Property of the Tangent Line for a Circle and an Ellipse

or

Circle-Ellipse Tangent Line Property

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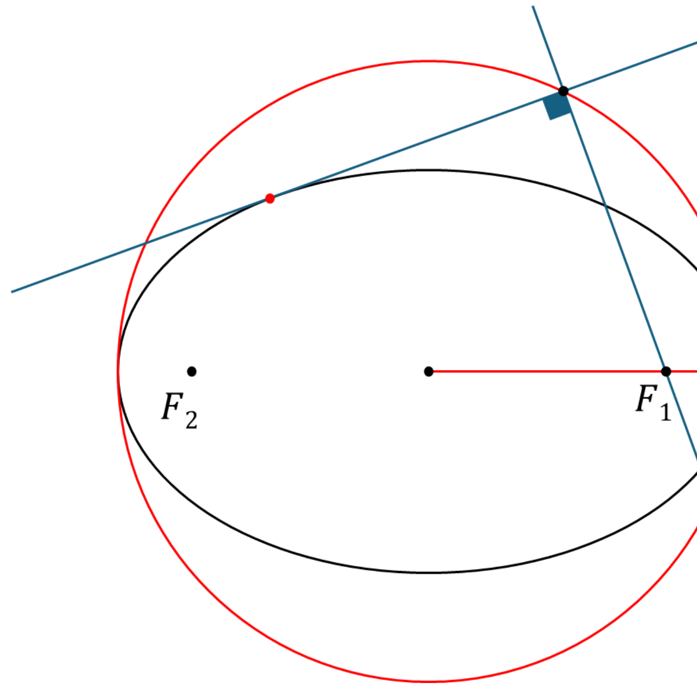
Property

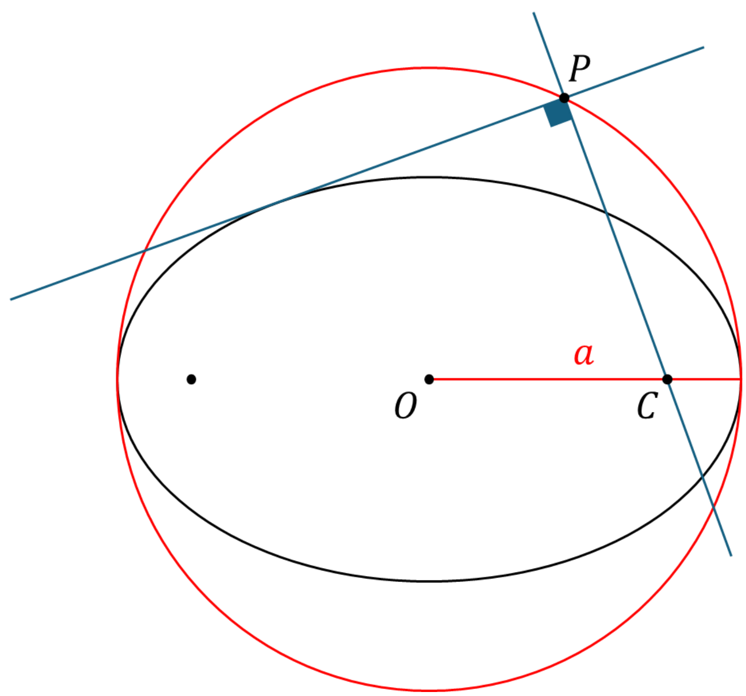
Let an ellipse be given with foci, and let a circle be centered at the center of the ellipse, with a radius equal to the semi-major axis of the ellipse. For any point on the circle, the line passing through that point and one of the foci of the ellipse has a perpendicular which, when drawn through the point on the circle, is tangent to the ellipse.



Demonstration

Given an ellipse with foci and a circle centered at the center ***O*** of the ellipse with a radius equal to the semi-major axis ***a*** of the ellipse, let ***P*** be any point on the circle.

Consider the line passing through one of the foci, ***C*** (*xc;yc*) and the point ***P*** (*xp;yp*), and the line perpendicular to it at point ***P***.



