$).

**Lemma 3.2.5.** *The map $\eta$ is injective.*

This lemma is the uniqueness half of an $\ell$-adic Torelli theorem, and we prove it by showing that a polarized isomorphism over $\overline{k}$ induces a complex Hodge isometry preserving the ample class. Hence by the standard Torelli theorem we get a unique isomorphism over $\mathbf{C}$ which descends uniquely to $\overline{k}$.

*Proof.* Let $\varphi \in \mathbf{Isom}_{\overline{k}}(Y_{\overline{k}}, X_{\overline{k}})$. By using our embedding $i$, we can base change to get an isomorphism $\overline{\varphi} : Y_{\mathbf{C}} \to X_{\mathbf{C}}$ and by pullback we get a Hodge isometry that preserves the ample class $\overline{\varphi}^* : H^2(X_{\mathbf{C}}, \mathbf{Z}) \to H^2(Y_{\mathbf{C}}, \mathbf{Z})$.

We could also produce an isomorphism by first taking $\eta(\varphi)$ and using the base change map and the canonical isomorphisms:

$$
\begin{array}{ccc}
H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_\ell) & \xrightarrow{\ \eta(\varphi)\ } & H^2_{\acute{e}t}(Y_{\overline{k}}, \mathbf{Z}_\ell) \\
\wr \downarrow & & \downarrow \wr \\
H^2_{\acute{e}t}(X_{\mathbf{C}}, \mathbf{Z}_\ell) & \xrightarrow{\ \sim\ } & H^2_{\acute{e}t}(Y_{\mathbf{C}}, \mathbf{Z}_\ell) \\
\wr \downarrow & & \downarrow \wr \\
H^2(X_{\mathbf{C}}, \mathbf{Z}) \otimes \mathbf{Z}_\ell & \xrightarrow{\ \sim\ } & H^2(Y_{\mathbf{C}}, \mathbf{Z}) \otimes \mathbf{Z}_\ell \\
\uparrow & & \uparrow \\
H^2(X_{\mathbf{C}}, \mathbf{Z}) & \xrightarrow{\ \overline{\varphi}^*\ } & H^2(Y_{\mathbf{C}}, \mathbf{Z})
\end{array}
$$

Now we have $H^2(X_{\mathbf{C}}, \mathbf{Z}) \subset H^2(X_{\mathbf{C}}, \mathbf{Z}) \otimes \mathbf{Z}_\ell$, and since the only non-canonical isomorphism was made with the same choice of embedding $i$ we see that the bottom isomorphism restricts to the one we labelled $\overline{\varphi}^*$ on the integral lattice.

To prove injectivity of $\eta$ we now note that by the above process two isomorphisms $\varphi$ and $\varphi'$ for which $\eta(\varphi) = \eta(\varphi')$ will give the same Hodge isometry $H^2(X_{\mathbf{C}}, \mathbf{Z}) \to H^2(Y_{\mathbf{C}}, \mathbf{Z})$. By Theorem 3.2.1 we must have that $\overline{\varphi} = \overline{\varphi}'$. Now we use Lemma 3.2.4 that an isomorphism over $\mathbf{C}$ descends uniquely to one over $\overline{k}$. Thus $\varphi = \varphi'$. $\qquad\square$

**Lemma 3.2.6.** *The map $\eta$ is Galois equivariant.*

*Proof.* First we define the Galois action on both sides. Let $\sigma \in Gal(\overline{k}/k)$. Given an isomorphism $\phi : Y_{\overline{k}} \to X_{\overline{k}}$ we think of this as a natural transformation between the functors of points. Thus, given a $\overline{k}$-algebra $B$, there is a map $\phi(B) : Y_{\overline{k}}(B) \to X_{\overline{k}}(B)$. The new isomorphism, $\sigma \cdot \phi : Y_{\overline{k}} \to X_{\overline{k}}$, is the functor defined by the diagram:

$$
\begin{array}{ccc}
Y_{\overline{k}}(B) & \xrightarrow{(\sigma \cdot \phi)(B)} & X_{\overline{k}}(B) \\
{\scriptstyle Y(\sigma)}\downarrow & & \uparrow{\scriptstyle X(\sigma^{-1})} \\
Y_{\overline{k}}(B^\sigma) & \longrightarrow & X_{\overline{k}}(B^\sigma).
\end{array}
$$

The map $\sigma : B^\sigma \to B$ is given by acting on coefficients $b \otimes x \mapsto b \otimes \sigma(x)$. This is a $k$-algebra homomorphism and a twisted $\overline{k}$-algebra homomorphism, so we must use the functors $X$ and $Y$ defined over $k$ to get the vertical arrows. The bottom arrow is just from $\phi$ being a natural transformation. Composition defines a natural transformation on the top horizontal arrow. Since all three arrows are isomorphisms, the top map is again an isomorphism. To see that it is polarized we only need to note that the ample classes lie in the invariants: $[\mathcal{L}] \in H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_\ell)^{Gal(\overline{k}/k)}$.

The action on cohomology is given similarly. Given an isomorphism

$$
g : H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_\ell) \xrightarrow{\sim} H^2_{\acute{e}t}(Y_{\overline{k}}, \mathbf{Z}_\ell)
$$

we have natural actions of the Galois group on $Y_{\overline{k}}$ and $X_{\overline{k}}$, so by pullback we get actions on étale cohomology. We can use $g$ to construct a new isomorphism by composing:

$$
\begin{array}{ccc}
H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_\ell) & \xrightarrow{\sigma \cdot g} & H^2_{\acute{e}t}(Y_{\overline{k}}, \mathbf{Z}_\ell) \\
{\scriptstyle \sigma^*}\downarrow & & \uparrow{\scriptstyle (\sigma^{-1})^*} \\
H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_\ell) & \xrightarrow{g} & H^2_{\acute{e}t}(Y_{\overline{k}}, \mathbf{Z}_\ell).
\end{array}
$$

The claim of the lemma is that $\eta$ is Galois equivariant, i.e. $\eta(\sigma \cdot \phi) = \sigma \cdot \eta(\phi)$. Note that $\eta$ is taking an isomorphism and sending it to the isomorphism on cohomology via pullback. Moreover, the action on cohomology by an element of the Galois group $\sigma^*$ is by definition given by the pullback of acting on the coefficients of $X_{\overline{k}}$ or $Y_{\overline{k}}$ respectively. Thus

$$
\begin{aligned}
\eta(\sigma \cdot \phi) &= \eta(X(\sigma^{-1}) \circ \phi \circ Y(\sigma)) \\
&= [X(\sigma^{-1}) \circ \phi \circ Y(\sigma)]^* \\
&= Y(\sigma)^* \circ \phi^* \circ X(\sigma^{-1})^* \\
&= \sigma^* \circ \phi^* \circ (\sigma^{-1})^* \\
&= \sigma \cdot \phi^* \\
&= \sigma \cdot \eta(\phi)
\end{aligned}
$$

$\square$

Now we prove Theorem 3.2.2 in characteristic 0.

*Proof.* Suppose we have a filtered Fourier-Mukai equivalence over $k$ preserving the ample class. This gives us an isomorphism on $\ell$-adic cohomology

$$
H^2_{\acute{e}t}(Y_{\overline{k}}, \mathbf{Z}_\ell) \xrightarrow{\sim} H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_\ell).
$$

Now via our embedding $i : \overline{k} \hookrightarrow \mathbf{C}$ we get a Hodge isometry preserving an ample class

$$
H^2(Y_{\mathbf{C}}, \mathbf{Z}) \xrightarrow{\sim} H^2(X_{\mathbf{C}}, \mathbf{Z}).
$$

By Theorem 3.2.1, we conclude there is a unique isomorphism $\phi : Y_{\mathbf{C}} \xrightarrow{\sim} X_{\mathbf{C}}$.

This isomorphism descends to an isomorphism $\overline{\phi} : Y_{\overline{k}} \xrightarrow{\sim} X_{\overline{k}}$ that is unique in the sense of inducing the isomorphism on $\ell$-adic cohomology via Lemma 3.2.4. The cohomological Fourier-Mukai transform has the property that

$$
\Phi^H_{\mathcal{P}} \in \mathbf{Isom}(H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_\ell), H^2_{\acute{e}t}(Y_{\overline{k}}, \mathbf{Z}_\ell)).
$$

Thus by Lemma 3.2.5, if there is a preimage under $\eta$ it is unique. But we just constructed $\overline{\phi}$ to be an isomorphism with the property that $\eta(\overline{\phi}) = \Phi_{\mathcal{P}}^H$. Since $\Phi_{\mathcal{P}}^H$ is the base change of a Fourier-Mukai transform over $k$ it is necessarily Galois equivariant. Now Lemma 3.2.6 tells us that $\overline{\phi}$ must be Galois equivariant. Thus it descends to a unique isomorphism $\phi : Y \to X$ over $k$. $\qquad\qquad\square$

Now we move on to proving the derived Torelli theorem for perfect fields, so fix a perfect field $k$ of characteristic $p > 2$. We begin by stating the following facts that will be needed in the proof.

**Proposition 3.2.7** ([LO11] Proposition 5.3). *If there is a filtered equivalence $\Phi : D(Y) \to D(X)$ of K3 surfaces over $k$, then there is an isomorphism of infinitesimal deformation functors $\delta : \mathrm{Def}_Y \to \mathrm{Def}_X$ such that*

1. *$\delta^{-1}(\mathrm{Def}_{(X,L)}) = \mathrm{Def}_{(Y,\Phi(L))}$ for all $L \in \mathrm{Pic}(Y)$;*

2. *for each augmented Artinian $W$-algebra $W \to A$ and each $(X_A \to A) \in \mathrm{Def}_{(X,H_X)}(A)$, there is an object $P_A \in D(X_A \times_A \delta(X_A))$ reducing to $P$ on $X \times Y$.*

This tells us that if we deform $Y$ with a polarization we can take a polarized deformation of $X$ that is derived equivalent, and moreover the kernel of the equivalence can be chosen so that its reduction is the original kernel.

We will also need a special case of a theorem of Matsusaka and Mumford:

**Theorem 3.2.8** ([MM64] Theorem 1). *If $X$ and $Y$ are K3 surfaces over a DVR and the generic fibers are isomorphic, then the special fibers are isomorphic.*

Fix a K3 surface $X/k$. We now prove Theorem 3.2.2 in the perfect field case.

*Proof.* We break the proof into two cases. First, suppose the height of $X$ is finite and let $\Phi : D(Y) \to D(X)$ be an equivalence with kernel $P$ that preserves the

polarization, i.e. $\Phi(H_Y) = H_X$. By [Del81], there is a polarized lift $(X_W, H_{X_W})$ over the Witt vectors $W = W(k)$. Let $K$ be the fraction field of $W$.

By Proposition 3.2.7, there is a polarized lift of $Y$ which we denote $(Y_n, H_{Y_n})$ together with a Fourier-Mukai kernel $P_n \in D(Y_n \times X_n)$ inducing a polarized equivalence such that $P_n \otimes k = P$. By Theorem 1.2.2 the lift algebraizes to a K3 surface $Y_W/W$. By the Grothendieck existence theorem for perfect complexes as stated in Proposition 3.6.1 of [Lie06], the system of complexes $(P_n)$ is the completion of a perfect complex $P_W \in D(Y_W \times X_W)$.

This complex restricted to the generic fiber induces a Fourier-Mukai equivalence $D(Y_K) \to D(X_K)$. By construction it is polarized and filtered. Since $K$ is of characteristic 0, Theorem 3.2.2 tells us that we have an isomorphism $Y_K \overset{\sim}{\to} X_K$. Thus by Theorem 3.2.8, we have an isomorphism $Y \overset{\sim}{\to} X$. This completes the proof if $X$ has finite height. Note the only subtlety preventing us from getting uniqueness is an analysis of how the specialization of a birational isomorphism acts on cohomology.

If $X$ does not have finite height, then we may not be able to lift over $W$. The same proof will still work if there is a lift over some ramified extension of $W$ that does not extend the base field $k$. But by Section 2 of [Ogu79] since $p > 2$ we know that supersingular K3 surfaces can be lifted over $W[\sqrt{p}]$. This completes the proof. $\qquad\square$

Now we prove the theorem over any field.

*Proof.* Fix two K3 surfaces $X$ and $Y$ over an arbitrary field $k$ with a filtered equivalence between them. Fix an algebraic closure $\bar{k}$ of $k$. We form the perfect closure of $k$ inside $\bar{k}$ and label this $k^{1/p^\infty}$. We need to prove the derived Torelli theorem over $k$.

Suppose our derived equivalence preserves an ample class. Then the base change to $k^{1/p^\infty}$ also preserves an ample class. Now $k^{1/p^\infty}$ is perfect, so by Theorem 3.2.2 we get that there is an isomorphism $\phi : X_{k^{1/p^\infty}} \overset{\sim}{\to} Y_{k^{1/p^\infty}}$ giving the isomorphism on étale cohomology.

