How to calculate an ellipse?

Proposition of a new method with demonstrations and explanations

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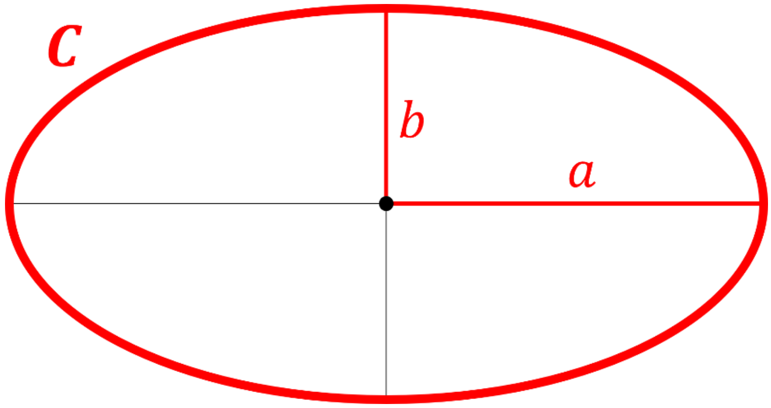
I- Proposed formula

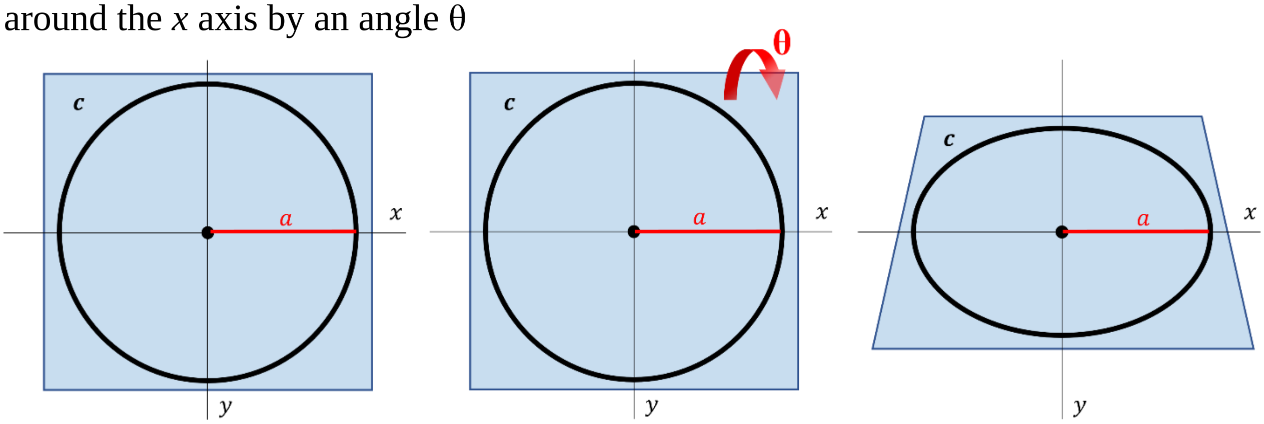
The circumference C of an ellipse with a semi-major axis a and a semi-minor axis b is

$$\boxed{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a}$$



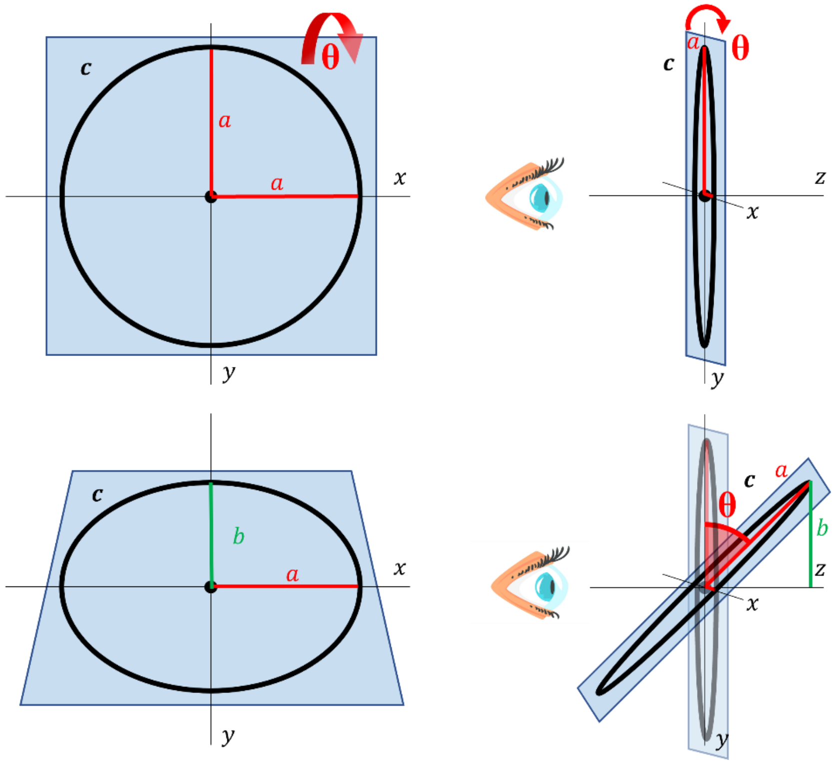
II- Demonstration & Explanation

If we draw a circle c with a radius a on a y plane, with the x axis crossing its center, and we observe from the z axis passing through the center of the circle, we obtain an ellipse with a semi major-axis a and a semi-minor axis b when rotating the y plane around the x axis by an angle θ



We can then say, that calculating the circumference of an ellipse, is calculating the observed circumference of a circle from a different angle.