First, let's determine the slope ***m*** of the line passing through ***P*** and ***C***.

$$m = \frac{yp - yc}{xp - xc}$$

Now, let's determine the slope ***n*** of the line perpendicular to line passing thought ***P*** and ***C*** at ***P***.

$$n = \frac{-1}{m}$$

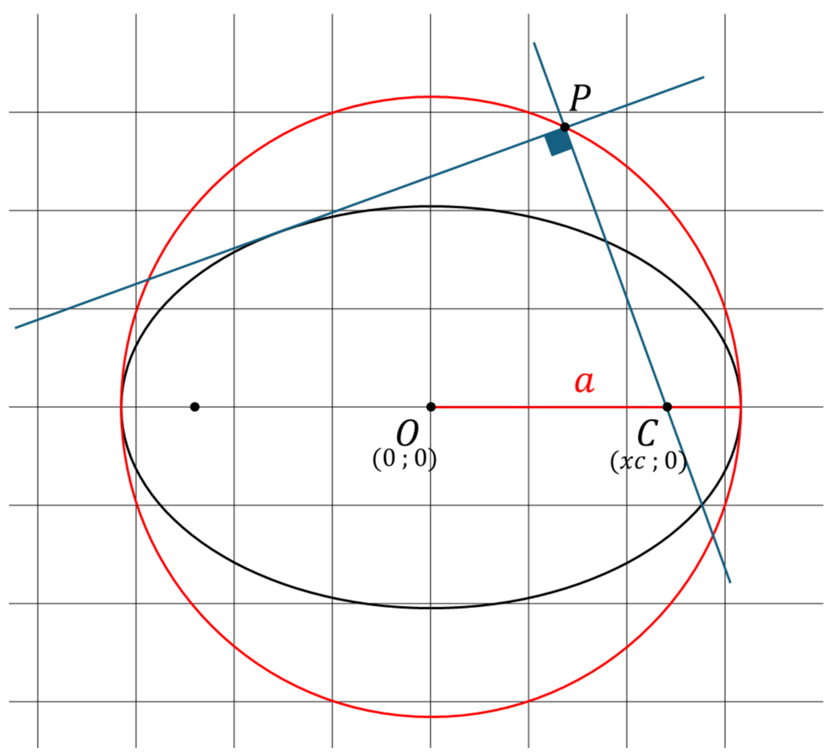
$$n = \frac{-1}{\left(\frac{yp - yc}{xp - xc}\right)}$$

$$n = \frac{-(xp - xc)}{yp - yc}$$

$$n = \frac{xc - xp}{yp - yc}$$

Let's now place ***O***, the center of the ellipse and of the circle with a radius equal to the semi-major axis ***a*** of the ellipse, at the origin of a cartesian plane (*0;0*), to obtain ***C*** (*xc;0*). The slope ***n*** becomes:

$$n = \frac{xc - xp}{yp}$$



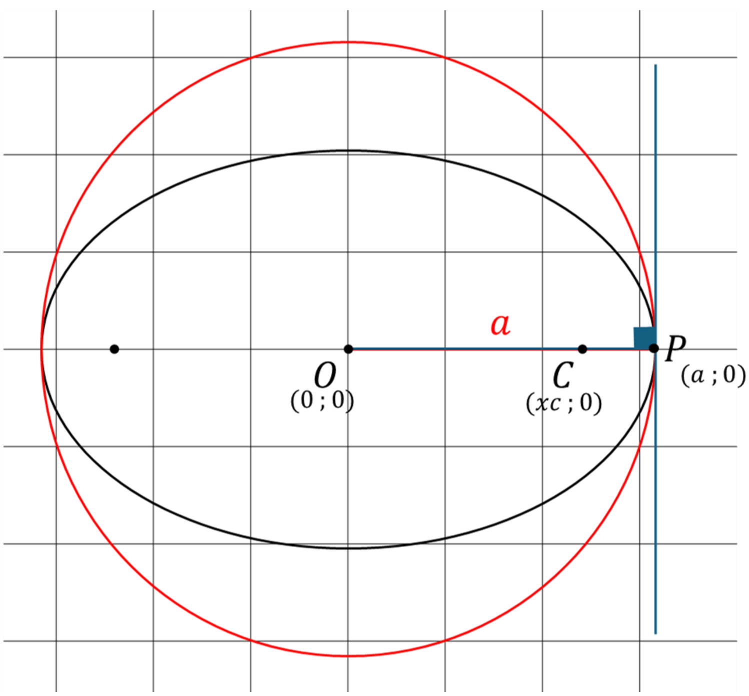
For this demonstration,

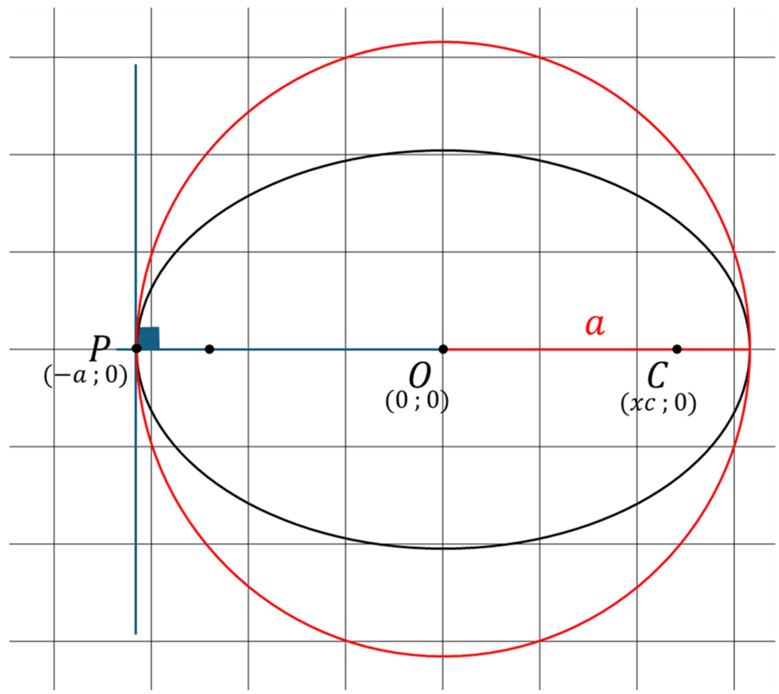
$$yp \neq 0$$

However, this theorem still works for

$$yp = 0$$

Which, in this situation, means that ***P*** has either the coordinates ***P*** (*a;0*) or ***P*** (-*a;0*), where in both cases, ***P*** belongs to the ellipse and to the circle that share the same center ***O***, with the circle having a radius equal to the semi-major axis ***a*** of the ellipse. Consequently, the line perpendicular to the line passing through ***P*** and ***C*** at ***P***, is tangent to the ellipse.





Now that we have the slope ***n***, we can determine the equation of the line perpendicular to the line passing through ***P*** and ***C*** at ***P***:

$$y - yp = n \cdot (x - xp)$$

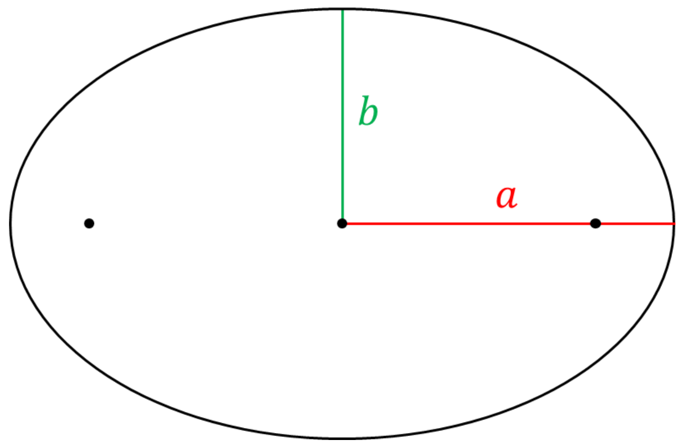
$$y = n \cdot (x - xp) + yp$$

Let's remind the equation of an ellipse with semi-major axis ***a*** and semi-minor ***b***:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2 \cdot \left(1 - \frac{x^2}{a^2}\right)$$

$$y = b \cdot \sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$



Let's review:

For the line perpendicular to the line passing through ***P*** and ***C*** at ***P*** to be tangent to the ellipse, there must be **one and only one** point where its coordinates satisfy these two equations:

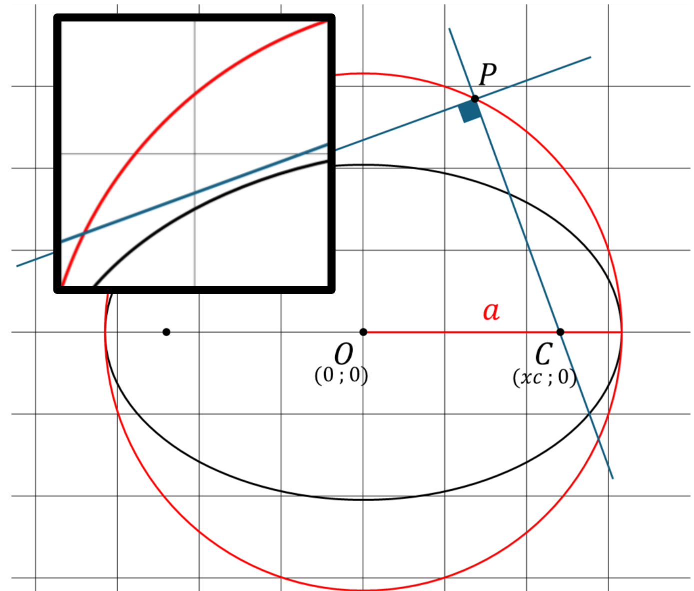
$$y = n \cdot (x - xp) + yp$$

and

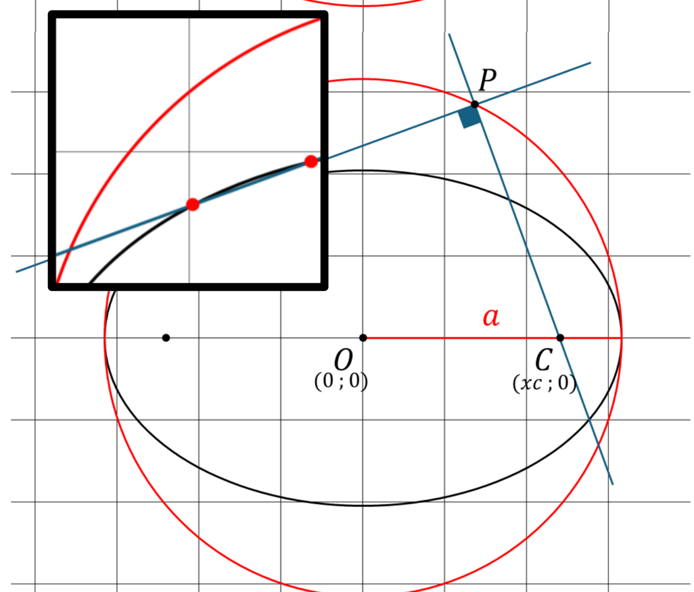
$$y = b \cdot \sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$

There are 3 cases:

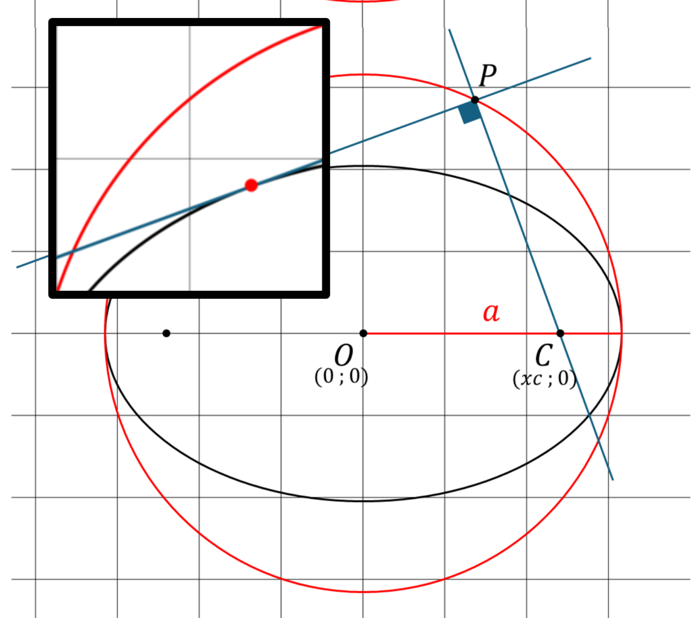
- the line perpendicular to the line passing through ***P*** and ***C*** at ***P*** DOES NOT INTERSECT the ellipse, so no point exists where its coordinates satisfy these two equations, thus; it is **not tangent** to the ellipse.