As in the Galois descent argument, we must now descend this isomorphism to

$k$. In this case we use Lemma 3.2.3 to prove that the canonical map $\mathbf{Isom}(k) \rightarrow \mathbf{Isom}(k^{1/p^\infty})$ is a bijection. The fact that $\mathbf{Isom}$ is étale over $k$ tells us that it is representable by an étale algebra $E$.

We must show that $T : \mathrm{Hom}_k(E, k) \rightarrow \mathrm{Hom}_k(E, k^{1/p^\infty})$ given by composing with the embedding $i : k \hookrightarrow k^{1/p^\infty}$, i.e. $(E \rightarrow k) \mapsto (E \rightarrow k \xrightarrow{i} k^{1/p^\infty})$ is a bijection. Since $T$ just composes with an inclusion, it is injective. Since $E$ is a product of separable field extensions, any homomorphism $E \rightarrow k^{1/p^\infty}$ must have separable image. The field extension of $k^{1/p^\infty}$ over $k$ is purely inseparable, so the image must land inside $k$. This shows that $T$ is surjective. Thus our map $\mathbf{Isom}(k) \rightarrow \mathbf{Isom}(k^{1/p^\infty})$ is a bijection. This says that there is a unique isomorphism descending $\phi$ to $k$ which proves the theorem. $\qquad\square$

## 3.3  Moduli of Sheaves on K3 Surfaces

This section will show how every Fourier-Mukai partner of a K3 surface is a moduli of sheaves. The proof is mostly a formal consequence of Theorem 3.2.2.

**Lemma 3.3.1.** *If $F : D(Y) \rightarrow D(X)$ is a filtered equivalence, then there exists an autoequivalence $G : D(Y) \rightarrow D(Y)$ such that the composition $F \circ G$ satisfies the hypothesis of Theorem 3.2.2.*

*Proof.* Since $F$ is an isometry under the Mukai pairing, we know there is some $b \in \mathrm{Pic}(Y)$ such that $F(1, 0, 0) = (1, b, \frac{1}{2}b^2)$. The autoequivalence $- \otimes (-b)$ allows us to assume $F(1, 0, 0) = (1, 0, 0)$ and $F(0, 0, 1) = (0, 0, 1)$. Now since $\mathrm{Pic}(Y) = (1, 0, 0)^\perp \cap (0, 0, 1)^\perp$ we see that we have the restriction $F : \mathrm{Pic}(Y) \rightarrow \mathrm{Pic}(X)$ is an isometry and hence an isomorphism of positive cones.

In order to send an ample class to an ample class we consider post-composing with tensoring by a high power of an ample. Choose any ample class $L$ in $\mathrm{Pic}(Y)$. The image $F(L)$ is coherent, so taking any ample $M$ in $\mathrm{Pic}(X)$ we know $F(L) \otimes M^N$ is generated by global sections for some $N > 0$. Since tensoring with a positive line

bundle preserves the positive cone, the Nakai-Moishezon criterion tells us $F(L) \otimes M^N$ is ample. Thus the composition $\mathrm{Pic}(Y) \to \mathrm{Pic}(X) \overset{-\otimes M^N}{\to} \mathrm{Pic}(X)$ is an isomorphism preserving an ample class.

Note that usually the Nakai-Moishezon criterion is only stated for algebraically closed fields, but ampleness can be checked after extension of scalars (use the cohomological criterion). Lastly, we need to find a Fourier-Mukai transform that induces our above transformations on the Mukai lattice. Such an algorithm is given in the proof of [LO11] Lemma 5.2. For example, reflection across $C$ is given by tensoring and composing with a spherical twist $(- \otimes \mathcal{O}(C)) \circ T_{\mathcal{O}_C}$. $\qquad \square$

**Theorem 3.3.2.** *Every Fourier-Mukai partner of a K3 surface is a moduli space of sheaves.*

*Proof.* Fix $X$ and $Y$ to be K3 surfaces defined over $k$. Suppose that there is some Fourier-Mukai equivalence $\Phi_\mathcal{P} : D(Y) \to D(X)$. The idea of the proof is to construct a particular moduli space of sheaves, $M$, on $X$ such that the composed equivalence $D(Y) \to D(X) \to D(M)$ is filtered, and hence by Lemma 3.3.1 and Theorem 3.2.2 we have $Y \overset{\sim}{\to} M$.

Let $v = (r, L_X, s)$ be the Mukai vector of $\mathcal{P}_y$ for some $y$. By composing with a twist and shift we can assume $r$ positive and $L_X$ very ample.

Let $M = M_X(v)$ be the moduli space of sheaves with Mukai vector $v$ with respect to $L_X$. This is fine and hence a K3 surface. The universal bundle on $X \times M$ induces a Fourier-Mukai equivalence $D(X) \to D(M)$. By construction the composition $D(Y) \to D(M)$ has the property that $(0, 0, 1) \mapsto (0, 0, 1)$ and hence the equivalence is filtered. Thus $Y \overset{\sim}{\to} M$. $\qquad \square$

As a consequence of this theorem we get the familiar result over any field of characteristic $p > 2$.

**Proposition 3.3.3.** *If $X$ is a Shioda supersingular K3 surface, then any Fourier-Mukai partner is isomorphic to $X$.*

The proof is contained in the proof of Proposition 7.2 in [LO11]. The argument goes through verbatim once the result is established that a Fourier-Mukai partner $Y$ of $X$ must have the form $Y \simeq M_X(r, \ell, s)$ for some primitive Mukai vector $v = (r, \ell, s)$. The idea of the proof is to consider the stack of sheaves, $\mathfrak{M}$, with Mukai vector $(r, \ell, s)$ over the universal deformation $\mathfrak{X} \to \operatorname{Spec} R$.

Now use the universal twisted sheaf and the original Fourier-Mukai kernel to give an equivalence between the geometric generic fibers $\mathfrak{X}_\eta$ and $\mathfrak{M}_\eta$. This means the two geometric generic fibers are isomorphic since they are rank 20 in characteristic 0. By uniqueness of specialization this tells us that the closed fibers $\mathfrak{X}_0 = X$ and $\mathfrak{M}_0 \simeq Y$ are isomorphic.

## 3.4  Application to a Lifting Problem

Over the past fifteen years, there has effectively only been two constructions of Calabi-Yau threefolds that have been discovered to not lift to characteristic 0. Hirokado [Hir99] produced one in characteristic 3, and Schröer [Sch04] produced a construction which gives examples in characteristic 2 and 3. More details on Calabi-Yau threefold and liftability considerations will follow in Chapter 4.

One of the original purposes of this thesis was to use moduli space techniques coupled with the theory of the derived category to produce new non-liftable Calabi-Yau threefolds. We can use Proposition 3.3.3 to prove that the most naive attempt in this direction will not work for Schröer's construction.

Recall that a Shioda supersingular K3 surface is one in which $\operatorname{rk} NS(X) = 22$. These have a finer invariant stratifying the moduli space called the Artin invariant. It is defined to be $\sigma(X)$ where $\operatorname{disc} NS(X) = -p^{2\sigma(X)}$. Note that this invariant is trivially preserved under derived equivalence because supersingular derived equivalent K3 surfaces are isomorphic.

Fix $k$, an algebraically closed field of characteristic 3. The relevant fact from Schröer's construction is that the end result is a supersingular K3 fibration $X \to \mathbf{P}^1$.

One way to construct such a family is to take Moret-Bailly's pencil of supersingular abelian varieties together with an automorphism of order 3 of each fiber. We quotient by this action and then take the minimal resolution. The result is a smooth Calabi-Yau threefold $X \to \mathbf{P}^1$ as desired. The generic fiber has Artin invariant 2.

We call such a family a *Moret-Bailly family* of K3 surfaces. It turns out these families play an extremely important role in the moduli space of supersingular K3 surfaces. Theorem 5.10 in [Eke04] shows that any fibration structure $X \to \mathbf{P}^1$ on a Calabi-Yau threefold whose fibers are supersingular K3 surfaces is a Moret-Bailly family. This shows that any relative Fourier-Mukai partner of a Moret-Bailly family is again a Moret-Bailly family. Hence no new *constructions* are obtained in this way. A priori, a relative Fourier-Mukai partner could be non-isomorphic as a threefold. This is what we explore next.

Recall the following beautiful characterization of the moduli space:

**Theorem 3.4.1** ([Ogu79] Theorem 7.15). *The moduli space of marked supersingular K3 surfaces of Artin invariant $\sigma \leq 2$ is a union of Moret-Bailly families.*

Now we fix one such non-liftable threefold admitting a Moret-Bailly family $X \to \mathbf{P}^1$. Since the example under consideration is a K3 fibration, the natural idea is to produce a relative Fourier-Mukai partner as done in [BM02]. This will produce a Calabi-Yau threefold which is also a K3 fibration over $\mathbf{P}^1$ and will have the property that the two are fiberwise derived equivalent.

**Lemma 3.4.2.** *If there exists a relative Fourier-Mukai equivalence $\Phi : D(X) \to D(Y)$, then there exists a fiber of each which is isomorphic.*

*Proof.* This follows immediately from Proposition 3.3.3, because the fibers are supersingular Fourier-Mukai partners. $\square$

**Proposition 3.4.3.** *If there exists a relative Fourier-Mukai equivalence $\Phi : D(X) \to D(Y)$, then $X \simeq Y$.*

*Proof.* If we restrict ourselves to the strata of the moduli space of marked supersingular K3 surfaces of Artin invariant $\sigma \leq 2$, then we get that it consists of a union of families of the form given in Schröer's example. In particular, any given Kummer K3 surface can only exist in a family over $\mathbf{P}^1$ in only one way (up to isomorphism). Since we know we can find two fibers that are isomorphic by Lemma 3.4.2, we know the total spaces must be isomorphic as well. $\qquad\square$

Note the stark contrast here with the case of an elliptically fibered K3 surface. For any $N > 0$ we can find a prime $p > 0$ such that the unique supersingular K3 surface of Artin invariant 1 in characteristic $p$ has greater than $N$ distinct pencils of elliptic curves on it. It is interesting to note that these distinct fibration structures are not even related by a relative Fourier-Mukai transform. For more details see [Shi13].

This strong rigidity seems related to the fact that there is a crystalline Torelli theorem for supersingular K3 surfaces. As Ogus points out in [Ogu79], the theorem is modelled on the corresponding theorem for abelian varieties of dimension $n \geq 2$. The crystalline Torelli theorem is actually false for elliptic curves! This seems to indicate that the natural idea of producing new examples of pathological spaces by generically twisting the fibration structure is not best studied by relative moduli spaces of sheaves in this case.

## Chapter 4

## ARITHMETIC DERIVED INVARIANTS OF CALABI-YAU THREEFOLDS

### 4.1  Introduction

The purpose of this chapter is to prove that many arithmetic invariants for Calabi-Yau threefolds are invariant under derived equivalence. Recently, the arithmetic theory of Calabi-Yau threefolds has been of interest. For example, in [GY11] it was proved that all rigid Calabi-Yau threefolds over $\mathbf{Q}$ are modular. Other non-rigid examples of modular Calabi-Yau threefolds have been found, but the conjecture as to whether or not they are all modular is still open. We prove here that modularity is a derived invariant.

There has been interest in positive characteristic as well. In characteristic zero, a Calabi-Yau threefold has no global 1-forms, but it is an open problem to determine this in positive characteristic. The $b$-number could be related to this phenomenon and we show it is a derived invariant. Every K3 surface is known to lift to characteristic 0, but there are examples of non-liftable Calabi-Yau threefolds. Liftability has been an essential tool in the study of K3 surfaces, so we would like to have a criterion for when they lift. The chapter ends with an explanation of Serre-Tate theory for ordinary Calabi-Yau threefolds that lift to characteristic 0. This gives liftable, ordinary Calabi-Yau threefolds a canonical lift.

### 4.2  Definitions and Background

In this section we will write down the basic definitions and describe the machinery that will be used throughout the chapter.

*Definitions*

Recall Definition 2.1.1: A *Calabi-Yau variety*, $X$, of dimension $n$ is defined to be a smooth, projective, geometrically connected variety over a field $k$ that satisfies $\omega_X \simeq \mathcal{O}_X$ and $H^j(X, \mathcal{O}_X) = 0$ for $0 < j < n$.

A *modular form* of weight $k$ and level $N$ is an element of the vector space $M_k(\Gamma_0(N))$. This consists of holomorphic functions on the upper half plane $f : \mathcal{H} \to \mathbf{C}$ satisfying the additional transformation property

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ such that $c \equiv 0 \mod N$ and $f$ extends to be holomorphic at all cusps.

A modular form has a Fourier expansion called a $q$-expansion denoted

$$f(q) = \sum_{n=0}^{\infty} a_n q^n \text{ where } q = e^{2\pi i z}.$$

The *cusp forms* are the subspace of the modular forms that vanish at all cusps. In particular, $f(0) = 0$ or $a_0 = 0$. It will also be assumed that our cusp forms are always normalized Hecke eigenforms.