After rotating the y plane around the x axis by the angle θ, observed from the z axis passing through the center of the circle c, the circle c with radius a becomes an ellipse with a semi-major axis a and a semi-minor axis b, where $b = a \cdot \cos\theta$

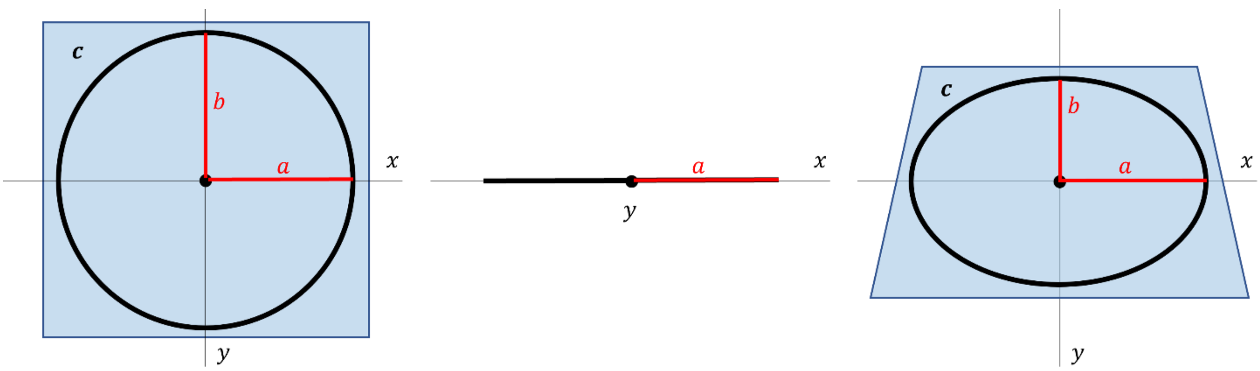


Then we can say that the circumference C of the ellipse depends on the rotation of the circle c by the angle θ of the y plane around the x axis crossing the center of the circle c, observed from the z axis passing through the center of the circle c.

If $\theta = 0°$,
then $b = a$,
and $C = 2\pi a$

If $\theta = 90°$,
then $b = 0$,
and $C = 4a$

If $0° < \theta < 90°$,
then $a > b > 0$,
and $2\pi a > C > 4a$

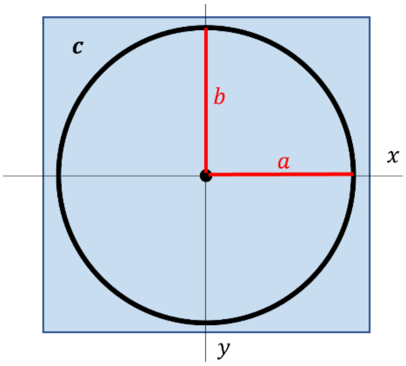


So, based on the value of θ where $\theta \in [0°; 90°]$, the circumference C of the ellipse is comprised between

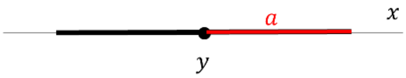
$$\boldsymbol{C} \in [\boldsymbol{4a}; \boldsymbol{2\pi a}] \text{ where } \boldsymbol{a} > \boldsymbol{0} \text{ and } \boldsymbol{b} \in [\boldsymbol{0};\ \boldsymbol{a}]$$

If we try to propose a common equation to the previous results, we can re-write their equations as follow:

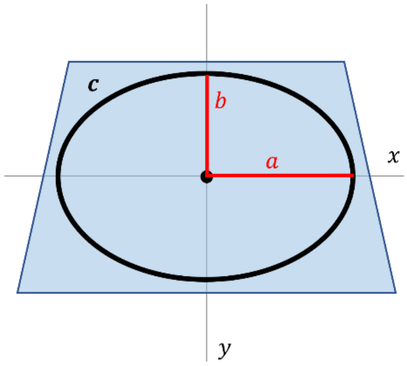
If $\theta = 0°$,
then $b = a$,
and $C = 4\left(\frac{\pi}{2}\right) a$



If $\theta = 90°$,
then $b = 0$,
and $C = 4(1)a$



If $0° < \theta < 90°$,
then $a > b > 0$,
and $C = 4(P)a$



So, based on the value of θ where $\theta \in [0°; 90°]$, the circumference C of the ellipse is

$$\boldsymbol{C = 4 \cdot P \cdot a} \quad \text{where } a > 0 \text{ and}$$

$P = \left(\frac{\pi}{2}\right)$ when $\theta = 0°$, $P = 1$ when $\theta = 90°$, and $1 < P < \left(\frac{\pi}{2}\right)$ when $0° < \theta < 90°$

In order to keep $P \in \left[1; \frac{\pi}{2}\right]$, we can use the exponential function by suggesting:

$$\boldsymbol{P = \left(\frac{\pi}{2}\right)^{q}} \qquad \text{where } \boldsymbol{q \in [0; 1]}$$

Which gives us:

$$\boldsymbol{C = 4 \cdot \left(\frac{\pi}{2}\right)^{q} \cdot a} \quad \text{where } \boldsymbol{a > 0} \text{ and } \boldsymbol{q \in [0; 1]}$$

Given that when,

- $\theta = 0°$, $P = \left(\frac{\pi}{2}\right)$, so $q = 1$
- $\theta = 90°$, $P = 1$, so $q = 0$
- $0° < \theta < 90°$, $1 < P < \left(\frac{\pi}{2}\right)$, so $0 < q < 1$