- the line perpendicular to the line passing through ***P*** and ***C*** at ***P*** INTERSECTS THE ELLIPSE IN 2 POINTS, so 2 points exist where their coordinates satisfy these two equations, thus; it is **not tangent** to the ellipse.



- the line perpendicular to the line passing through ***P*** and ***C*** at ***P*** INTERESCTS THE ELLIPSE IN 1 POINT, so 1 point exists where its coordinates satisfy these two equations, thus; it is **tangent** to the ellipse.



Now let's solve for

$$n \cdot (x - xp) + yp = b \cdot \sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$

$$n \cdot (x - xp) + \ yp = b \cdot \sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$

$$(n \cdot (x - xp) + \ yp)^2 = \left(b \cdot \sqrt{(1 - \frac{x^2}{a^2})}\right)^2$$

$$n^2 \cdot (x - xp)^2 + 2 \cdot n \cdot (x - xp) \cdot yp + yp^2 = b^2 \cdot (1 - \frac{x^2}{a^2})$$

$$n^2 \cdot (x^2 - 2 \cdot x \cdot xp + xp^2) + 2 \cdot n \cdot (x - xp) \cdot yp + yp^2 = b^2 - \frac{b^2 \cdot x^2}{a^2}$$

$$n^2 \cdot x^2 - 2 \cdot n^2 \cdot x \cdot xp + n^2 \cdot xp^2 + 2 \cdot n \cdot (x - xp) \cdot yp + yp^2 = b^2 - \frac{b^2}{a^2} \cdot x^2$$

$$n^2 \cdot x^2 - 2 \cdot n^2 \cdot x \cdot xp + n^2 \cdot xp^2 + 2 \cdot n \cdot x \cdot yp - 2 \cdot n \cdot xp \cdot yp + yp^2 = b^2 - \left(\frac{b}{a}\right)^2 \cdot x^2$$

At this point, we will identify the x's and the coefficients to see clearer.

$$x^2 \cdot [n^2] + x \cdot [2 \cdot n \cdot yp - 2 \cdot n^2 \cdot xp] + [n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2]$$

$$= [b^2] - x^2 \cdot [\left(\frac{b}{a}\right)^2]$$

$$x^2 \cdot [n^2 + \left(\frac{b}{a}\right)^2] + x \cdot [2 \cdot n \cdot yp - 2 \cdot n^2 \cdot xp] + [n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2] = 0$$

We end up with a polynomial of the second degree, where we can identify the coefficients A, B and C.

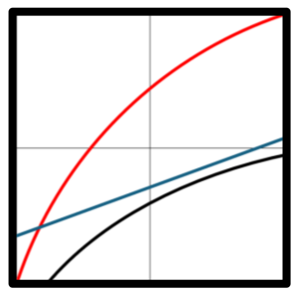
$$\underbrace{x^2 \cdot [n^2 + \left(\frac{b}{a}\right)^2]}_{A} + x \cdot \underbrace{[2 \cdot n \cdot yp - 2 \cdot n^2 \cdot xp]}_{B} + \underbrace{[n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2]}_{C} = 0$$

We can calculate the discriminant Δ to find the solutions to this polynomial:

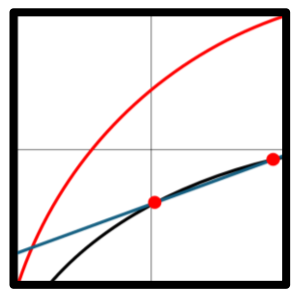
$$\Delta = \boldsymbol{B^2 - 4 \cdot A \cdot C}$$

Let's review:

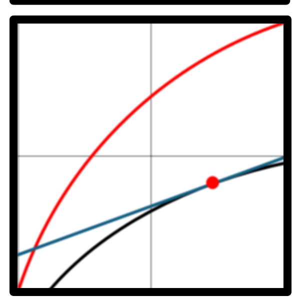
- If $\Delta < 0$, the equation has no real solution, so the perpendicular to the line passing through ***P*** and ***C*** at ***P*** does not intersect the ellipse and, therefore, is not tangent to the ellipse.