*Cohomology of Rigid Calabi-Yau Threefolds*

Define the Hodge numbers of $X$ to be $h^{i,j} = \dim_k H^j(X, \Omega^i)$. We will assume our variety satisfies Hodge symmetry which says that $h^{i,j} = h^{j,i}$. This is always the case over a characteristic 0 field or if $X$ arises as the reduction to a finite field of characteristic $p > 3$ from an integral model.

By repeated application of Serre duality, Hodge symmetry, and triviality of the canonical bundle we get that $h^{0,0} = h^{3,0} = h^{0,3} = h^{3,3} = 1$. The numbers $h^{1,2} = h^{2,1}$ and $h^{1,1} = h^{2,2}$ can vary. Lastly, all other Hodge numbers are 0. This means that there are only two unknowns among the Hodge numbers. The Hodge diamond is

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{1,2} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

A *rigid* Calabi-Yau threefold is one that has no non-trivial infinitesimal deformations. Theorem 1.2.3 says that infinitesimal deformations are parametrized by $H^1(X, \mathcal{T}) \simeq H^1(X, \Omega^2)$. Thus $h^{1,2} = 0$ for a rigid Calabi-Yau threefold, and we are only left with one unknown Hodge number.

Suppose that $X$ is a rigid Calabi-Yau threefold defined over $\mathbf{Q}$. Let $X_{\overline{\mathbf{Q}}}$ be the base change of $X$ to $\overline{\mathbf{Q}}$. Note that $X_{\overline{\mathbf{Q}}}$ is also rigid. Since we are over an algebraically closed field of characteristic 0 the Hodge-de Rham spectral sequence degenerates at $E_1$. This implies that

$$
H^3_{dR}(X_{\overline{\mathbf{Q}}}/\overline{\mathbf{Q}}) \simeq \bigoplus_{i+j=3} H^j(X_{\overline{\mathbf{Q}}}, \Omega^i).
$$

All of the Hodge numbers are known and given above, so we see that $\dim_{\overline{\mathbf{Q}}} H^3_{dR}(X_{\overline{\mathbf{Q}}}/\overline{\mathbf{Q}}) = 2$.

Similarly, since we are over an algebraically closed field of characteristic 0 we get that for any prime $\ell$ the middle $\ell$-adic cohomology

$$
H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) := \varprojlim H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}}, \mathbf{Z}/\ell^n) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell
$$

is also two-dimensional.

*Derived Equivalence of Calabi-Yau Threefolds*

We now turn back to properties of the derived category of Calabi-Yau threefolds. We will begin by examining some basic well-known properties that are preserved under

derived equivalence.

**Proposition 4.2.1.** *If $X$ is a Calabi-Yau threefold over $\mathbf{Q}$ and $Y$ is a smooth, projective variety over $\mathbf{Q}$ such that there is a triangulated equivalence $F : D(Y) \overset{\sim}{\to} D(X)$, then $Y$ is a Calabi-Yau threefold.*

*Proof.* First, we can conclude that $\omega_Y \simeq \mathcal{O}_Y$ using Proposition 4.1 of [Huy06] which says the canonical bundles have the same order. Next we check that $Y$ must be a threefold. Theorem 2.2.1 tells us that $F = \Phi_{\mathcal{P}}$ for some kernel. The left and right adjoints to this Fourier-Mukai transform are given explicitly by the kernels $\mathcal{P}_L = \mathcal{P}^{\vee} \otimes p^*\omega_Y[\dim(Y)]$ and $\mathcal{P}_R = \mathcal{P}^{\vee} \otimes q^*\omega_Y[\dim(X)]$.

By triviality of the canonical bundles, the fact that the adjoints are quasi-inverses, and uniqueness of the kernel we get that $\mathcal{P}^{\vee} \simeq \mathcal{P}^{\vee} \otimes (\mathcal{O}[3-\dim(Y)])$. Thus, $\dim(Y) = 3$. The last property we need is to check the vanishing cohomology conditions. We know $b_1(X) = 0$ from the Hodge diamond. The main theorem of [PS11] tells us that derived equivalence preserves the first Betti number for threefolds, so $b_1(Y) = 0$. Thus $H^1(Y, \mathcal{O}_Y) = 0$, and by Serre duality $H^2(Y, \mathcal{O}_Y) = 0$. $\square$

**Corollary 4.2.2.** *If $X$ and $Y$ are Calabi-Yau threefolds and $Y$ is derived equivalent to $X$ via $\Phi_{\mathcal{P}} : D(Y) \to D(X)$, then the cohomological Fourier-Mukai transform induces an isomorphism on the middle cohomology $\Phi_{\mathcal{P}}^H : H^3_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) \overset{\sim}{\to} H^3_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$.*

*Proof.* This follows immediately, because the cohomological Fourier-Mukai transform gives an isomorphism on the odd étale realization of the Mukai motive (see Chapter 2, equation 2.4), but for Calabi-Yau threefolds we have $H^j_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) = 0$ for all odd integers $j \neq 3$. $\square$

For this thesis we will use the terminology "$\ell$-adic cohomology" to mean the standard Weil cohomology theory obtained by base change to $\overline{\mathbf{Q}}$ as above. The above corollary is probably false if one wants to examine the cohomology before base change.

## 4.3 Positive Characteristic Invariants

For this section we suppose that $k$ is an algebraically closed field of characteristic $p >$ 3. The purpose of this section is to prove that several arithmetic properties of Calabi-Yau threefolds in positive characteristic are preserved under derived equivalence. We first prove that the height (see Definition 1.3.2) is a derived invariant. As a corollary, we see that the properties of being ordinary, supersingular, and Frobenius split are preserved under derived equivalence. We then prove that the Zeta function is a derived invariant for Calabi-Yau threefolds. It follows that the number of $\mathbf{F}_{p^n}$-rational points are the same.

For technical reasons, we make the assumption that all Calabi-Yau threefolds have trivial first Betti number. This arises in many natural settings such as when we know it is the reduction of a variety from characteristic 0. Alternatively, we could assume the variety satisfies Hodge symmetry. It is still an interesting open conjecture as to whether or not this always holds and is related to the existence of global 1-forms. We will return to this later. Joshi in [Jos04] has proved that if the crystalline cohomology is torsion-free and the Hodge-de Rham spectral sequence degenerates at $E_1$ we will have Hodge symmetry.

We begin by formalizing a main tool into a lemma.

**Lemma 4.3.1.** *If $X$ and $Y$ are derived equivalent Calabi-Yau threefolds over $k$, then we have an isomorphism of F-isocrystals $H^3_{crys}(X/K) \xrightarrow{\sim} H^3_{crys}(Y/K)$.*

*Proof.* Let $X$ and $Y$ be derived equivalent Calabi-Yau threefolds over $k$. From the above discussion $H^1_{crys}(X/K) \simeq H^5_{crys}(X/K) = 0$ and $H^1_{crys}(Y/K) \simeq H^5_{crys}(Y/K) = 0$. This tells us the only non-zero part of the odd crystalline realization of the Mukai motive (see Chapter 2, equation 2.2) is the middle one. But we have an isomorphism $\tilde{H}_o(X/K) \simeq \tilde{H}_o(Y/K)$ as F-isocrystals and hence an isomorphism $H^3_{crys}(X/K) \xrightarrow{\sim} H^3_{crys}(Y/K)$ of F-isocrystals. $\qquad\square$

Our first consequence is that the height is preserved under derived equivalence.

**Theorem 4.3.2.** *If $X$ is a Calabi-Yau threefold over $k$ and $Y$ is derived equivalent to $X$, then $Y$ has the same height as $X$.*

*Proof.* Suppose $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ is the Fourier-Mukai transform giving the equivalence. By Lemma 4.3.1, the kernel induces an isomorphism of F-isocrystals $H^3_{crys}(X/K) \to H^3_{crys}(Y/K)$. In particular, all slopes of Frobenius are the same. Thus we get an isomorphism

$$H^3_{crys}(X/K)_{[0,1)} \xrightarrow{\sim} H^3_{crys}(Y/K)_{[0,1)}.$$

Since the height is just the dimension of this vector space (see Section 1.3 for a brief discussion) this completes the theorem. $\square$

**Corollary 4.3.3.** *The b-number is a derived invariant for Calabi-Yau threefolds.*

*Proof.* Proposition 6.1 of [vdGK13] states that $b(X) = \mathrm{ht}(X) + p_g(X)$. But for Calabi-Yau threefolds $p_g(X) = 1$. $\square$

This corollary is quite interesting, since in positive characteristic we cannot use Hodge symmetry to prove that there are no global 1-forms on a Calabi-Yau. This fact for K3 surfaces plays a role in proving they all lift. It is still only a conjecture that there are no global 1-forms on a Calabi-Yau threefold. The $b$-number gives some information in this direction.

**Corollary 4.3.4.** *The a-number is a derived invariant for finite height Calabi-Yau threefolds.*

*Proof.* Katsura and van der Geer [vdGK02] have shown that $a(X) = 0$ when $X$ has height 1, $a(X) = 1$ when $X$ has finite height other than 1 and that $a(X) = 2$ or $3$ when $X$ has infinite height. Thus the invariant is less refined for finite height Calabi-Yau varieties and hence preserved. $\square$

**Corollary 4.3.5.** *The properties of being supersingular, ordinary, and Frobenius split are derived invariants for Calabi-Yau threefolds.*

*Proof.* The corollary follows from the fact that supersingular is equivalent to having infinite height and Lemma 2.1.6. □

Now we prove that the number of $\mathbf{F}_{q^n}$-rational points are the same. This shows that the zeta function is a derived invariant.

**Theorem 4.3.6.** *If $X$ is a Calabi-Yau threefold over $k = \mathbf{F}_q$ and $Y$ is derived equivalent to $X$, then $X$ and $Y$ have the same zeta function.*

*Proof.* The idea is to show that all traces of Frobenius are the same and then count points using the Lefschetz trace formula. Consider $F : X_{\overline{k}} \to X_{\overline{k}}^{(q^n)}$ the $\mathbf{F}_{q^n}$-linear Frobenius. The fixed points are exactly the set $X(\mathbf{F}_{q^n})$. For notation define

$$t_X^i := \mathrm{Tr}(F^* : H_{\acute{e}t}^i(X_{\overline{k}}, \mathbf{Q}_\ell))$$

and similarly for $Y$.

Since the only odd cohomology of a Calabi-Yau is in degree three, the Fourier-Mukai kernel induces a Frobenius equivariant isomorphism $H_{\acute{e}t}^3(X_{\overline{k}}, \mathbf{Q}_\ell) \xrightarrow{\sim} H_{\acute{e}t}^3(Y_{\overline{k}}, \mathbf{Q}_\ell)$. This immediately tells us that $t_X^3 = t_Y^3$.

The étale realization of the motive in even degree is more subtle. From Chapter 2, equation 2.5, we have a Frobenius equivariant isomorphism

$$\bigoplus_{n=0}^{3} H_{\acute{e}t}^{2n}(X_{\overline{k}}, \mathbf{Q}_\ell(n)) \xrightarrow{\sim} \bigoplus_{n=0}^{3} H_{\acute{e}t}^{2n}(Y_{\overline{k}}, \mathbf{Q}_\ell(n)).$$

First, the Weil conjectures tell us that $t_X^0 = 1$ and $t_X^6 = q^3$. Then by taking into account the twists, we get the relation

$$1 + \frac{t_X^2}{q} + \frac{t_X^4}{q^2} + 1 = 1 + \frac{t_Y^2}{q} + \frac{t_Y^4}{q^2} + 1$$

Now we use Poincaré duality to check that $t_X^4 = qt_X^2$ and $t_Y^4 = qt_Y^2$. This will complete the proof. Poincaré duality states that $H_{\acute{e}t}^4(X_{\overline{k}}, \mathbf{Q}_\ell) \simeq H_{\acute{e}t}^2(X_{\overline{k}}, \mathbf{Q}_\ell(3))^\vee$. Suppose $e_1, \cdots, e_r$ are the eigenvalues of Frobenius acting on the second (non-twisted, non-dualized) cohomology. Since the trace is the sum of the eigenvalues we get that

$$
\begin{aligned}
t_X^4 &= q^3 \left( \frac{1}{e_1} + \cdots + \frac{1}{e_r} \right) \\
&= q^3 \left( \frac{\overline{e_1}}{q^2} + \cdots + \frac{\overline{e_r}}{q^2} \right) = qt_X^2
\end{aligned}
$$

The second equality comes from the Weil conjectures which tells us that all the eigenvalues have complex absolute value $q$. The last equality comes from the fact that the eigenvalues come in conjugate pairs, so taking complex conjugates everywhere does not change the trace.

Thus we see that $t_X^2 = t_Y^2$ and $t_X^4 = t_Y^4$. Since all the traces are the same and the Lefschetz trace formula tells us

$$
|X(\mathbf{F}_{q^n})| = \sum_{i=0}^{6} (-1)^i t_X^i
$$

we have that $|X(\mathbf{F}_{q^n})| = |Y(\mathbf{F}_{q^n})|$. This proves the theorem, because the zeta function of a variety is defined to be

$$
Z(X, t) = \exp \left( \sum_{n>0} \frac{|X(\mathbf{F}_{q^n})| t^n}{n} \right).
$$

$\square$

It is interesting to note that even though this same argument works in dimensions 1, 2, and 3, there is not an obvious way to extend it to higher dimensions because there are too many traces. Poincaré duality will not reduce it down to a single trace.