We can then suggest that: $\boldsymbol{q = \cos\theta}$

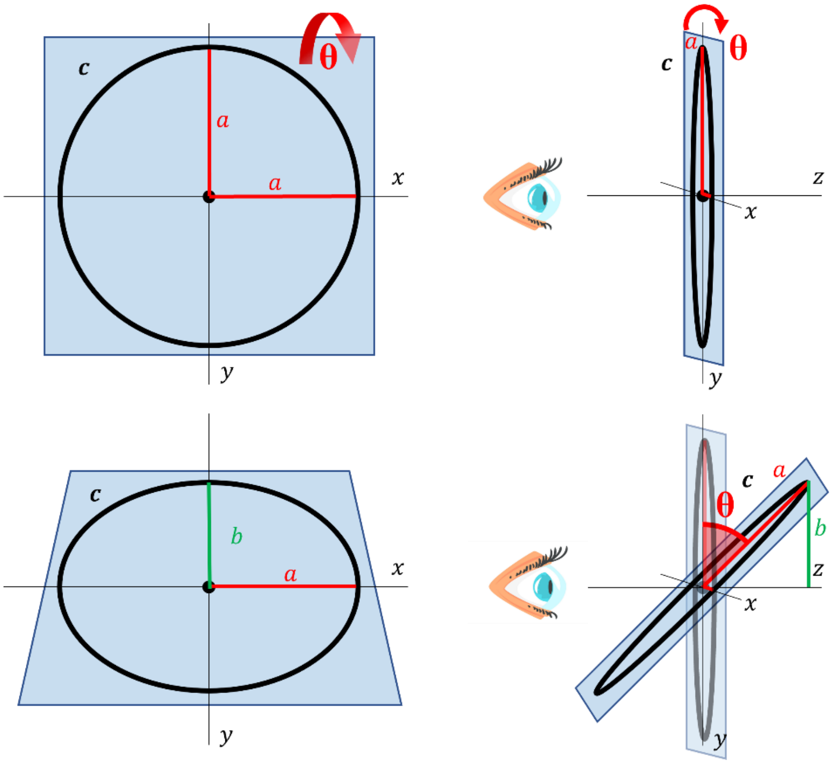
Which gives us:

$$\boldsymbol{C = 4 \cdot \left(\frac{\pi}{2}\right)^{\cos\theta} \cdot a} \quad \text{where } \boldsymbol{a > 0} \text{ and } \boldsymbol{\theta \in [0°; 90°]}$$

Remember that after rotating the y plane around the x axis by the angle θ, observed from the z axis passing through the center of the circle c, the circle c with radius a becomes an ellipse with a semi-major axis a and a semi-minor axis b, where $b = a \cdot \cos\theta$

So,

$$\cos\theta = \frac{b}{a}$$



We can then re-write our equation as:

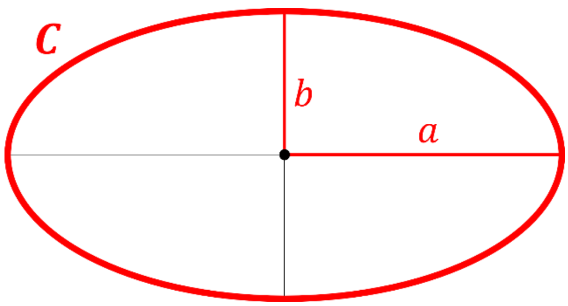
$$C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a$$

Where,
C is the circumference of the ellipse,
a is the semi-major axis of the ellipse,
b is the semi-minor axis of the ellipse.

III- Conclusion

The circumference C of an ellipse with a semi-major axis a and a semi-minor axis b is

$$\boxed{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a}$$



IV- Analyzes & Limits

➔ Let's analyze the formula

The circumference C of an ellipse with a semi-major axis a and a semi-minor axis b is

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a}$$

with $\boldsymbol{a > 0}$, and $\boldsymbol{b} \in [\boldsymbol{0}; \boldsymbol{a}]$

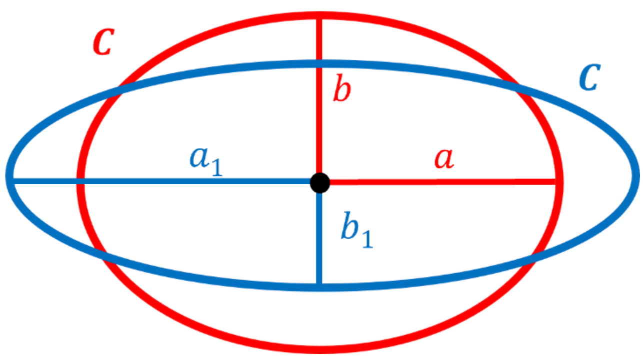
Let's try to find the variations of the semi-major axis a and a semi-minor axis b for all the ellipses with the same circumference C such as

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1}$$

With:
$a_1 = n \cdot a$
$b_1 = m \cdot b$

Where, if a changes by a factor n, so b changes by a factor m, and vice-versa.

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{m\cdot b}{n\cdot a}} n \cdot a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1}$$

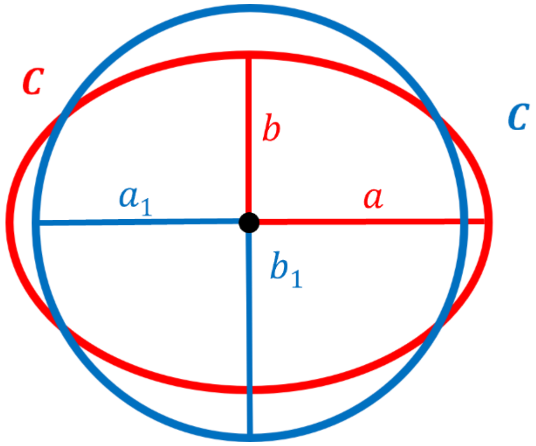


All ellipses with the same circumference C, with varied semi-major axis a and semi-minor axis b are comprises between:

- A ‘circular’ ellipse (circle) with

$n \cdot a = m \cdot b$
$a_1 = b_1$
Where

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{m \cdot b}{n \cdot a}} n \cdot a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1 = 4\left(\frac{\pi}{2}\right) a_1}$$

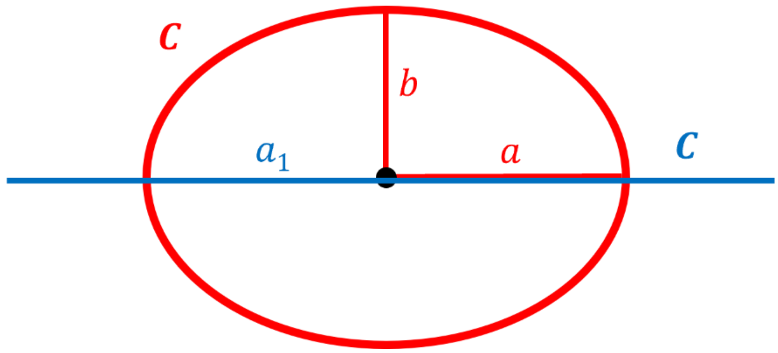


And

- A ‘flat’ ellipse with

$n \cdot a = a_1$
$m \cdot b = b_1 = 0$
Where

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{m \cdot b}{n \cdot a}} n \cdot a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1 = 4\left(\frac{\pi}{2}\right)^{\frac{0}{a_1}} a_1 = 4\, a_1}$$



➔ From this observation we can/must define the limits of $\boldsymbol{n}$ and $\boldsymbol{m}$.

When the ellipse is ‘circular’ (circle)
$n \cdot a = m \cdot b$
$a_1 = b_1$

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{m \cdot b}{n \cdot a}} n \cdot a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1 = 4\left(\frac{\pi}{2}\right) a_1}$$

$\Leftrightarrow$

$$\boldsymbol{4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right) a_1}$$

$\Leftrightarrow$

$$\boldsymbol{4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right) n \cdot a}$$

$\Leftrightarrow$

$$\boldsymbol{\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = \left(\frac{\pi}{2}\right) n}$$

$$\boldsymbol{n = \left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}}$$

Given
$n \cdot a = m \cdot b$
So

$$\boldsymbol{m = \frac{a}{b} \cdot n}$$

$$\boldsymbol{m = \frac{a}{b} \cdot \left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}}$$

So, when an ellipse with circumference *C* and a semi-major axis *a* and a semi-minor axis *b* becomes an a ‘circular’ ellipse with a constant circumference *C*

$\boldsymbol{n = \left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}}$ and $\boldsymbol{m = \frac{a}{b} \cdot \left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}}$

When the ellipse is ‘flat’

$n \cdot a = a_1$

$m \cdot b = b_1 = 0$

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{m \cdot b}{n \cdot a}} n \cdot a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1 = 4\left(\frac{\pi}{2}\right)^{\frac{0}{a_1}} a_1 = 4\, a_1}$$

$\leftrightarrow$

$$\boldsymbol{4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\, a_1}$$

$\leftrightarrow$

$$\boldsymbol{4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\, n \cdot a}$$

$\leftrightarrow$

$$\boldsymbol{\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = n}$$

Given

$m \cdot b = b_1 = 0$

So

$$\boldsymbol{m = 0}$$

So, when an ellipse with circumference *C* and a semi-major axis *a* and a semi-minor axis *b* becomes an a ‘flat’ ellipse with a constant circumference *C*

$$\boldsymbol{n = \left(\frac{\pi}{2}\right)^{\frac{b}{a}}} \text{ and } \boldsymbol{m = 0}$$

So, we can say that the limits of $\boldsymbol{n}$ and $\boldsymbol{m}$ are

$$\boldsymbol{n} \in [\left(\frac{\pi}{2}\right)^{\boldsymbol{\frac{b}{a}-1}}; \left(\frac{\pi}{2}\right)^{\boldsymbol{\frac{b}{a}}}] \text{ and } \boldsymbol{m} \in [\boldsymbol{0}; \boldsymbol{\frac{a}{b}} \cdot \left(\frac{\pi}{2}\right)^{\boldsymbol{\frac{b}{a}-1}}]$$

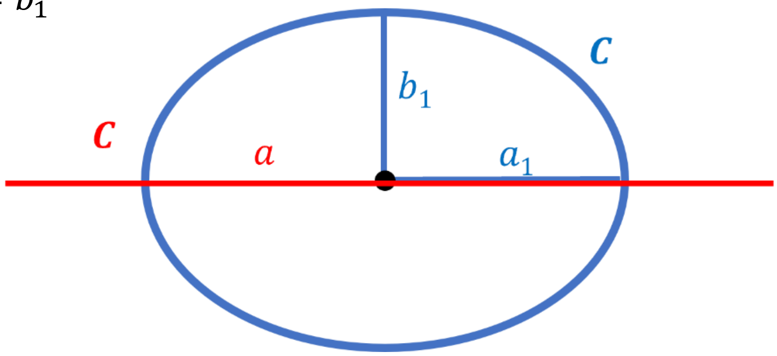
When, $\boldsymbol{n} = \left(\frac{\pi}{2}\right)^{\boldsymbol{\frac{b}{a}-1}}$, $\boldsymbol{m} = \boldsymbol{\frac{a}{b}} \cdot \left(\frac{\pi}{2}\right)^{\boldsymbol{\frac{b}{a}-1}}$**,** and when $\boldsymbol{n} = \left(\frac{\pi}{2}\right)^{\boldsymbol{\frac{b}{a}}}$, $\boldsymbol{m = 0}$

