Identification with External Instruments in Structural VARs under Partial Invertibility

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Abstract

This paper discusses the conditions for identification in SVAR-IVs when only the shock of interest or a subset of the structural shocks can be recovered as a linear combination of the VAR residuals. This condition of partial invertibility is very general, often of empirical relevance, and less stringent than the standard full invertibility that is routinely assumed in the SVAR literature. We show that, under partial invertibility, the dynamic responses can be correctly recovered using an external instrument even when this correlates with leads and lags of other invertible shocks. We call this a limited lead-lag exogeneity condition. We evaluate our results in a simulated environment, and provide an empirical application to the case of monetary policy shocks.

Keywords: Identification with External Instruments; Structural VAR; Invertibility; Monetary Policy Shocks.
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1 Introduction

A central endeavour in empirical macroeconomics is the study of the dynamic causal effects that structural shocks have on macroeconomic variables. Since Sims (1980), this has been typically accomplished with Structural VARs (SVARs). An almost always maintained assumption in the SVAR literature is that of 'invertibility', or 'fundamentalness' of the structural shocks, given the chosen model. If this assumption holds, all the structural shocks can be recovered from the current and lagged values of the observables included in the VAR. In fact, under invertibility the VAR innovations are a linear combination of all the contemporaneous structural shocks and, given the variance-covariance matrix of the residuals, causal effects are identified up to an orthogonal matrix that defines the contemporaneous relationships. A lot of creativity in the SVAR literature has been devoted to the formulation of appropriate identifying assumptions to inform the choice of this orthogonal matrix. The structural moving average, obtained by inverting the identified SVAR, allows inference on the dynamic causal effects of the structural shocks, represented in the form of impulse response functions (IRFs).

In contrast with standard statistical identifications, an important advancement in the more recent practice has seen the adoption of instrumental variables for the identification of structural shocks.[1] These instruments – that can be thought of as noisy observations of the shocks of interest –, can be used either in conjunction with Structural VARs – as external instruments (SVAR-IV, also called Proxy-SVARs) or as internal instruments and part of the endogenous information set (sometimes referred to as Hybrid VARs) –, or with direct regression methods, such as Jordà (2005)'s Local Projections (LP-IV), with or without controls.

This paper introduces the conditions for identification with external instruments in Structural VARs under the assumption of partial invertibility of the shock of interest. The concept of partial invertibility has been discussed in Sims and Zha (2006), Stock

[1]This rapidly expanding research programme, surveyed in Ramey (2016), has produced, among other applications, a number of instruments for the identification of the effects of monetary policy (e.g. Romer and Romer, 2004; Gürkaynak et al., 2005; Gertler and Karadi, 2015; Cloyne and Hürtgen, 2016; Miranda-Agrippino and Ricco, 2017; Paul, 2017; Hansen et al., 2019; Altavilla et al., 2019), fiscal spending (e.g. Ramey, 2011; Ricco et al., 2016; Ramey and Zubairy, 2018), tax (e.g. Romer and Romer, 2010; Mertens and Ravn, 2012; Cloyne, 2013; Leeper et al., 2013; Mertens and Montiel-Olea, 2018), government asset purchases (Fieldhouse and Mertens, 2017; Fieldhouse et al., 2018), oil (e.g. Hamilton, 2003; Kilian, 2008; Känzig, 2019), productivity news shocks Arezki et al. (2017), and technology news shocks (e.g. Miranda-Agrippino et al., 2018; Cascaldi-Garcia and Vukotic, 2019).

and Watson (2018) and in Forni et al. (2019) and is the empirically relevant condition when only one or a subset of the structural shocks are the object of the study ('partial' identification). Our results generalise the approach of Stock and Watson (2018), that discuss the conditions for identification in both SVAR-IV and LP-IV under the assumption of full invertibility, but also observe that identification can be achieved with IV methods under partial invertibility.[2]

The central contribution of this paper is to show that, in general, fairly weak conditions are required to achieve identification in SVAR-IVs. In particular, other than the standard relevance and contemporaneous exogeneity conditions, under partial invertibility the instrument has to fulfil a limited lead-lag exogeneity condition that ensures that the VAR innovations and the instrument are related only via the contemporaneous structural shock of interest. Importantly, the condition allows the instrument to be contaminated by leads and lags of other partially invertible shocks without compromising the correct identification of the shock of interest. Our results allow to extend the application of SVAR-IV (and LP-IV with controls) to the many empirically relevant cases in which while some of the structural disturbances may be non-invertible, the shock of interest is arguably invertible. This is, for example, the case of monetary policy shocks – i.e. the residuals to a Taylor rule – that are identified in small VAR models for which, arguably, several types of shocks are non fundamental, viz. financial shocks or news shocks. Interestingly, the narrative and high-frequency instruments used in identifying monetary disturbances are reported to be autocorrelated, and hence potentially contaminated by past shocks.

Specifically, we make three contributions. First, we show that under partial invertibility a covariance stationary stochastic vector process admits a 'semi-structural' representation that is the sum of two terms, orthogonal to one another.[3] The first one only depends on the current realisations of the partially invertible shocks. The second instead combines leads and lags of the remaining non-invertible shocks. This result implies that if the VAR lag order correctly captures the autocorrelation structure of the Wold repre-

[2] Stock and Watson (2018) note that direct methods, such as local projections, do not need to explicitly assume invertibility of the system under strict exogeneity of the instrument at all leads and lags. However, if lagged observables are required as control variables for an instrument that violates the lead-lag exogeneity condition, then, in general, the same invertibility conditions of a structural VAR are required. Plagborg-Møller and Wolf (2018b) discuss the cases in which invertibility can be dispensed with for identification of LP-IV with controls and Hybrid VARs.