### 4.4 Characteristic 0 Invariants

In this section we first show how the properties of the previous section spread out over integral models so that they are preserved at almost all reductions. We observe

several consequences of this. We then show that modularity is a derived invariant which shows that the Euler product expansion of the L-series must match at every prime.

*Spreading out the Kernel*

Let $X$ and $Y$ be Calabi-Yau threefolds over $\mathbf{Q}$. Choose models for each over $\mathbf{Z}$ and denote them $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. Suppose $X$ and $Y$ are derived equivalent. From this we get a Fourier-Mukai kernel giving the equivalence $\Phi_{\mathcal{P}} : D(X) \to D(Y)$. Note that the kernel $\mathcal{P} \in D(X \times Y)$. In particular, it is a complex on the generic fiber of the scheme $\mathfrak{X} \times \mathfrak{Y} \to \operatorname{Spec} \mathbf{Z}$. Since the map is flat this complex spreads out over an open set of the base. We denote this spread out complex $\mathfrak{P}$. Let $S$ be the finite set of primes for which the kernel does not spread out together with primes of bad reduction for $X$ and $Y$.

Given some prime $p \notin S$ we can look at the restriction of $\mathfrak{P}$ as a complex on the fiber over $p$ to get a complex $\mathcal{Q}_p \in D(X_p \times Y_p)$. Since the original complex gave an equivalence, this also gives an equivalence $\Phi_{\mathcal{Q}_p} : D(X_p) \to D(Y_p)$. Since $X_p$ and $Y_p$ are Calabi-Yau threefolds over a field of positive characteristic, we can now apply all the results of the previous section. This immediately shows us that for all but finitely many primes, the reductions of $X$ and $Y$ have the same height and local Zeta functions.

**Proposition 4.4.1.** *If $X$ and $Y$ are derived equivalent Calabi-Yau threefolds over a number field, then $X$ and $Y$ have the same Hodge numbers.*

*Proof.* One could prove this by modifying the arguments in [PS11] to work over a non-algebraically closed field which would tell us the Betti numbers are the same. Then we could use various dualities to extrapolate that all Hodge numbers are the same. Instead, we can use the spreading out technique above. A theorem of Ito [Ito03], via the use of $p$-adic Hodge theory, says that any two varieties over a number

field that have the same local zeta functions at all but finitely many primes must have the same Hodge numbers. □

*Modularity*

For the definition of the Galois action we do not need any special considerations except that our variety $X$ is defined over $\mathbf{Q}$. We continue to write $X_{\overline{\mathbf{Q}}}$ for the base change to $\overline{\mathbf{Q}}$. Consider the absolute Galois group $G := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Any element $\sigma \in G$ is an automorphism of $\overline{\mathbf{Q}}$, so it induces an automorphism $\mathrm{Spec}(\overline{\mathbf{Q}}) \to \mathrm{Spec}(\overline{\mathbf{Q}})$. Thus we can form a commuting diagram

$$
\begin{array}{ccc}
X_{\overline{\mathbf{Q}}} & \xrightarrow{\ \sigma\ } & X_{\overline{\mathbf{Q}}} \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\overline{\mathbf{Q}}) & \longrightarrow & \mathrm{Spec}(\overline{\mathbf{Q}}).
\end{array}
$$

Since it is an isomorphism it will induce an isomorphism on cohomology

$$
H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) \xrightarrow{\ \sigma^*\ } H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell).
$$

This works for any element of $G$. When we assume that $X$ is a rigid Calabi-Yau threefold, the action on cohomology induces a two-dimensional representation

$$
\rho'_X : G \to \mathrm{Aut}(H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)) \simeq GL_2(\mathbf{Q}_\ell).
$$

We will define the actual Galois representation $\rho_X$ on the dual space via the contragredient representation. One notable difference is that this will invert eigenvalues.

This is the convention used in [GY11], but since we will only care in the future about $\rho_X(Frob_p)$ often the convention is to not take the contragredient and instead use the geometric rather than arithmetic Frobenius.

We will first define the L-series via the Galois representation. Fix $p$ a prime not equal to $\ell$ and of good reduction for $X$. The representation $\rho_X$ is unramified at $p$, so we can lift a topological generator of $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ to get a conjugacy class $\mathrm{Frob}_p$ whose image under $\rho_X$ has well-defined trace and determinant.

We define

$$L(X, s) = L(H^3_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell), s)$$

$$= (*) \prod_{p \text{ good}} \frac{1}{1 - \text{tr}(\rho_X(\text{Frob}_p))p^{-s} + \det(\rho_X(\text{Frob}_p))p^{-2s}}$$

where $(*)$ is a product of terms at the bad primes.

The denominator is just a certain characteristic polynomial. It is formed by plugging in $p^{-s}$ to

$$P_p(t) = \det(1 - t\rho_X(\text{Frob}_p)|H^3_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)).$$

Since the representation is unramified at the good primes, we have no issues in forming such an object. At the primes of bad reduction we can still make this object by just considering the action on the part of the cohomology fixed by the inertia subgroup:

$$P_p(t) = \det(1 - t\rho_X(\text{Frob}_p)|H^3_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)^{I_p}).$$

Thus the full $L$-series can succinctly be written in this form as:

$$L(X, s) = \prod_{p \text{ prime}} \frac{1}{P_p(p^{-s})}.$$

The determinant term turns out to always be $p^3$ since it can be checked to be the third power of the $\ell$-adic cyclotomic character. This tells us the representation is odd.

The last piece we need to state the modularity conjecture is the L-function of a modular form. Let $f \in S_4(\Gamma_0(N))$ be a weight 4 cuspidal Hecke eigenform with $q$-expansion $f = \sum_{n=1}^{\infty} a_n q^n$. The L-series associated to the Mellin transform is given by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and this has an Euler product expansion given by

$$L(f, s) = \prod_{p|N}(*) \prod_{p \nmid N} \frac{1}{1 - a_p p^{-s} + p^{3-2s}}.$$

We say that a rigid Calabi-Yau threefold is *modular* if $L(X, s)$ coincides with $L(f, s)$ up to finitely many primes for some $f \in S_4(\Gamma_0(N))$.

**Theorem 4.4.2.** *Every rigid Calabi-Yau threefold defined over* $\mathbf{Q}$ *is modular.*

The proof given in [GY11] is essentially a corollary to Serre's Conjecture. The full modularity conjecture predicts that all Calabi-Yau threefolds should be modular. Our current definition of modular does not make sense in the non-rigid case. All proposed definitions of what the non-rigid definition should be are completely determined by the Galois representation. For example, one could ask for the representation to split into a two-dimensional part and some other part for which the two-dimensional part is modular.

**Proposition 4.4.3.** *If* $X$ *and* $Y$ *are derived equivalent Calabi-Yau threefolds over* $\mathbf{Q}$, *then the induced isomorphism that comes from the odd $\ell$-adic realization of the Mukai motive* $H^3_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) \xrightarrow{\sim} H^3_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$ *is Galois equivariant.*

*Proof.* First note that the statement that we have an isomorphism on the odd $\ell$-adic realization (see Chapter 2, equation 2.4) already contains the content of the proposition, but we give more details here to show exactly how this isomorphism works. Since our derived equivalence is defined over $\mathbf{Q}$, the Fourier-Mukai kernel giving an equivalence $D(X_{\overline{\mathbf{Q}}}) \to D(Y_{\overline{\mathbf{Q}}})$, say $\mathcal{F}^\bullet$, is pulled back from one given on $D(X \times Y)$, say $\mathcal{E}^\bullet$.

Let $f : X_{\overline{\mathbf{Q}}} \times Y_{\overline{\mathbf{Q}}} \to X \times Y$ be the pullback morphism. Since the construction of characteristic classes commute with smooth pullback, this tells us that we have $v(\mathcal{F}^\bullet) = v(f^*\mathcal{E}^\bullet) = f^*(v(\mathcal{E}^\bullet))$. But now we use the fact that the image of $f^*$ on cohomology is contained in $H^*_{\text{ét}}(X_{\overline{\mathbf{Q}}} \times Y_{\overline{\mathbf{Q}}})^{\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})}$. This means that the Mukai vector

$$v(\mathcal{E}^\bullet) = \alpha \in \bigoplus_{i=0}^{d} H^{2i}_{\text{ét}}(X_{\overline{k}}, \mathbf{Q}_\ell)(i)$$

giving the isomorphism on cohomology is Galois equivariant.

Recall that if $p$ and $q$ are the two projections, the map on cohomology that we are referring to is $\Phi_\alpha^H(\beta) = p_*(\alpha.q^*\beta)$. This produces an isomorphism $H^3_{\acute{e}t}(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) \to H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$. We cup with a class that lies only in even degrees, and hence in order to actually be an isomorphism we only need to check what happens in the case that we cup with something in degree 6. Thus we may suppose $\alpha \in H^6_{\acute{e}t}(X_{\overline{\mathbf{Q}}} \times Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell(3))$.

Define $\Lambda = \mathbf{Z}/\ell^n$. The map $\Phi_\alpha^H$ is a composition of three maps. First

$$q^* : H^3_{\acute{e}t}(Y_{\overline{\mathbf{Q}}}, \Lambda) \to H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}} \times Y_{\overline{\mathbf{Q}}}, \Lambda),$$

then the cup product with the class $\alpha$ introduces a twist

$$H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}} \times Y_{\overline{\mathbf{Q}}}, \Lambda) \to H^9_{\acute{e}t}(X_{\overline{\mathbf{Q}}} \times Y_{\overline{\mathbf{Q}}}, \Lambda(3)).$$

Finally the pushforward twists back to give

$$p_* : H^9_{\acute{e}t}(X_{\overline{\mathbf{Q}}} \times Y_{\overline{\mathbf{Q}}}, \Lambda(3)) \to H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}}, \Lambda).$$

Note first that $p_*$ and $q^*$ are Galois equivariant maps (the action on the product is the diagonal one). The only thing needed to make sure the vector space isomorphism $H^3_{\acute{e}t}(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) \to H^3_{\acute{e}t}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$ is Galois equivariant is to make sure the class $\alpha$ is Galois invariant. This was the content of the first paragraph, so we have proved the proposition. $\qquad\square$

**Corollary 4.4.4.** *Modularity is a derived invariant for Calabi-Yau threefolds.*

*Proof.* Since modularity can be defined entirely in terms of the Galois action on the third $\ell$-adic cohomology (even in the non-rigid case), we see that modularity is a derived invariant and moreover the modular forms attached to derived equivalent Calabi-Yau threefolds are exactly the same.

A simpler proof of this fact is that modularity, as was defined for rigid Calabi-Yau threefolds, is defined in terms of all but finitely many factors in the $L$-series. The $L$-series involves characteristic polynomials of Frobenius. Thus by spreading out and

reducing at good primes we see that derived equivalent Calabi-Yau threefolds have the same factors in the $L$-series up to finitely many terms. □

**Corollary 4.4.5.** *If $X$ and $Y$ are birational modular Calabi-Yau threefolds over $\mathbf{Q}$, then $X$ and $Y$ have the same modular form attached up to a twist.*

*Proof.* Bridgeland proved that Calabi-Yau threefolds related by a flop are derived equivalent (see e.g. [Bri02]). Since Calabi-Yau threefolds are minimal, any two birational ones are related by a sequence of flops after base change to $\overline{\mathbf{Q}}$. Thus they are derived equivalent. Since everything was originally defined over $\mathbf{Q}$, $X$ and $Y$ are derived equivalent after a finite extension to $K$ (take the field of definition of all maps involved). So by Corollary 4.4.4 they have the same modular form attached up to a possible twist coming from $K/\mathbf{Q}$. □

Note the above was already known by different means. Namely, one can take the closure of the graph of the birational morphism. This defines an algebraic class in the cohomology of the product. The associated correspondence will induce an isomorphism of the Galois representations. We have seen several examples now where the arithmetic theory of the Fourier-Mukai transform produces a uniform approach to proving many seemingly unrelated results. This is one of the main themes of this chapter.

**Corollary 4.4.6.** *If $X$ and $Y$ are derived equivalent rigid Calabi-Yau threefolds over $\mathbf{Q}$, then every term of Euler product expansion of the L-series is the same.*

*Proof.* This follows because the L-series is defined entirely from the Galois representation which was shown to be isomorphic. □

The main reason for pointing out this corollary is that it is still an open problem to determine how the conductor of the modular form relates to the primes of bad reduction. The reduction type ought to have something to do with the form the

Euler factor takes. It is also interesting that the spreading out technique requires us to throw out finitely many primes, but here we get a result at all primes.