⚠ However, there is a problem with this limit when the ellipse is ‘flat’:⚠

In fact, if we start with a 'flat' ellipse, with the circumference C and semi-major axis a and semi-minor axis b, where $b = 0$, and we want to find a varied ellipse with the same circumference C and semi-major axis a_1 and semi-minor axis b_1 such as

$n \cdot a = a_1$

$m \cdot b = m \cdot 0 = b_1$



There is no solution where $b_1 > 0$, given that for any value of m, $m \cdot b = m \cdot 0 = 0$

So then, we have to exclude the 'flat' ellipse from the group of varied ellipses with the same circumference, and use the 'flat' ellipse only as a limit where:

$$\boldsymbol{m \to 0} \text{ as } \boldsymbol{n} \to \left(\frac{\pi}{2}\right)^{\frac{b}{a}}$$

So, the limits of $\boldsymbol{n}$ and $\boldsymbol{m}$ become

$$\boldsymbol{n} \in \left[\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}; \left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right[\text{ and } \boldsymbol{m} \in \left]\boldsymbol{0}; \frac{a}{b}\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}\right]$$

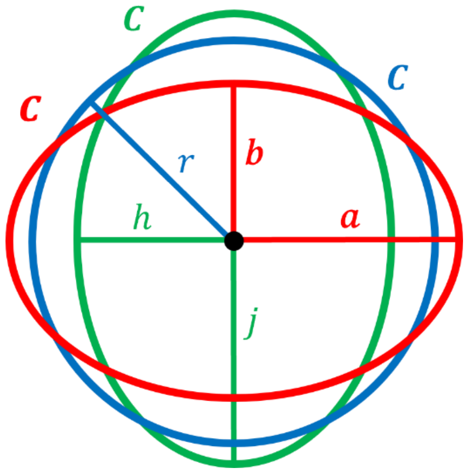
So, an ellipse with circumference C, and semi-major axis a and semi-minor axis b, where $a > 0$ and $b > 0$, becomes an ellipse with a semi-major axis a_1 and a semi-minor axis b_1 and with the same circumference C with,

$$n \cdot a = a_1 \text{ and } m \cdot b = b_1, \text{ where } \boldsymbol{n} \in \left[\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}; \left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right[\text{ and } \boldsymbol{m} \in \left]\boldsymbol{0}; \frac{a}{b}\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}\right]$$

Also, by definition, in an ellipse the semi-major axis is longer than the semi-minor axis, so in an ellipse with circumference C, and semi-major axis a and semi-minor axis b, $\boldsymbol{a > b}$.

However, if we accept that a circle with circumference C can be called a 'circular' ellipse since it can be transformed to an ellipse with the same circumference C, then we can re-define a as $\boldsymbol{a \geq b}$.

V- Confirmation of limits



It might be attempting to think that all ellipses with the same circumference C, with varied semi-major axis a and semi-minor axis b are NOT limited by the 'circular' ellipse, and so it exists another ellipse with the same circumference C, with a semi-major axis h and semi-minor axis j such as

$n \cdot a = h$ where $0 < h < b$

$m \cdot b = j$ where $j > a$

$j > h$

So, the length of the semi-major axis becomes shorter than the semi-minor axis.

⚠ However, this proposition is wrong the following reasons:⚠

- By definition, in an ellipse the semi-major axis distance is longer than the semi-minor axis distance, except in a 'circular' ellipse where they are equal, and these distances are entered as follow in the proposed formula:

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{[\text{semi}-\text{minor axis distance}]}{[\text{semi}-\text{major axis distance}]}}} [\text{semi} - \text{major axis distance}]$$

- If we accept that among all ellipses with the same circumference C, with varied semi-major axis a and semi-minor axis b, it exists another ellipse with the same circumference C, with a semi-major axis h and semi-minor axis j such as $j > h$

Then we can imagine the following scenario:

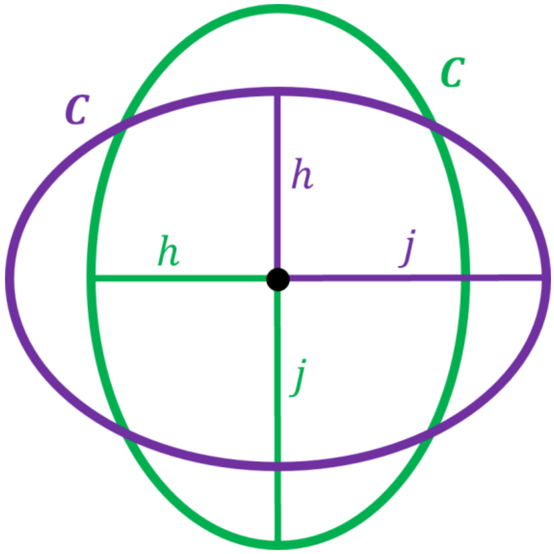
$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{m \cdot b}{n \cdot a}} n \cdot a = 4\left(\frac{\pi}{2}\right)^{\frac{j}{h}} h}$$

However, among the ellipses with varied semi-major axis a and semi-minor axis b and with the same circumference C comprised between a 'flat' ellipse excluded and a 'circular' ellipse included, it exists an ellipse with a semi-major axis j and semi-minor axis h such as

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{h}{j}} j}$$

So, **if this is true**, then

$$\boldsymbol{4\left(\frac{\pi}{2}\right)^{\frac{j}{h}} h = 4\left(\frac{\pi}{2}\right)^{\frac{h}{j}} j}$$



$\leftrightarrow$

$$\boldsymbol{\frac{4\left(\frac{\pi}{2}\right)^{\frac{j}{h}} h}{4\left(\frac{\pi}{2}\right)^{\frac{h}{j}} j} = 1}$$