- If $\Delta > 0$, the equation has two real solutions, so the perpendicular to the line passing through ***P*** and ***C*** at ***P*** intersects the ellipse at two points and, therefore, is not tangent to the ellipse.



- If $\Delta = 0$, the equation has one real solution, so the perpendicular to the line passing through ***P*** and ***C*** at ***P*** intersects the ellipse at only one point and, therefore, is tangent to the ellipse.



Let's calculate Δ:

As there are many terms, we will calculate Δ slowly to keep it clear, first let's develop $\boldsymbol{B^2}$:

$$\Delta = [2 \cdot n \cdot yp - 2 \cdot n^2 \cdot xp]^2 - 4 \cdot [n^2 + \left(\frac{b}{a}\right)^2] \cdot [n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2]$$

$$\Delta = [4 \cdot n^2 \cdot yp^2 - 8 \cdot n^3 \cdot xp \cdot yp + 4 \cdot n^4 \cdot xp^2]$$

$$-4 \cdot [n^2 + \left(\frac{b}{a}\right)^2] \cdot [n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2]$$

We can factorize the entire product by 4:

$$\Delta = 4 \cdot [n^2 \cdot yp^2 - 2 \cdot n^3 \cdot xp \cdot yp + n^4 \cdot xp^2]$$

$$-4 \cdot [n^2 + \left(\frac{b}{a}\right)^2] \cdot [n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2]$$

$\boldsymbol{B^2}$ and $[\boldsymbol{A} \cdot \boldsymbol{C}]$ both have 4 as a common factor:

$$\Delta = 4 \cdot ([n^2 \cdot yp^2 - 2 \cdot n^3 \cdot xp \cdot yp + n^4 \cdot xp^2]$$

$$-[n^2 + \left(\frac{b}{a}\right)^2] \cdot [n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2])$$

We can now develop $[\boldsymbol{A} \cdot \boldsymbol{C}]$, however, for conveniency, we will only distribute $\boldsymbol{n^2}$ to $[\boldsymbol{n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2}]$, which gives us:

$$\Delta = 4 \cdot ([n^2 \cdot yp^2 - 2 \cdot n^3 \cdot xp \cdot yp + n^4 \cdot xp^2] - [n^4 \cdot xp^2 - 2 \cdot n^3 \cdot xp \cdot yp + n^2 \cdot yp^2$$

$$- n^2 \cdot b^2 + \left(\frac{b}{a}\right)^2 \cdot (n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2)])$$

We can remove the inner brackets, and carefully change the signs inside since we have a subtraction:

$$\Delta = 4 \cdot [n^2 \cdot yp^2 - 2 \cdot n^3 \cdot xp \cdot yp + n^4 \cdot xp^2 - n^4 \cdot xp^2 + 2 \cdot n^3 \cdot xp \cdot yp - n^2 \cdot yp^2 + n^2 \cdot b^2 - \left(\frac{b}{a}\right)^2 \cdot (n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2)]$$

We can identify three common terms that cancel each other out:

$$\Delta = 4 \cdot [n^2 \cdot yp^2 - 2 \cdot n^3 \cdot xp \cdot yp + n^4 \cdot xp^2 - n^4 \cdot xp^2 + 2 \cdot n^3 \cdot xp \cdot yp - n^2 \cdot yp^2 + n^2 \cdot b^2 - \left(\frac{b}{a}\right)^2 \cdot (n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2)]$$

We obtain:

$$\Delta = 4 \cdot [n^2 \cdot b^2 - \left(\frac{b}{a}\right)^2 \cdot (n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2)]$$

For conveniency, we will multiply the term $[\boldsymbol{n^2 \cdot b^2}]$ by $\boldsymbol{\frac{a^2}{a^2}}$, we obtain:

$$\Delta = 4 \cdot [\frac{a^2}{a^2} \cdot n^2 \cdot b^2 - \left(\frac{b}{a}\right)^2 \cdot (n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2)]$$

$$\Delta = 4 \cdot [a^2 \cdot n^2 \cdot \frac{b^2}{a^2} - \left(\frac{b}{a}\right)^2 \cdot (n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2)]$$

$$\Delta = 4 \cdot [a^2 \cdot n^2 \cdot \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^2 \cdot (n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2)]$$