**Proposition 4.4.7.** *If $X$ is a rigid Calabi-Yau threefold, then its completed L-series is the same as the L-series of the attached modular form for all Euler factors.*

*Proof.* First note that modularity tells us that $L(X, s)$ extends to an entire function with the property that $L(X, s) = \omega L(X, 4-s)$ where $\omega = \pm 1$. Since the two functions agree at all but finitely many factors we can multiply and divide by the appropriate factors to get that $L(f, s) = R(s)L(X, s)$ where without loss of generality we assume it has the form

$$R(s) = \frac{\prod(1 - \alpha_i p_i^{-s})}{\prod(1 - \beta_j q_j^{-s})} = \frac{L(f, s)}{L(X, s)}.$$

A similar argument will work if there are quadratic terms.

Thus $R(s) = \varepsilon R(4-s)$ where $\varepsilon = \pm 1$. Now we induct on the number of factors in the numerator and denominator to prove that $R(s) \equiv 1$. This reduces us to checking the case of a single factor in the numerator and denominator. Suppose

$$\frac{1 - \alpha p^{-s}}{1 - \beta p^{-s}} = \frac{1 - \alpha p^{s-4}}{1 - \beta p^{s-4}}.$$

Rearranging terms gives us $p^{s-4} - p^{-s} \equiv 0$ which is a contradiction. Thus $R(s)$ must be constant. If the constant is anything other than 1, then $L(X, s)$ would not consist of Euler factors of the right form. Thus $L(X, s) = L(f, s)$. □

Since it is an open problem to relate the primes of bad reduction to the conductor of the L-series we would like to be able to say the primes of bad reduction are preserved under derived equivalence. On the one hand, we can say this for all primes that the kernel spreads out over, but this may give us no new information. For example, it may be the case that the places of bad reduction are completely contained within the set of primes we throw out. The corresponding result is true for abelian varieties.

**Proposition 4.4.8.** *If $A$ and $B$ are derived equivalent abelian varieties over $k$, then the places $\nu$ of good reduction are the same.*

*Proof.* Suppose $D(A) \simeq D(B)$ over $k$. Then [Orl02] shows that there is a $k$-isogeny $f : A \to B$. Let $\nu$ be a place of $k$ and denote by $k_\nu$ the completion at $\nu$. Since Fourier-Mukai transforms are compatible with base change we get that $D(A_{k_\nu}) \simeq D(B_{k_\nu})$. Choose $\ell$ a prime that does not divide the degree of the isogeny or the characteristic of the residue field of $\mathcal{O}_{k_\nu}$. The isogeny induces an isomorphism $T_\ell(A) \simeq T_\ell(B)$ as Galois modules and hence the inertia $I_\nu$ acts trivially on $T_\ell(A)$ if and only if it acts trivially on $T_\ell(B)$. But the criterion of Néron-Ogg-Shafarevich tells us that $A$ and $B$ have good reduction if and only if this happens. $\square$

**Proposition 4.4.9.** *Modularity is a derived invariant for K3 surfaces.*

It is interesting to note that there is something a little subtle about this proposition. A K3 surface has the property that

$$L(X, s) = L(NS(X), s)L(T, s),$$

but note that examples of Oguiso (Corollary 1.8 [Ogu02]) show that $NS(X)$ is in general not preserved under derived equivalence. All that is going on is that the derived equivalence does not preserve lattice structure in general. It also won't preserve the individual Galois representation on $\ell$-adic cohomology, because the equivalence packages the cohomology together with some twisting. Fortunately, in these low dimensional examples the Weil conjectures let us untwist enough to recover the representation.

*Proof.* Let $k = \mathbf{Q}$. Suppose $X$ and $Y$ are derived equivalent K3 surfaces over $k$. The $\ell$-adic realization of the Mukai motive shows that

$$H^0(X_{\overline{k}}, \mathbf{Q}_\ell(-1)) \oplus H^2(X_{\overline{k}}, \mathbf{Q}_\ell) \oplus H^4(X_{\overline{k}}, \mathbf{Q}_\ell(1))$$

and

$$H^0(Y_{\overline{k}}, \mathbf{Q}_\ell(-1)) \oplus H^2(Y_{\overline{k}}, \mathbf{Q}_\ell) \oplus H^4(Y_{\overline{k}}, \mathbf{Q}_\ell(1))$$

are isomorphic as Galois modules. Since the outer pieces are appropriately twisted characters we see that the Galois representations on the middle $\ell$-adic cohomology are the same. □

*Examples and Counterexamples*

We will conclude this section with an application. Consider the rigid Calabi-Yau threefold defined over $\mathbf{Q}$ as a double cover of $\mathbf{P}^3$ branched over an octic, say $y^2 = f_8(x_1, x_2, x_3, x_4)$. By Theorem 4.4.2 we know this is modular. Given any square-free $D$ we can form a quadratic twist given by the equation $Dy^2 = f_8(x_1, x_2, x_3, x_4)$. This again is rigid and hence modular. These two threefolds become isomorphic upon base change to $\mathbf{Q}(\sqrt{D})$, but are not isomorphic over $\mathbf{Q}$.

The new observation is that these cannot be derived equivalent over $\mathbf{Q}$ either. Suppose the original threefold has the modular form $f \in S_4(\Gamma_0(N))$ attached. In [GKY13] it is checked that the twist has a modular form of a different level attached. If they were derived equivalent over $\mathbf{Q}$, then by Proposition 4.4.3 they would have the same modular form attached. This tells us that we can produce many distinct derived equivalence classes over $\mathbf{Q}$ by twisting.

One might hope that the other direction holds. Elliptic curves have the nice property that if they have the same modular form attached, then they must be isogenous. In analogy, it might be the case that Calabi-Yau threefolds with the same modular form attached must be derived equivalent. This statement is false, since [Mey05] contains examples with different Hodge numbers, but the same modular form. It is a conjecture of the author that a qualified statement might be true: If $X$ and $Y$ are rigid Calabi-Yau threefolds over $\mathbf{Q}$ that are quadratic twists of each other, and they have the same modular form attached, then $X$ and $Y$ must be derived equivalent.

Note that one cannot hope for much more than this conjectural relationship (in the specific instance of quadratic twists of rigid examples), because in [Sch11] it is shown that there are two double covers branched over an octic that are rigid (arrangements

240 and 245 in the paper's notation) that give the same modular form. But the Hodge numbers $h^{1,1}$ are calculated to be 40 and 38 respectively. Thus they cannot be derived equivalent.

*Invariants not Preserved by Derived Equivalence*

It is well-known that the rational Hodge structures $H^1(X, \mathbf{Q})$ are preserved under derived equivalence of Calabi-Yau threefolds over $\mathbf{C}$. This seems to suggest that $\pi_1(X)$ should be preserved since $H^1(X, \mathbf{Q}) = Hom(\pi_1(X), \mathbf{Q})$.

Christian Schnell [Sch12] gave an example to show that this is not the case. The example is given quite explicitly, and the reason behind the counterexample is simple. The original threefold is simply connected and has a nice action of $G = \mathbf{Z}/8 \times \mathbf{Z}/8$ on it. One can check that $D(X) \simeq D(X/G)$, but clearly $\pi_1(X/G) \simeq \mathbf{Z}/8 \times \mathbf{Z}/8$. This shows that the étale fundamental groups are also different, because the étale fundamental group of a variety over $\mathbf{C}$ is just the profinite completion of the topological fundamental group.

This same example was analyzed by Nicholas Addington [Add13] to show that the Brauer group is also not a derived invariant.

### 4.5 Serre-Tate Theory for Ordinary Calabi-Yau Threefolds

*Introduction*

Suppose $X_0/k$ is an ordinary Calabi-Yau threefold over a perfect field of characteristic $p > 3$. Ordinary is taken in the Bloch-Kato sense, so all $H^j(X, B^i) = 0$. Suppose that $X_0$ lifts to characteristic 0. By Schröer's crystalline $T^1$-lifting theorem (Theorem 4.1 in [Sch03]) we know that the universal formal deformation $\mathcal{X}/S$ over $W(k)$ is formally smooth. We assume some technical conditions (which may be possible to remove by showing that a liftable, ordinary Calabi-Yau threefold always satisfies them).

First, we assume that the crystalline cohomology is torsion-free. This is used in

three places: for Hodge symmetry, formal smoothness of the universal deformation, and that the Hodge and Newton polygons coincide. The reader uncomfortable with that assumption can alternately assume these stronger conditions, but as stated the condition should be considered mild. For example, if $X$ is a projective variety over a number field, then for all but finitely many primes the reduction will have torsion-free crystalline cohomology.

For the rest of this subsection, we will sketch the idea behind the proof of the main theorem of this section. By [Jos04], we know that $X_0$ satisfies Hodge symmetry. Let $d = \frac{1}{2}(b_3(X_0) - 2)$. We know that $S \simeq \operatorname{Spf} W(k)[[t_1, \dots, t_d]]$, because of formal smoothness and the fact that $d = h^{1,2}$ which classifies deformations. Recall (Corollary II.2.12 in [AM77]) that the functor $\Phi : \operatorname{Art}_k \to \operatorname{Grp}$ given by

$$\Phi(A) = \ker(H^3_{\acute{e}t}(X_0 \otimes A, \mathbf{G}_m) \to H^3_{\acute{e}t}(X_0, \mathbf{G}_m))$$

is prorepresentable by a formal group which we again call $\Phi$. Since $X_0$ satisfies the vanishing cohomology condition, by Theorem 2.1.2 it is a $p$-divisible group of height 1. We will denote the enlarged formal group by $\Psi$ (details on its existence and properties are given in the next section). There is a natural map

$$\gamma : \operatorname{Def}_{X_0} \to \operatorname{Def}_\Psi$$

given by $X/A \mapsto \Psi_{X/A}$, the enlarged formal group of the lifted variety.

Since $\Psi$ is $p$-divisible its universal deformation space is also formally smooth. By standard theory and the fact that $\dim(\Psi) = 1$ we get that the dimension of this space is $\dim(\Psi^D)$.

**Lemma 4.5.1.** *The height of $\Psi$ is $b_3(X_0)/2$.*

*Proof.* Let $K = Frac(W(k))$. The theory of Artin-Mazur tells us that this enlarged formal group has height $\dim_K H^3_{crys}(X_0/W) \otimes K_{[0,1]}$ (we will actually prove this in Proposition 4.5.8). By [BK86] Proposition 7.3, the Newton polygon equals the Hodge

polygon. The Hodge polygon is easily seen to have exactly half the slopes less than or equal to 1. Thus

$$\dim_K H^3_{crys}(X_0/W) \otimes K_{[0,1]} = b_3(X_0)/2$$

which is what we wanted to show. □

The theory of $p$-divisible groups tells us that $\text{ht}(\Psi) = \dim(\Psi) + \dim(\Psi^D)$. Thus we see that $\dim(\Psi^D) = b_3(X_0)/2 - 1 = \frac{1}{2}(b_3(X_0) - 2)$. Thus $\gamma$ is a morphism between smooth formal schemes of the same dimension. The main theorem of this section is

**Theorem 4.5.13.** *The map $\gamma$ is an isomorphism.*

This will give a Serre-Tate theory for ordinary Calabi-Yau threefolds. The proof is organized as follows. In Section 4.5.1 we will prove that the enlarged formal group exists and has a known connected-étale decomposition involving the Artin-Mazur formal group and a certain flat cohomology. In Section 4.5.2 we will prove the main result that there is a natural isomorphism between the universal deformation spaces of the variety and its enlarged formal group.

*4.5.1   The Enlarged Formal Group*

In this section we review the theory from [AM77] for the existence of an enlarged formal group with the correct properties for proving the natural isomorphism between the deformation spaces. We go back to the hypotheses made on the Calabi-Yau threefold $X/k$ at the start of the section that $X$ is ordinary, liftable, and has torsion-free crystalline cohomology. The subscript is intentionally dropped for ease of notation for this section only, because deformations will not be considered.

**Proposition 4.5.2.** *A liftable, ordinary Calabi-Yau threefold has no p-power torsion in the Brauer group.*

*Proof.* This follows immediately from the truncated long exact sequence associated to the $p$-Kummer sequence. Consider

$$0 \to \mathrm{Pic}(X)/p\,\mathrm{Pic}(X) \to H^2_{fl}(X, \boldsymbol{\mu}_p) \to Br(X)[p] \to 0.$$

Since $X$ satisfies Kato's criterion we get that

$$H^2_{fl}(X, \boldsymbol{\mu}_p) \simeq \mathbf{Z}/p \oplus \cdots \oplus \mathbf{Z}/p$$

where there are $h^{1,1}$ copies (a more complete proof of this will be given during the proof of Theorem 4.5.13 after the appropriate tools have been developed). By [KvdG03] Proposition 4.3 there is no $p$-power torsion in $\mathrm{Pic}(X)$. Thus if $\mathrm{rk}\,\mathrm{Pic}(X) = h^{1,1}$ the natural injection will also be surjective implying that $Br(X)[p] = 0$.

We may check this on the lifted variety, because if $\mathfrak{X}/W(k)$ is a lift to characteristic 0 and $X_n = \mathfrak{X} \otimes W_n(k)$, then $\mathrm{Pic}(X_n) = \mathrm{Pic}(X)$ for all $n$ by our Calabi-Yau conditions $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Let $X_\eta = \mathfrak{X} \otimes K$ be the generic fiber. Smooth proper base change for $\ell$-adic cohomology tells us that $b_2(X) = b_2(X_\eta)$.