$\leftrightarrow$

$$\boldsymbol{\frac{\left(\frac{\pi}{2}\right)^{\frac{j}{h}}}{\left(\frac{\pi}{2}\right)^{\frac{h}{j}}} \times \frac{h}{j} - 1 = 0}$$

Let's write

$$\boldsymbol{\frac{h}{j} = k}$$

Which gives us

$$\boldsymbol{\frac{4\left(\frac{\pi}{2}\right)^{\frac{1}{k}}}{\left(\frac{\pi}{2}\right)^{k}} k - 1 = 0}$$

$\leftrightarrow$

$$\boldsymbol{\left(\frac{\pi}{2}\right)^{\frac{1}{k}-k} k - 1 = 0}$$

So, **if** for **all values** of k, with $k > 0$ and $k = \frac{h}{j}$,

$$\boldsymbol{\left(\frac{\pi}{2}\right)^{\frac{1}{k}-k} k - 1 = 0}$$

Then the following equation would be true

$$\boldsymbol{4\left(\frac{\pi}{2}\right)^{\frac{j}{h}} h = 4\left(\frac{\pi}{2}\right)^{\frac{h}{j}} j}$$

Which would mean that all the ellipses with the same circumference C, with varied semi-major axis a and semi-minor axis b are NOT limited by the 'circular' ellipse, and so it exists other ellipses with the same circumference C, with a semi-major axis h and semi-minor axis j such as

$n \cdot a = h$ where $0 < h < b$

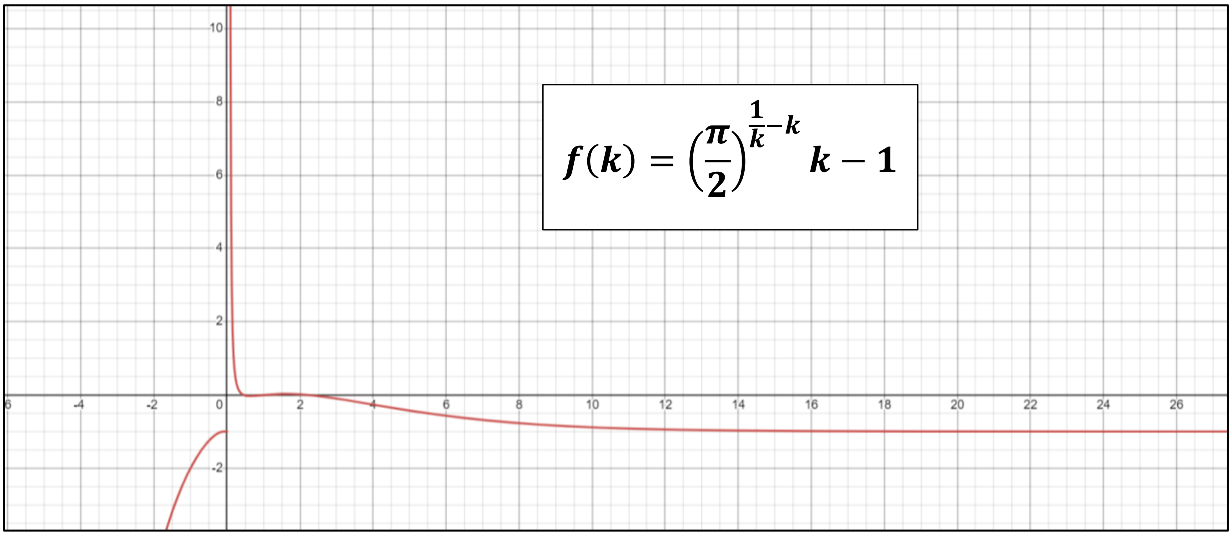
$m \cdot b = j$ where $j > a$

$j > h$

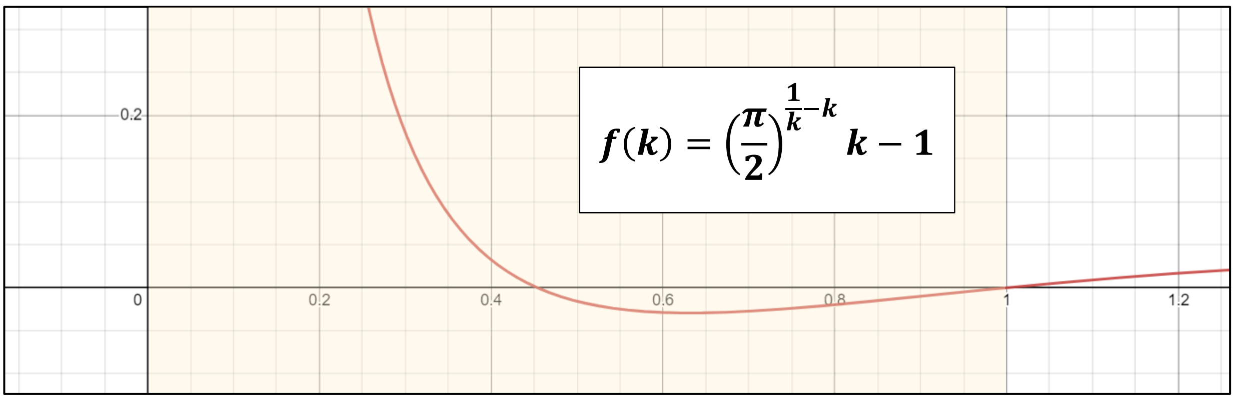
We can suggest to analyze graphically the function

$$\boldsymbol{f(k) = \left(\frac{\pi}{2}\right)^{\frac{1}{k}-k} k - 1}$$

and find when $\boldsymbol{f(k) = 0}$



Remember that $\boldsymbol{k = \frac{h}{j}}$ with $\boldsymbol{j > h > 0}$, so $\boldsymbol{k \in \,]0; 1[}$, so we can observe within the domain of k.