As $\left(\frac{b}{a}\right)^2$ is a common factor, we can re-write our calculation as:

$$\Delta = 4 \cdot \left(\frac{b}{a}\right)^2 \cdot [a^2 \cdot n^2 - (n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2)]$$

To simplify our calculation, let

$$k = 4 \cdot \left(\frac{b}{a}\right)^2$$

our calculations for Δ becomes:

$$\Delta = k \cdot [a^2 \cdot n^2 - (n^2 \cdot xp^2 - 2 \cdot n \cdot xp \cdot yp + yp^2 - b^2)]$$

We can remove the inner brackets, and carefully change the signs inside since we have a subtraction:

$$\Delta = k \cdot [a^2 \cdot n^2 - n^2 \cdot xp^2 + 2 \cdot n \cdot xp \cdot yp - yp^2 + b^2]$$

We can factorize $\boldsymbol{a^2}$ and $\boldsymbol{xp^2}$ by $\boldsymbol{n^2}$, which gives us:

$$\Delta = k \cdot [n^2 \cdot (a^2 - xp^2) + 2 \cdot n \cdot xp \cdot yp - yp^2 + b^2]$$

At this stage, we need to remind the equation of a circle:

$$x^2 + y^2 = r^2$$

We can apply this equation to our circle with a radius ***a***:

$$x^2 + y^2 = a^2$$

Since ***P*** (*xp; yp*) is a point of the circle with a radius ***a***, then,

$$xp^2 + yp^2 = a^2$$
$$yp^2 = a^2 - xp^2$$

We can now substitute $[\boldsymbol{a^2 - xp^2}]$ by $\boldsymbol{yp^2}$, in our calculations for Δ:

$$\Delta = k \cdot [n^2 \cdot yp^2 + 2 \cdot n \cdot xp \cdot yp - yp^2 + b^2]$$

Remember:

$$n = \frac{xc - xp}{yp}$$

We can now substitute $\boldsymbol{n}$ by $\frac{\boldsymbol{xc-xp}}{\boldsymbol{yp}}$:

$$\Delta = k \cdot [\left(\frac{xc - xp}{yp}\right)^2 \cdot yp^2 + 2 \cdot \frac{xc - xp}{yp} \cdot xp \cdot yp - yp^2 + b^2]$$

$$\Delta = k \cdot [\frac{(xc^2 - 2.xc.xp + xp^2)}{yp^2} \cdot yp^2 + 2 \cdot \frac{xc - xp}{yp} \cdot xp \cdot yp - yp^2 + b^2]$$

$$\Delta = k \cdot [\frac{(xc^2 - 2.xc.xp + xp^2)}{\cancel{yp^2}} \cdot \cancel{yp^2} + 2 \cdot \frac{xc - xp}{\cancel{yp}} \cdot xp \cdot \cancel{yp} - yp^2 + b^2]$$

$$\Delta = k \cdot [xc^2 - 2.xc.xp + xp^2 + 2 \cdot (xc - xp) \cdot xp - yp^2 + b^2]$$

$$\Delta = k \cdot [xc^2 - 2.xc.xp + xp^2 + 2 \cdot xc \cdot xp - 2 \cdot xp^2 - yp^2 + b^2]$$

$$\Delta = k \cdot [xc^2 \cancel{- 2.xc.xp} + xp^2 \cancel{+ 2 \cdot xc \cdot xp} - 2 \cdot xp^2 - yp^2 + b^2]$$

$$\Delta = k \cdot [xc^2 + xp^2 - 2 \cdot xp^2 - yp^2 + b^2]$$

$$\Delta = k \cdot [xc^2 - xp^2 - yp^2 + b^2]$$

Remember:

$$xp^2 + yp^2 = a^2$$

So,

$$-xp^2 - yp^2 = -a^2$$

Our calculations for Δ becomes:

$$\Delta = k \cdot [xc^2 - a^2 + b^2]$$

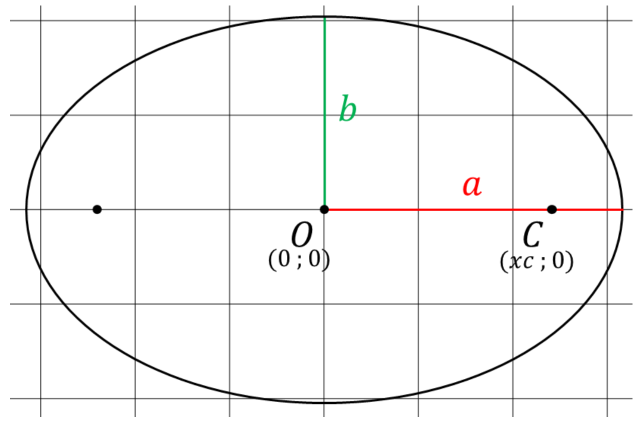
Let's review how to calculate the distance between the center of an ellipse and its focus point:

In an ellipse with center ***O*** (*0;0*), semi-major axis ***a***, and semi-minor axis ***b***, the distance between the center ***O*** and a focus point ***C*** (*xc;0*) is:

$$a^2 - b^2 = [OC]^2$$

In this case,

$$[OC]^2 = xc^2$$



So,

$$xc^2 = a^2 - b^2$$

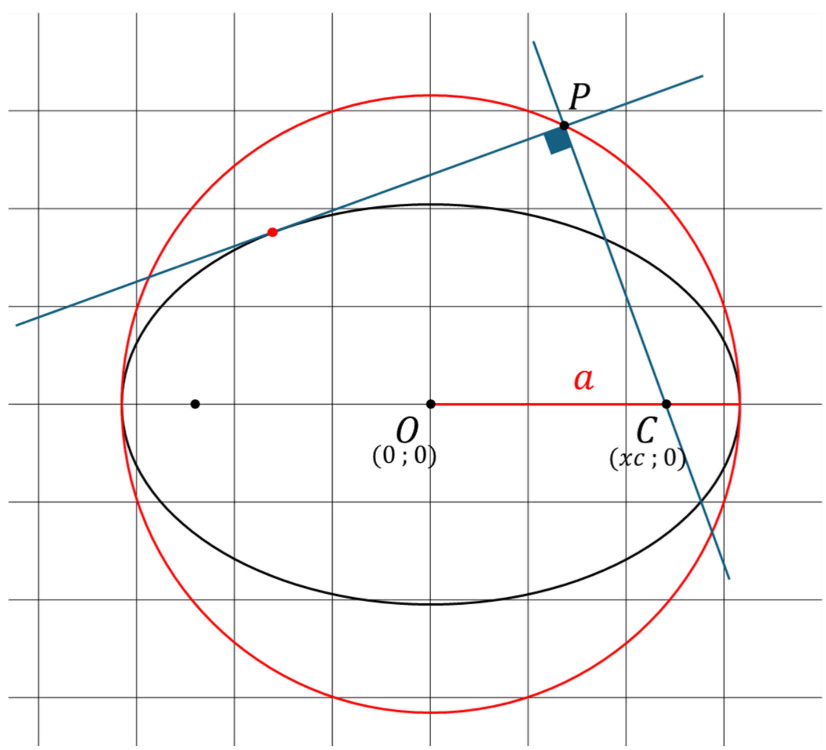
We can substitute $\boldsymbol{xc^2}$ by $[\boldsymbol{a^2 - b^2}]$ in our calculation for Δ:

$$\Delta = k \cdot [a^2 - b^2 - a^2 + b^2]$$

$$\Delta = k \cdot 0$$

$$\boxed{\Delta = 0}$$

As $\Delta = 0$, the equation has one real solution, so the perpendicular to the line passing through ***P*** and ***C*** at ***P*** intersects the ellipse at only one point; thus, is tangent to the ellipse.



Conclusion

This confirms the property:

Let an ellipse be given with foci, and let a circle be centered at the center of the ellipse, with a radius equal to the semi-major axis of the ellipse. For any point on the circle, the line passing through that point and one of the foci of the ellipse has a perpendicular which, when drawn through the point on the circle, is tangent to the ellipse.