Since the crystalline cohomology is torsion-free we get the comparison $\dim_k H^2_{dR}(X/k) = b_2(X)$. Degeneration of the Hodge-de Rham spectral sequence plus the vanishing of $H^2(X, \mathcal{O}_X)$ shows us that $h^{1,1}(X) = h^{1,1}(X_\eta)$.

Thus it now suffices to check that $\mathrm{rk}\,\mathrm{Pic}(X_\eta) = h^{1,1}$. This follows from the Lefschetz principle and Hodge theory. This is equivalent to checking that the first Chern class map is surjective. Consider

$$0 \to 2\pi i \mathbf{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1.$$

The map under consideration is the connecting homomorphism

$$0 \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbf{Z}(1)) \to H^2(X, \mathcal{O}_X) = 0.$$

Thus we have surjectivity. This completes the proof that there is no $p$-torsion. This implies the result by induction, because if $\alpha \in Br(X)[p^n]$, then $p\alpha \in Br(X)[p^{n-1}]$. $\square$

This has two interesting consequences:

**Corollary 4.5.3.** *If $X/k$ is an ordinary Calabi-Yau threefold over a field of characteristic $p > 0$ and $Br(X)[p^n] \neq 0$ for some $n$, then $X$ does not lift to characteristic $0$.*

The proof is immediate and gives a nice, possibly computable obstruction to lifting a Calabi-Yau threefold. Just changing points of view we get another consequence:

**Corollary 4.5.4.** *If $X/K$ is a Calabi-Yau threefold over a field of characteristic $0$ and $Br(X)[p^n] \neq 0$ for some $n$, then the reduction of $X$ mod $p$ will not be ordinary.*

One should keep in mind that here we are referring to the honest Brauer group of $X$. Just as in the classical Serre-Tate theory for elliptic curves, we will eventually need to consider a certain related $p$-divisible group. In the case of a supersingular elliptic curve $E[p^n] = 0$ for all $n$ as well. Even though the corresponding group scheme is geometrically a single point, it carries non-trivial infinitesimal information. The way to capture this information is to work with the Picard scheme rather than the Picard group and consider the scheme-theoretic kernel of multiplication-by-$p$.

We now turn to representability issues. To give an analogy for the enlarged formal group we remind the reader that by using Schlessinger's criterion [Sch68], Artin-Mazur obtain the following simple sufficient conditions to check that $\Phi$ is pro-representable by a formal Lie group (written in the specific case of a threefold):

**Proposition 4.5.5.** *If $H^4(X, \mathcal{O}_X) = 0$ and $\widehat{Br}_X$ is formally smooth, then $\Phi$ is pro-representable by a formal Lie group.*

This is [AM77] Corollary II.2.12. Note that since $X$ is a threefold, it is a standard fact that $H^4(X, \mathcal{O}_X) = 0$. Since $H^2(X, \mathcal{O}_X) = 0$, the Brauer group is discrete and hence the formal completion at the identity is trivial. Thus it is formally smooth. This shows that the Artin-Mazur formal group of our Calabi-Yau threefold is pro-representable by a formal Lie group.

In order to show that that enlarged formal group is a smooth $p$-divisible group whose connected component is $\Phi$, we will need to check similar conditions on a certain $R^4$ for formal smoothness and $R^2$ for being pro-representable by a $p$-divisible group.

First, we need to describe the functor defining the enlarged group. Consider the complex $\mathbf{G}_m \xrightarrow{p^n} \mathbf{G}_m$ which we denote $\mathbf{G}_m[p^n]$. Since this is a complex of smooth group schemes quasi-isomorphic to $\boldsymbol{\mu}_{p^n}$ in the fppf topology we get an isomorphism:

$$\mathbf{H}^q_{\acute{e}t}(X, \mathbf{G}_m[p^n]) \xrightarrow{\sim} H^q_{fl}(X, \boldsymbol{\mu}_{p^n}).$$

Suppose we have a lift to characteristic 0 given by a proper map $f : X \to \mathrm{Spec}(R)$. We get that $R^q_{\acute{e}t} f_* (\mathbf{G}_m[p^n])$ is the étale sheaf associated to the presheaf defined by $S' \mapsto H^q_{fl}(X_{S'}, \boldsymbol{\mu}_{p^n})$.

**Definition 4.5.6.** *The* enlarged formal group, $\Psi$, *is defined to be the subsheaf of* $\mathcal{R} := \varinjlim R^3_{\acute{e}t} f_* \mathbf{G}_m[p^n]$ *which to any Artinian local $R$-algebra $A$ with residue field $k$ associates the subgroup $\Psi(A) \subset \mathcal{R}(A)$ of sections which map to the divisible part of $H^3_{fl}(X, \boldsymbol{\mu}_{p^\infty})$.*

**Lemma 4.5.7.** *The functor $R^2_{\acute{e}t} f_* \mathbf{G}_m[p^\infty]$ is formally smooth.*

*Proof.* We use [AM77] IV Proposition 1.5. Since the Picard group discrete, the proposition implies that $R^2_{\acute{e}t} f_* \mathbf{G}_m[p^\infty]$ is representable by a formal group. Since the Brauer group is discrete, we have $\widehat{Br_X} = R^2_{\acute{e}t} f_* \mathbf{G}_m[p^\infty]^0 = 0$. This shows that this formal group is étale and hence formally smooth. $\qquad\square$

**Proposition 4.5.8.** *If $X$ is ordinary, then $\Psi$ is the restriction to the category of Artinian local $R$-algebras of the functor associated to a $p$-divisible group over $R$.*

*Proof.* We first show that $\Psi$ is formally smooth. We define the complex:

$$\mathbf{G}_m[p^\infty] := \varinjlim \mathbf{G}_m[p^n] = \left( \mathbf{G}_m \to \mathbf{G}_m \otimes \mathbf{Z}\left[\frac{1}{p}\right] \right).$$

Consider $X_1 \subset X_1'$, an infinitesimal extension of $X$-schemes by a square-zero ideal $J$. This gives us an exact sequence (using the fact that $\mathrm{Lie}(\mathbf{G}_m) = \mathbf{G}_a$):

$$0 \to \mathbf{G}_a \otimes J \to \mathbf{G}_m|_{X_1'} \to \mathbf{G}_m|_{X_1} \to 0.$$

Since $\mathbf{G}_a \otimes J$ is $p$-torsion we get by tensoring by $\mathbf{Z}[1/p]$ an isomorphism

$$\mathbf{G}_m \otimes \mathbf{Z}[1/p]\Big|_{X_1'} \overset{\sim}{\to} \mathbf{G}_m \otimes \mathbf{Z}[1/p]\Big|_{X_1}.$$

This gives us an exact triangle of complexes

$$\mathbf{G}_a \otimes J \to \mathbf{G}_m[p^\infty]\Big|_{X_1'} \to \mathbf{G}_m[p^\infty]\Big|_{X_1}.$$

All sheaves and derived functors will be on the étale site unless there is a subscript indicating otherwise. Now given any proper flat $f : X \to S$ we can consider a square-zero infinitesimal extension $Z \subset Z'$ of $S$-schemes with ideal $I$ and apply the total derived functor $\mathbf{R}f_*$ to the above triangle to get a long exact sequence

$$\cdots \to R^3 f_* \mathbf{G}_a \otimes I\Big|_Z \to R^3 f_* \mathbf{G}_m[p^\infty]\Big|_{Z'} \to R^3 f_* \mathbf{G}_m[p^\infty]\Big|_Z \to R^4 f_* \mathbf{G}_a \otimes I\Big|_Z \to \cdots$$

The $R^4 f_*$ term vanishes since $\mathbf{G}_a \otimes I$ is coherent and $X$ is a threefold. Thus we have a surjection on the right which proves the formal smoothness of $\mathcal{R}$. It follows that the subfunctor $\Psi$ is formally smooth, because by definition $\Psi(A)$ is the subgroup of $\mathcal{R}(A)$ that maps to $\mathrm{Div}H^3_{fl}(X, \boldsymbol{\mu}_{p^\infty})$. Given a section $\xi \in \Psi(Z)$ it is an element of $\mathcal{R}(Z)$ which lifts to an element $\xi' \in \mathcal{R}(Z')$. But by compatibility of restriction $\xi'$ must be in $\Psi(Z')$.

By definition we know that we have an exact sequence

$$0 \to \Phi \to \Psi \to \mathrm{Div}H^3_{fl}(X, \boldsymbol{\mu}_{p^\infty}) \to 0.$$

We have that $R^2 f_* \mathbf{G}_m$ is formally smooth by Lemma 4.5.7. This gives us that $R^3 f_* \mathbf{G}_m[p^\infty]$ is a torsor under $\Phi = R^3 f_* \mathbf{G}_m$. That is, the connected-étale sequence for the enlarged formal group is exactly:

$$0 \to \Phi \to \Psi \to \mathrm{Div}H^3_{fl}(X, \boldsymbol{\mu}_{p^\infty}) \to 0.$$

Note that $\mathrm{Div}H^3_{fl}(X, \boldsymbol{\mu}_{p^\infty})$ is $p$-divisible (in fact, in the ordinary case without using anything about the existence of $\Psi$ we know by Proposition 4.5.11 that $H^3_{fl}(X, \boldsymbol{\mu}_{p^\infty})$ is $p$-divisible), and by assumption $\Phi$ is $p$-divisible of height 1. This tells us that $\Psi$ is $p$-divisible of height $1 + \mathrm{corank}_{\mathbf{Z}_p} H^3_{fl}(X, \boldsymbol{\mu}_{p^\infty})$. By [Ber75] 4.5,

$$\mathrm{corank}_{\mathbf{Z}_p} H^3_{fl}(X, \boldsymbol{\mu}_{p^\infty}) = \dim H^3_{crys}(X/K)_{[1,1]}.$$

Thus $\mathrm{ht}(\Psi) = \dim H^3_{crys}(X/K)_{[0,1]}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In fact, we now see that the complicated definition of $\Psi$ is equivalent to the functor

$$\Psi : \mathrm{Art}_R \to \mathrm{Grp}$$

given by

$$\Psi(A) = H^3_{fl}(X \otimes A, \boldsymbol{\mu}_{p^\infty})$$

(this will be proved as Corollary 4.5.12). It is a 1-dimensional $p$-divisible group of height $b_3(X)/2$ and satisfies $\Psi^0 = \Phi$ and $\Psi^{et} = H^3_{fl}(X_{\overline{k}}, \boldsymbol{\mu}_{p^\infty})$.

### 4.5.2   Serre-Tate Theory

In this section we establish that the natural map of formal schemes from Theorem 4.5.13 is an isomorphism. In the last section we dropped the subscript on $X_0/k$, because we were proving general facts about formal groups. In this section it will be important to keep track of the special fiber, so we switch back to this notation.

**Lemma 4.5.9.** *There is an isomorphism $H^i_{fl}(X_0, \boldsymbol{\mu}_p) \xrightarrow{\sim} H^{i-1}_{\acute{e}t}(X_0, \mathbf{G}_m/\mathbf{G}_m^p)$ for any $i$.*

*Proof.* First note that on the fppf site there is a quasi-isomorphism of complexes $\boldsymbol{\mu}_p \to [\mathbf{G}_m \xrightarrow{p} \mathbf{G}_m]$. This is just a restatement of the standard $p$-Kummer sequence.

Thus we have an isomorphism of hypercohomology:

$$H^i_{fl}(X_0, \boldsymbol{\mu}_p) \simeq \mathbf{H}^i_{fl}(X_0, [\mathbf{G}_m \xrightarrow{p} \mathbf{G}_m]).$$

Since $\mathbf{G}_m$ is a smooth group scheme, a theorem of Grothendieck (appendix of [Gro68]) lets us compute the hypercohomology on the étale site. We do this via the hypercohomology spectral sequence:

$$E_2^{i,j} = H_{\acute{e}t}^j(X_0, \mathcal{H}^i([\mathbf{G}_m \xrightarrow{p} \mathbf{G}_m])) \Rightarrow \mathbf{H}_{\acute{e}t}^{i+j}(X_0, [\mathbf{G}_m \xrightarrow{p} \mathbf{G}_m])$$

But the complex only has cohomology at $i = 1$ which is isomorphic to $\mathbf{G}_m/\mathbf{G}_m^p$. This means the spectral sequence collapses at $E_2$ and hence $\mathbf{H}_{\acute{e}t}^i(X_0, [\mathbf{G}_m \xrightarrow{p} \mathbf{G}_m]) \simeq H_{\acute{e}t}^{i-1}(X_0, \mathbf{G}_m/\mathbf{G}_m^p)$. This proves the lemma. $\qquad\square$

**Lemma 4.5.10.** *If $X_0$ is ordinary, then $H_{fl}^4(X_0, \mathbf{Z}_p(1)) = 0$.*

*Proof.* We have an exact sequence in the flat topology

$$0 \to \boldsymbol{\mu}_p \to \boldsymbol{\mu}_{p^n} \xrightarrow{p} \boldsymbol{\mu}_{p^{n-1}} \to 0.$$

Taking the long exact sequence on cohomology we get:

$$\to H_{fl}^4(X_0, \boldsymbol{\mu}_p) \to H_{fl}^4(X_0, \boldsymbol{\mu}_{p^n}) \to H_{fl}^4(X_0, \boldsymbol{\mu}_{p^{n-1}}) \to .$$

We will check that $H_{fl}^4(X_0, \boldsymbol{\mu}_p) = 0$ so that we get injections

$$H_{fl}^4(X_0, \boldsymbol{\mu}_{p^n}) \hookrightarrow H_{fl}^4(X_0, \boldsymbol{\mu}_{p^{n-1}})$$

for all $n \geq 1$. Induction will imply that

$$H_{fl}^4(X_0, \mathbf{Z}_p(1)) := \varprojlim H_{fl}^4(X_0, \boldsymbol{\mu}_{p^n}) = 0$$

which will prove the lemma.