We can see that $f(k)$ is not equal to 0 for all values of k. So, $\boldsymbol{4\left(\frac{\pi}{2}\right)^{\frac{j}{h}} h = 4\left(\frac{\pi}{2}\right)^{\frac{h}{j}} j}$ is not true all the time.

Therefore, all ellipses with the same circumference C with varied semi-major axis a and semi-minor axis b, are **comprised** between a ‘flat’ ellipse excluded and ‘circular’ ellipse included, such as,

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{m\cdot b}{n\cdot a}} n \cdot a}$$

Where

$$\boldsymbol{a \geq b}$$

$$\boldsymbol{b > 0}$$

$$\boldsymbol{n \in [\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1} ; \left(\frac{\pi}{2}\right)^{\frac{b}{a}} [}$$

$$\boldsymbol{m \in]0; \frac{a}{b}\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}]}$$

VI- Variations of the semi-major axis and of the semi-minor axis

We can transform an ellipse with a circumference C and a with semi-major axis a and semi-minor axis b into another ellipse with the same circumference C with a semi-major axis a_1 and semi-minor axis b_1 such as

$n \cdot a = a_1$ with $\boldsymbol{n \in [\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1} ; \left(\frac{\pi}{2}\right)^{\frac{b}{a}} [}$

$m \cdot b = b_1$ with $\boldsymbol{m \in]0; \frac{a}{b}\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}]}$

Where

$\boldsymbol{m \to 0}$ as $\boldsymbol{n \to \left(\frac{\pi}{2}\right)^{\frac{b}{a}}}$ and $\boldsymbol{m \to \frac{a}{b}\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}}$ as $\boldsymbol{n \to \left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}}$

or

$\boldsymbol{n \to \left(\frac{\pi}{2}\right)^{\frac{b}{a}}}$ as $\boldsymbol{m \to 0}$ and $\boldsymbol{n \to \left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}}$ as $\boldsymbol{m \to \frac{a}{b}\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}}$

We can try to find the value of m for a given value of n, and vice versa, such as

$$\boldsymbol{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1 = 4\left(\frac{\pi}{2}\right)^{\frac{m\cdot b}{n\cdot a}} n \cdot a}$$

➔ Let's find the value of $\boldsymbol{m}$ for a given factor $\boldsymbol{n}$, where

$a_1 = n \cdot a$ with $\boldsymbol{n} \in \left[\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1} ; \left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right[$

$b_1 = m \cdot b$

$$4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1 = 4\left(\frac{\pi}{2}\right)^{\frac{m \cdot b}{n \cdot a}} n \cdot a$$

$\Leftrightarrow$

$$4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{m \cdot b}{n \cdot a}} n \cdot a$$

$\Leftrightarrow$

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = \left(\frac{\pi}{2}\right)^{\frac{m \cdot b}{n \cdot a}} n$$

$\Leftrightarrow$

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = \left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)^{\frac{m}{n}} n$$

$\Leftrightarrow$

$$\frac{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}}{n} = \left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)^{\frac{m}{n}}$$

$\Leftrightarrow$

$$\frac{m}{n} = \log_{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}} \frac{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}}{n}$$

$\Leftrightarrow$

$$\frac{m}{n} = \log_{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}} \left(\frac{\pi}{2}\right)^{\frac{b}{a}} - \log_{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}} n = 1 - \log_{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}} n = 1 - \frac{a}{b} \log_{\left(\frac{\pi}{2}\right)} n$$

$\Leftrightarrow$

$$\frac{m}{n} = 1 - \frac{a}{b} \log_{\left(\frac{\pi}{2}\right)} n$$

$\Leftrightarrow$

$$m = n\left(1 - \frac{a}{b} \log_{\left(\frac{\pi}{2}\right)} n\right)$$

➔ Let's find the value of $\boldsymbol{n}$ for a given factor $\boldsymbol{m}$, where

$b_1 = m \cdot b$ with $\boldsymbol{m} \in \left]\mathbf{0}; \frac{\boldsymbol{a}}{\boldsymbol{b}}\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}-\mathbf{1}}\right]$

$a_1 = n \cdot a$

$$4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1 = 4\left(\frac{\pi}{2}\right)^{\frac{m\cdot b}{n\cdot a}} n \cdot a$$

$\Leftrightarrow$

$$4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{m\cdot b}{n\cdot a}} n \cdot a$$

$\Leftrightarrow$

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = \left(\frac{\pi}{2}\right)^{\frac{m\cdot b}{n\cdot a}} n$$

$\Leftrightarrow$

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = \left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)^{\frac{m}{n}} n$$

At this step, we must prepare the equation for the Lamber W function format

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = e^{\ln\left(\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)^{\frac{m}{n}}\right)} n$$

$\Leftrightarrow$

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = e^{\frac{m}{n}\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)} n$$

$\Leftrightarrow$

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = e^{\frac{m\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{n}} n$$

Let's write

$$-u = \frac{m\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{n} \quad \text{and} \quad n = -\frac{m\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{u}$$

$\Leftrightarrow$

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = e^{-u}\left(-\frac{m\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{u}\right)$$

$\Leftrightarrow$

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = \frac{1}{e^{u}}\left(-\frac{m\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{u}\right)$$

$\Leftrightarrow$

$$\left(\frac{\pi}{2}\right)^{\frac{b}{a}} = -\frac{m\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{e^{u}u}$$

We can re-write the equation with the Lambert format

$$e^{u}u = -\frac{m\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}}$$

So now we can solve for:

$$\mathbf{W}(e^{u}u) = \mathbf{W}\left(-\frac{m\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}}\right) = u$$

$\Leftrightarrow$

$$\mathbf{W}\left(-\frac{m\ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}}\right) = u$$

Remember that

$$-\boldsymbol{u} = \frac{\boldsymbol{m}\ \mathbf{ln}\left(\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}\right)}{\boldsymbol{n}}$$

So

$$\boldsymbol{u} = -\frac{\boldsymbol{m}\ \mathbf{ln}\left(\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}\right)}{\boldsymbol{n}}$$

We can then replace $\boldsymbol{u}$ and then solve for $\boldsymbol{n}$

$$\mathbf{W}\left(-\frac{\boldsymbol{m}\ \mathbf{ln}\left(\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}\right)}{\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}}\right) = -\frac{\boldsymbol{m}\ \mathbf{ln}\left(\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}\right)}{\boldsymbol{n}}$$

$\leftrightarrow$

$$\boldsymbol{n} = -\frac{\boldsymbol{m}\ \mathbf{ln}\left(\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}\right)}{\mathbf{W}\left(-\frac{\boldsymbol{m}\ \mathbf{ln}\left(\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}\right)}{\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}}\right)}$$

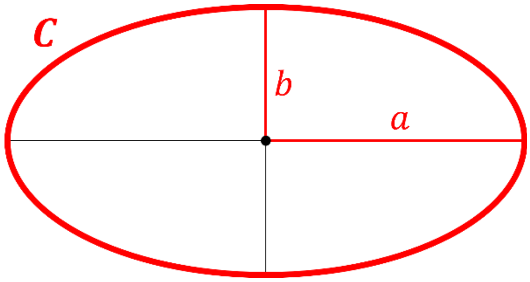
Review

- for a given factor $\boldsymbol{n}$, where $\boldsymbol{n} \in [\left(\frac{\pi}{2}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}-\mathbf{1}} ; \left(\frac{\pi}{2}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}} [$, $\boldsymbol{m} = \boldsymbol{n}\left(\mathbf{1} - \frac{\boldsymbol{a}}{\boldsymbol{b}}\mathbf{log}_{\left(\frac{\pi}{2}\right)}\boldsymbol{n}\right)$

- for a given factor $\boldsymbol{m}$, where $\boldsymbol{m} \in]\mathbf{0}; \frac{\boldsymbol{a}}{\boldsymbol{b}}\left(\frac{\pi}{2}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}-\mathbf{1}}]$, $\boldsymbol{n} = -\frac{\boldsymbol{m}\ \mathbf{ln}\left(\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}\right)}{\mathbf{W}\left(-\frac{\boldsymbol{m}\ \mathbf{ln}\left(\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}\right)}{\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right)^{\frac{\boldsymbol{b}}{\boldsymbol{a}}}}\right)}$

VII- Summary

The circumference C of an ellipse with a semi-major axis a and a semi-minor axis b is

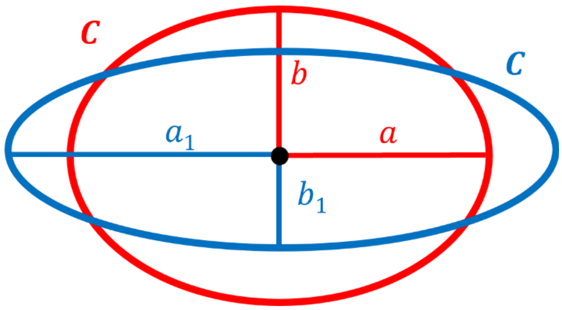


$$\boxed{C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a} \quad \text{with } a \geq b \text{ and } b > 0$$

All ellipses with the same circumference C, with varied semi-major axis a and semi-minor axis b are comprised between a ‘flat’ ellipse excluded and a ‘circular ellipse’ included.

An ellipse with circumference C and a semi-major axis a and a semi-minor axis b, can be transformed into an ellipse with the same circumference C and a semi-major axis a_1 and semi-minor axis b_1 such as

$$C = 4\left(\frac{\pi}{2}\right)^{\frac{b}{a}} a = 4\left(\frac{\pi}{2}\right)^{\frac{b_1}{a_1}} a_1 = 4\left(\frac{\pi}{2}\right)^{\frac{m \cdot b}{n \cdot a}} n \cdot a$$



With

$a_1 = n \cdot a$ where $\boldsymbol{n} \in \left[\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}; \left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right[$

$b_1 = m \cdot b$ where $\boldsymbol{m} \in \left]0; \frac{a}{b}\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}\right]$

If a changes by a factor n, so b changes by a factor m, and vice-versa.

- for a given factor $\boldsymbol{n}$, where $\boldsymbol{n} \in \left[\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}; \left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right[$

$$m = n\left(1 - \frac{a}{b}\log_{\left(\frac{\pi}{2}\right)} n\right)$$

- for a given factor $\boldsymbol{m}$, where $\boldsymbol{m} \in \left]0; \frac{a}{b}\left(\frac{\pi}{2}\right)^{\frac{b}{a}-1}\right]$

$$n = -\frac{m \ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{\mathrm{W}\left(-\frac{m \ln\left(\left(\frac{\pi}{2}\right)^{\frac{b}{a}}\right)}{\left(\frac{\pi}{2}\right)^{\frac{b}{a}}}\right)}$$

VIII- Other application – The observable circumference of a circle

The ‘observable circumference’ of a circle is the circumference of a circle observed from a different angle or point of view

If we draw a circle *c* with a radius *r* on the *y* plane, and rotate the *y* plane by an angle θ around the *x* axis, the observable circumference *C* of the circle *c* observed from the *z* axis passing through its center is:

$$C = 4\left(\frac{\pi}{2}\right)^{|\cos\theta|} a$$

where, θ is the angle in degrees, and $a > 0$