To check this we use the relation from Lemma 4.5.9 which says that $H_{fl}^4(X_0, \boldsymbol{\mu}_p) = H_{\acute{e}t}^3(X_0, \mathbf{G}_m/\mathbf{G}_m^p)$.

To prove the lemma we need only show that $H_{\acute{e}t}^3(X_0, \mathbf{G}_m/\mathbf{G}_m^p) = 0$. Consider the exact sequence of étale sheaves on $X_0$ (see equation I (2.1.23) in [Ill79]):

$$0 \to \mathbf{G}_m/\mathbf{G}_m^p \xrightarrow{d\log} Z^1 \xrightarrow{i-C} \Omega^1 \to 0. \tag{4.1}$$

We examine the long exact sequence on cohomology:

$$H^2(X_0, Z^1) \xrightarrow{i-C} H^2(X_0, \Omega^1) \to H^3(X_0, \mathbf{G}_m/\mathbf{G}_m^p) \xrightarrow{d\log} H^3(X_0, Z^1)$$

To prove the lemma it suffices to check that $H^3(X_0, Z^1) = 0$ and the first map $i - C$ is surjective. By definition we have

$$0 \to Z^1 \xrightarrow{i} \Omega^1 \to d\Omega^1 \to 0. \tag{4.2}$$

Thus we get an exact sequence

$$\to H^2(X_0, d\Omega^1) \to H^3(X_0, Z^1) \to H^3(X_0, \Omega^1) \to$$

By Hodge symmetry and the Calabi-Yau property

$$H^3(X_0, \Omega^1) \simeq H^1(X_0, \Omega^3) \simeq H^1(X_0, \mathcal{O}_{X_0}) = 0.$$

Since $X_0$ is ordinary, the defintion includes $H^2(X_0, d\Omega^1) = 0$. Thus $H^3(X_0, Z^1) = 0$. Lastly, we check that $i - C$ is surjective.

Since $X_0$ is ordinary, the induced map $i$ on cohomology acts as the identity. The map $C$ is $p^{-1}$-linear. More generally, for any $p^{-1}$-linear endomorphism $\lambda$ on a $k$-vector space $V$, it is well-known that $1 - \lambda$ is surjective. For a proof see Corollary 1.1.4. $\square$

This lemma is used to prove the next proposition, but the utility of the next proposition is that it shows that $H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty})$ is $p$-divisible. Without any explicit knowledge of this group we can already conclude this as follows. In the proof of the lemma we saw that $H^4_{fl}(X_0, \boldsymbol{\mu}_{p^r}) = 0$ for all $r$. We consider the part of the long exact sequence immediately preceding the above part

$$\to H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}) \xrightarrow{p^r} H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}) \to H^4_{fl}(X_0, \boldsymbol{\mu}_{p^r}) \to$$

to get the desired $p$-divisibility.

**Proposition 4.5.11.** *If $X_0$ is ordinary, then there is a canonical isomorphism $H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}) \xrightarrow{\sim} H^3_{fl}(X_0, \mathbf{Z}_p(1)) \otimes \mathbf{Q}_p/\mathbf{Z}_p$*

*Proof.* We are taking as a definition

$$H^n_{fl}(X_0, \mathbf{Z}_p(1)) := \varprojlim H^n_{fl}(X_0, \boldsymbol{\mu}_{p^n}).$$

This definition together with Lemma 4.5.10 gives us the exact sequence

$$\to H^3_{fl}(X_0, \mathbf{Z}_p(1)) \xrightarrow{p^n} H^3_{fl}(X_0, \mathbf{Z}_p(1)) \to H^3_{fl}(X_0, \boldsymbol{\mu}_{p^n}) \to 0.$$

In particular, we get that

$$H^3_{fl}(X_0, \mathbf{Z}_p(1)) \otimes \mathbf{Z}/p^n \simeq H^3_{fl}(X_0, \boldsymbol{\mu}_{p^n}) \tag{4.3}$$

Thus

$$
\begin{aligned}
H^3_{fl}(X_0, \mathbf{Z}_p(1)) \otimes \mathbf{Q}_p/\mathbf{Z}_p &= \varinjlim H^3_{fl}(X_0, \mathbf{Z}_p(1)) \otimes \mathbf{Z}/p^n \\
&\simeq \varinjlim H^3_{fl}(X_0, \boldsymbol{\mu}_{p^n}) \\
&= H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}).
\end{aligned}
$$

$\square$

**Corollary 4.5.12.** *If $X/A$ is a proper smooth lifting of $X_0$, then $\Psi_{X/A}$ is the functor defined by*

$$B \mapsto H^3_{fl}(X \otimes_A B, \boldsymbol{\mu}_{p^\infty}).$$

*Proof.* Let $f : X \to \mathrm{Spec}(A)$ be the structure morphism. By definition $\Psi_{X/A} : \mathrm{Art}_A \to$ Grp is the functor whose group of sections over $B$ is the subgroup of $R^3_{\acute{e}t} f_* \mathbf{G}_m[p^\infty](B)$ consisting of elements that map to the divisible part of $H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty})$. But [AM77] Corollary IV.1.7 tells us that $R^3_{\acute{e}t} f_* \mathbf{G}_m[p^\infty]$ is the sheaf associated to the presheaf defined by $B \mapsto H^3_{fl}(X \otimes_A B, \boldsymbol{\mu}_{p^\infty})$.

Thus there is an inclusion $\Psi_{X/A}(B) \to H^3_{fl}(X \otimes_A B, \boldsymbol{\mu}_{p^\infty})$, since $\Psi_{X/A}(B)$ is the subgroup of elements of $H^3_{fl}(X \otimes_A B, \boldsymbol{\mu}_{p^\infty})$ mapping to the divisible part of $H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty})$. By Proposition 4.5.11, we see that this whole cohomology group is $p$-divisible and hence the natural inclusion is an isomorphism. $\square$

We abuse notation and denote the constant $p$-divisible group by $H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty})$. As in the introduction, we know the dimension of the universal deformation is the product of the dimension of $H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty})$ with the dimension of its Cartier dual. Since the first is étale, it has dimension 0 and this shows the $p$-divisible group is rigid. Thus there is a unique lift to $A$.

Considering the connected-étale sequence for $\Psi_{X/A}$ we get an extension of $p$-divisible groups over $A$:

$$0 \to \Phi_{X/A} \to \Psi_{X/A} \to H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}) \to 0.$$

Call this extension $E_{\Psi_{X/A}}$. Height 1 groups are also rigid, so $\Phi_{X/A}$ is also determined up to unique isomorphism from $\Phi_{X_0/k}$. Define a functor $\mathrm{Art}_W \to \mathrm{Grp}$ by

$$A \mapsto \mathrm{Ext}^1_A(H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}), \Phi_{X/A}).$$

This gives us a natural transformation between the deformation functors

$$\gamma : \{\text{Isomorphism classes of liftings } X/A\} \to \mathrm{Ext}^1_A(H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}), \Phi_{X/A})$$

given by $\gamma(X/A) = E_{\Psi_{X/A}}$.

**Theorem 4.5.13.** *The map $\gamma$ is an isomorphism.*

The functor $A \mapsto \mathrm{Ext}^1_A(H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}), \Phi_{X/A})$ is the deformation functor of $\Psi$. Thus calling this natural transformation $\gamma$ matches the old map $\gamma : \mathrm{Def}_X \to \mathrm{Def}_\Psi$. The proof of this theorem will show that the universal deformation spaces are isomorphic. Let $U$ and $S$ be the (formally smooth) universal deformations over $W$ respectively. We now prove the theorem by showing the map on tangent spaces is an isomorphism:

$$T_\gamma : T_{S/W} \otimes k \to T_{U/W} \otimes k.$$

*Proof.* Consider the functorial isomorphism as given in Proposition 2.5 of the Appendix to [Mes72]:

$$\delta : \mathrm{Ext}^1_A(H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}), \Phi_{X/A}) \to \mathrm{Hom}_{\mathbf{Z}_p}(H^3_{fl}(X, \mathbf{Z}_p(1)), \Phi_{X/A}(A)).$$

At this point in Nygaard's proof we are examining facts about formal groups as given in Messing and the fact that these were formed out of a threefold rather than a K3 surface is irrelevant to the discussion. Following the exact same argument as Theorem 1.6 and Proposition 1.7 in [Nyg83] we find that we can compute $\delta(E_{\Psi_{X/A}})$ as follows.

Given some extension of sheaves $E_{\Psi_{X/A}}$ we can evaluate on $A$ to get

$$0 \to \Phi_{X/A}(A) \to H^3_{fl}(X_A, \boldsymbol{\mu}_{p^\infty}) \to H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}) \to 0.$$

For sufficiently large $r$ we can form well-defined maps

$$p^r : H^3_{fl}(X_0, \boldsymbol{\mu}_{p^r}) \to \Phi_{X/A}(A)$$

which are compatible, so by taking $\varprojlim$ they will give a morphism

$$\delta(E_{\Psi_{X/A}}) : H^3_{fl}(X_0, \mathbf{Z}_p(1)) \to \Phi_{X/A}(A).$$

This can be easily described on elements as follows. Given $x_0 \in H^3_{fl}(X_0, \boldsymbol{\mu}_{p^r})$ we can consider it as an element of $H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty})$. Choose some lift $x \in H^3_{fl}(X_A, \boldsymbol{\mu}_{p^\infty})$. The element $p^r x$ by definition restricts to $p^r x_0 = 0$ and hence is an element of $\Phi(A)$.

Alternately, we have an exact sequence of sheaves

$$0 \to 1 + \mathfrak{m}\mathcal{O}_X \to \mathbf{G}_{m,X}/\mathbf{G}^{p^r}_{m,X} \to \mathbf{G}_{m,X_0}/\mathbf{G}^{p^r}_{m,X_0} \to 1$$

where the second map is restricting to $X_0$. The long exact sequence on cohomology gives us a connecting homomorphism

$$\beta_r : H^2(X_0, \mathbf{G}_m/\mathbf{G}^{p^r}_m) \to H^3(X, 1 + \mathfrak{m}\mathcal{O}_X) \simeq \Phi(A).$$

Now we use Lemma 4.5.9 and the fact that these are also compatible to give a morphism $\beta : H^3_{fl}(X_0, \mathbf{Z}_p(1)) \to \Phi(A)$. Proposition 1.7 of [Nyg83] proves that $\beta = \delta(E_{\Psi_{X/A}})$.

The Kodaira-Spencer map gives us an isomorphism:

$$T_{S/W} \otimes k \xrightarrow{\sim} H^1(X_0, \mathcal{T}).$$

From [AM77] II.1.7, we have that $T_e\Phi_{X_0/k} \simeq H^3(X_0, \mathcal{O}_X)$. This gives us

$$
\begin{aligned}
T_{U/W} \otimes k &\xrightarrow{\sim} \quad \text{Lie } \text{Ext}^1(H^3_{fl}(X_0, \boldsymbol{\mu}_{p^\infty}), \Phi_X) \\
&\xrightarrow{\delta} \quad Hom_{\mathbf{Z}_p}(H^3_{fl}(X_0, \mathbf{Z}_p(1)) \otimes k, T_0\Phi) \\
&\simeq \quad Hom_k(H^3_{fl}(X_0, \mathbf{Z}_p(1)) \otimes k, H^3(X_0, \mathcal{O}_{X_0})).
\end{aligned}
$$

Using Kato's criterion and the short exact sequence 4.2, we get a bijection $i$ : $H^2(X_0, Z^1) \to H^2(X_0, \Omega^1)$. Kato's criterion again applied to

$$
0 \to B^1 \to Z^1 \xrightarrow{C} \Omega^1 \to 0
$$

shows that $C : H^2(X_0, Z^1) \to H^2(X_0, \Omega^1)$ is a $p^{-1}$-linear automorphism and hence so is $C \circ i^{-1} : H^2(X_0, \Omega^1) \to H^2(X_0, \Omega^1)$. By abuse of notation (justified by the fact that $i^{-1}$ acts as the identity), we will call this $C$, the Cartier operator on $H^2(X_0, \Omega^1)$.

But the identification under $i$ of the kernel of $i - C$ consists of exactly the fixed points of $C$, so we get an isomorphism:

$$
\text{dlog} : H^2_{\acute{e}t}(X_0, \mathbf{G}_m/\mathbf{G}_m^p) \xrightarrow{\sim} H^1(X_0, \Omega^2)^C.
$$

The elements of $H^1(X_0, \Omega^2)^C$ will be an $\mathbf{F}_p$-vector space abstractly isomorphic to $\mathbf{F}_p \oplus \cdots \oplus \mathbf{F}_p$ where the number of factors is the same as the dimension of $H^1(X_0, \Omega^2)$. We can explicitly produce a $k$-basis $\{\omega_1, \ldots, \omega_n\}$ for all of $H^1(X_0, \Omega^2)$ such that $\text{Span}_{\mathbf{F}_p}\langle \omega_1, \ldots, \omega_n \rangle$ is exactly $H^1(X_0, \Omega^2)^C$ often called a *basis of logarithmic forms*. This was the content of Corollary 1.1.3. A more concrete element-wise construction is given in Section 4 of [KvdG03]. Hence tensoring back up to $k$ gives the desired last equality in the following:

$$
\begin{aligned}
H^3_{fl}(X_0, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} k \quad &= \quad H^3_{fl}(X_0, \boldsymbol{\mu}_p) \otimes_{\mathbf{Z}_p} k \qquad\qquad \text{equation 4.3} \\
&= \quad H^2_{\acute{e}t}(X_0, \mathbf{G}_m/\mathbf{G}_m^p) \otimes_{\mathbf{Z}_p} k \\
&\xrightarrow{d\log} \quad H^2(X_0, \Omega^1)^C \otimes_{\mathbf{Z}_p} k \\
&= \quad H^2(X_0, \Omega^1).
\end{aligned}
$$

Now Serre duality gives us a natural isomorphism:

$$H^1(X_0, \mathcal{T}) \overset{\sim}{\to} \operatorname{Hom}_k(H^2(X_0, \Omega^1), H^3(X_0, \mathcal{O}_{X_0})).$$

To prove the theorem it suffices to check that the following diagram commutes since we have now shown that the left, bottom, and right arrows are all isomorphisms:

$$
\begin{array}{ccc}
T_{X/W} \otimes k & \overset{T_\gamma}{\longrightarrow} & \operatorname{Hom}(H^3_{fl}(X_0, \mathbf{Z}_p(1) \otimes k, H^3(X_0, \mathcal{O}_{X_0})) \\
\downarrow & & \uparrow {\scriptstyle (d\log, id)} \\
H^1(X_0, \mathcal{T}) & \longrightarrow & \operatorname{Hom}(H^2(X_0, \Omega^1), H^3(X_0, \mathcal{O}_{X_0}))
\end{array}
$$

This commutes because the maps are defined exactly as in Nygaard. In particular, given some $s \in S(k[\varepsilon])$ corresponding to $X/k[\varepsilon]$ we know that $T_\gamma(s)$ is the map

$$
\begin{aligned}
H^3_{fl}(X_0, \mathbf{Z}_p(1)) \otimes k \quad &= \quad H^2_{et}(X_0, \mathbf{G}_m/\mathbf{G}_m^p) \otimes k \\
&\overset{\beta_1}{\longrightarrow} \quad H^3(X, 1 + \varepsilon \mathcal{O}_{X_0}) \\
&\overset{\log}{\longrightarrow} \quad \varepsilon H^3(X_0, \mathcal{O}_{X_0}) \\
&\overset{\varepsilon^{-1}}{\longrightarrow} \quad H^3(X_0, \mathcal{O}_{X_0})
\end{aligned}
$$

$\square$

We call the lift corresponding to the split extension the *canonical lift* and denote it $X_{can}$. Since line bundles always deform on a Calabi-Yau threefold it follows that $X_{can}$ always algebraizes. It satisfies some nice properties:

**Proposition 4.5.14.** *All Brauer classes on $X_0$ lift uniquely to $X_{can}$.*

*Proof.* Let $X_n = X_{can} \otimes W_n$. This is by definition the lift over $W_n$ corresponding to the split extension class. We have an exact sequence coming from restriction to the closed fiber:

$$0 \to 1 + p\mathcal{O}_{X_n} \to \mathcal{O}_{X_n}^* \to \mathcal{O}_{X_0}^* \to 0.$$

The long exact sequence on cohomology gives us

$$H^2(X_n, 1 + p\mathcal{O}_{X_n}) \to Br(X_n) \to Br(X_0) \to H^3(X_n, 1 + p\mathcal{O}_{X_n})$$

The uniqueness part of the theorem comes from the fact that we have an isomorphism log : $H^2(X_n, 1 + p\mathcal{O}_{X_n}) \xrightarrow{\sim} pH^2(X_n, \mathcal{O}_{X_n}) = 0$. This is true for all lifts of all Calabi-Yau threefolds. The surjection is a special property of the canonical lift.

We have encountered this connecting homomorphism before. It factors as

$$H^2(X_0, \mathbf{G}_m) \to \varprojlim H^2(X_0, \mathbf{G}_m/\mathbf{G}_m^{p^r}) \xrightarrow{\beta_r} H^3(X_n, 1 + p\mathcal{O}_{X_n}).$$

Thus the connecting homomorphism is exactly $\varprojlim \beta_r = \delta(E_{\Psi_n/W_n}) = 0$. Thus all Brauer classes on $X_0$ lift uniquely to $X_{can}$.

$\square$

### 4.5.3  Applications

We will now examine how Serre-Tate theory is useful in the study of the derived category of Calabi-Yau threefolds. Let $X$ and $Y$ be two ordinary, liftable Calabi-Yau threefolds over a perfect field of characteristic $p > 3$ satisfying the hypotheses of this chapter. The first main utility is the following proposition.

**Proposition 4.5.15.** *If $D(X) \simeq D(Y)$, then there is a canonical isomorphism of the deformation functors $\mathrm{Def}_X \xrightarrow{\sim} \mathrm{Def}_Y$ that sends the canonical lift of $X$ to the canonical lift of $Y$.*

*Proof.* First, it is known that the Fourier-Mukai kernel induces an isomorphism of $F$-isocrystals:

$$H^3_{crys}(X/K) \xrightarrow{\sim} H^3_{crys}(Y/K).$$

Since $\Psi_{X/k}$ is a $p$-divisible group with Dieudonné module $D(\Psi_{X/k}) = H^3_{crys}(X/K)_{[0,1]}$ and likewise for $\Psi_{Y/k}$, we see that the $p$-divisible groups are isogenous. The isogeny

$\phi : \Psi_{X/k} \to \Psi_{Y/k}$ induces a map on the deformation functors $\mathrm{Def}_{\Psi_{X/k}} \to \mathrm{Def}_{\Psi_{Y/k}}$ which preserves the deformation corresponding to the trivial extension.

Now using Theorem 4.5.13 and the definition of the canonical lift we fill in the top arrow:

$$\begin{array}{ccc} \mathrm{Def}_X & \longrightarrow & \mathrm{Def}_Y \\ \downarrow & & \uparrow \\ \mathrm{Def}_{\Psi_X} & \longrightarrow & \mathrm{Def}_{\Psi_Y} \end{array}$$

Since every arrow is an isomorphism the top arrow is as well. The canonical lift is preserved by definition. $\qquad\qquad\square$

## Chapter 5

## OPEN PROBLEMS AND FUTURE DIRECTIONS

Since the general topic of studying questions in arithmetic geometry via Fourier-Mukai equivalences has not been investigated much, this thesis seems to raise more questions than it answers. We will categorize them for convenience.

### 5.1 Calabi-Yau Threefolds

We will start with questions surrounding liftability of Calabi-Yau threefolds since this was one of the original motivating questions. In light of [Jos07] and the fact that liftability seems to be less well-behaved as the height of the formal group gets bigger, the following question seems tractable:

**Question 5.1.1.** *Under some nice hypotheses, does every ordinary Calabi-Yau threefold lift to characteristic* $0$?

More abstractly, if $X$ and $Y$ are ordinary, liftable Calabi-Yau threefolds that are derived equivalent, then by Proposition 4.5.15 there is a natural isomorphism of the deformation functors sending the canonical lift of $X$ to the canonical lift of $Y$.

**Question 5.1.2.** *Are the canonical lifts also derived equivalent?*

At a more basic and fundamental level

**Question 5.1.3.** *Is liftability a derived invariant for Calabi-Yau threefolds?*

See [LO11] for a proof of the corresponding result for K3 surfaces. We also saw that any relative Fourier-Mukai partner of Schröer's non-liftable threefold is isomorphic to it. There is still hope for constructing a new non-liftable Calabi-Yau threefold using a

moduli space of sheaves and derived equivalences, but this will depend on the answer to the following question

**Question 5.1.4.** *What is the Fourier-Mukai number of Schröer's nonliftable three-fold?*

Turning back to [LO11], it is found that there is a filtered equivalence between two K3 surfaces if and only if the K3 surfaces are isomorphic. This fact makes use of the Torelli theorem which is false for Calabi-Yau threefolds.

**Question 5.1.5.** *Are there non-isomorphic Calabi-Yau threefolds with a filtered derived equivalence between them?*

There is still a lot of mystery surrounding the relationship between the primes dividing the conductor of a rigid modular Calabi-Yau threefold and the type of reduction at those primes. It would be interesting to know

**Question 5.1.6.** *Are the primes of bad reduction for a Calabi-Yau threefold preserved under derived equivalence?*

If so, does the fact that modularity is preserved under derived equivalence shed any light on the conductor issue? This isn't stated as an official question, because not all rigid, modular Calabi-Yau threefolds with the same modular form attached are derived equivalent.

It would be interesting to know in general

**Question 5.1.7.** *Is being a K3 fibration preserved under derived equivalence of Calabi-Yau threefolds?*

Lastly, we will end with a fundamental question that still seems to be unknown for Calabi-Yau threefolds over $\mathbf{C}$.

**Question 5.1.8.** *Given some Calabi-Yau threefold $X/\mathbf{C}$, are there only finitely many Fourier-Mukai partners for $X$?*

## 5.2 General Questions

Now we move on to some more general questions. Not much is known about higher dimensional modular varieties. Since moduli spaces of sheaves seem to bear some relation to preserving modularity, consider the following setup. Fix a polarization and primitive Mukai vector $v$ for a rank 20 K3 surface, $X/\mathbf{Q}$, with $v^2 > 0$. General theory tells us that $X$ is modular. The moduli space of stable sheaves with Mukai vector $v$ has dimension strictly larger than 2.

**Question 5.2.1.** *Is $M_X(v)$ also modular?*

In light of Corollary 2.3.12, if $C/\mathbf{Q}$ a genus 1 curve, is $Stab(C)$ realizable as an algebraic space over $\mathbf{Q}$? If so, are there detectable walls that allow us to produce a family of stability conditions with respect to some numerical condition $v = (r, d)$ to some $\sigma \in Stab(C)$ such that $M_\sigma(v) = C'$ is a genus one curve over $\mathbf{Q}$ not isomorphic to $C$ but still derived equivalent? In other words, can the counterexamples be realized in some Bridgeland stability framework?

Another question that seems plausible is whether or not the "strong" derived Torelli theorem for K3 surfaces holds over all fields:

**Question 5.2.2.** *If $\Phi_{\mathcal{P}} : D(Y) \to D(X)$ is a filtered equivalence whose associated cohomological Fourier-Mukai transform satisfies $\Phi_{\mathcal{P}}^H(\mathcal{L}') = \mathcal{L}$, then is there is a unique isomorphism $Y \xrightarrow{\sim} X$ defined over $k$ inducing $\Phi_{\mathcal{P}}^H$ on cohomology?*

The proof given in Chapter 2 is very close to proving this, because it works in characteristic 0. It also shows the isomorphism descends uniquely once the result is known for perfect fields. A careful analysis of either lifting automorphisms with the K3 surface or understanding how specializations of birational isomorphisms interact with cohomology in mixed characteristic will complete the proof.

In terms of some general arithmetic invariants about any class of varieties

**Question 5.2.3.** *Is there a Brauer-pairing type obstruction to liftability? Can such pairing ideas be seen on the derived category to better understand good and bad reduction?*

Or

**Question 5.2.4.** *Does derived equivalence preserve the Hasse principle?*

In light of Proposition 4.4.1

**Question 5.2.5.** *If $X$ and $Y$ are derived equivalent varieties over some number field, are their Hodge numbers equal?*

A promising line of attack for this was laid out in the threefold case. It is subtle to extrapolate information about individual traces of Frobenius on the $\ell$-adic cohomology, because the cohomology preserved under derived equivalence is packaged together and twisted. If one could unravel these traces in the same way to show the alternating sum is the same over finite fields, then the two varieties would have the same Zeta functions at primes of good reduction. The same argument involving $p$-adic Hodge theory would prove the answer to the question is yes.

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