[3] Forni et al. (2019) provide an independently derived result and show that, under partial invertibility of one shock, the data has a moving average representation in the shock of interest and the residual Wold innovations.

sentation, the impulse response functions obtained from the partially identified structural moving average are the dynamic causal effects of the shock of interest.

Second, we show that under partial invertibility SVAR-IV methods (and LP-IV with controls) achieve identification under much weaker conditions on the external instrument than LP-IV without controls. The existence of a semi-structural representation allows the instrument to be contaminated by leads or lags (but not contemporaneous realisations) of any of the other invertible shocks in the system. We call this requirement a limited lead-lag exogeneity condition. We also derive an explicit formula for the bias in the IRFs that arises when the instrument violates the conditions for identification. Finally, we extend results in Stock and Watson (2018) to show that given an instrumental variable for the shock of interest, Structural VARs and Local Projection methods achieve identification under the same set of conditions, albeit in different settings. Hence, the choice of the empirical specification has to depend on the bias-variance trade-off of the specific application and sample at hand.

Third, we discuss identification of causal effects in the empirically likely cases in which the VAR is misspecified along some dimensions – e.g. inappropriate lag order, missing moving average components, missing variables, and missing higher order terms –, and hence fails to correctly capture the data generating process. While in these cases the dynamic responses will generally be biased, if one can still assume that the VAR is partially invertible in the shock of interest, the impact effects are correctly identified provided that the limited lead-lag exogeneity of the instrument hold. This result provides empirical researchers with a simple heuristic to gauge the contamination of an instrument versus the misspecification of the chosen model. If one can assume partial invertibility across different specifications of an empirical model, an instrument that fulfils the conditions for identification delivers stable impact responses but unstable IRFs across models. In this case, increasing the number of lags and/or using a larger information set can help stabilising the dynamics responses by providing a better approximation of the Wold representation. Conversely, an instrument that violates the lead-lag exogeneity condition is likely to deliver also unstable impact responses across different models. We formalise this intuition with a Hausman type test, along the lines of Lu and White (2014). Under the maintained assumption of partial invertibility of the shock of interest, the test can be used to assess the robustness of the impact response to changes in the VAR specification, a necessary but not sufficient condition for valid IV inference.

We provide an application of our results using artificial data from a stylised standard New-Keynesian DSGE model with price stickiness and four shocks – monetary policy, government spending, technology, and a cost push shock. The simulated system is by construction partially invertible in the monetary policy shock – i.e. the residuals of the Taylor rule. However, due to the introduction of technology news (see e.g. Beaudry and Portier, 2006; Barsky and Sims, 2011), and fiscal foresight (see Ramey, 2011; Leeper et al., 2013), a VAR in output growth, inflation, government spending and the policy interest rate fails the 'poor man's invertibility condition' of Fernandez-Villaverde et al. (2007), and is hence unable to recover all the four shocks. We use this simulated environment to study the identification of monetary policy shocks with external instruments. Our results validate our discussion. Under partial invertibility, an instrument contaminated by leads or lags of an invertible shock correctly recovers impacts and dynamic responses to the shock of interest, provided that the VAR correctly captures the variables' dynamics. If, instead, the instrument is contaminated by a non-invertible shock, the degree of distortion in the estimated IRFs depends on how pervasive the contaminating shock is, that is, on how much of the variance in the system it accounts for.

Lastly, we provide an empirical application of our results by examining popular instruments for the identification of monetary policy shocks in monthly VARs for US data. We consider two VARs for which we assume partial invertibility of conventional monetary policy shocks, and three variants of the high-frequency instruments popularised by Gürkaynak et al. (2005). We show that two of these are likely to fail the limited lead-lag exogeneity condition, and as a consequence recover impact responses of output and prices that are strongly dependent on the VAR of choice. The third instrument, constructed as in Miranda-Agrippino and Ricco (2017) with a pre-whitening step to remove correlation with other shocks, recovers impact responses that are invariant to the VAR specification and composition.

This paper builds and expands on the econometric literature supporting the use of IV in macroeconomics. The SVAR-IV techniques were first introduced by Stock (2008), and then explored in Stock and Watson (2012) and Mertens and Ravn (2013). The use of instrumental variables for identification in direct regressions (LP-IV), with or without controls, has been proposed independently by Jordà et al. (2015) and Ramey and Zubairy (2018). The econometric conditions for instruments' validity in the direct regression without control variables have first appeared in lecture notes by Mertens (2014). Stock and Watson (2018) have provided a unified discussion of the use of external instruments in macroeconomics, discussed the conditions for instruments validity with control variables and relation to full invertibility, and explored the connections between SVAR-IV and LP-IV methods. Recently, Arias et al. (2018) have proposed algorithms for exact finite sample inference for SVAR-IV when multiple instruments are employed to identify more than one shock.

This paper adds to the small but important econometric literature that has strived to clarify the conditions and limits under which macroeconomic structural shocks and their dynamic effects can be identified in empirical reduced form models (for a recent discussion see Canova and Ferroni, 2019). A strand of this literature has focused on the link between the conditions for invertibility of structural shocks and the information included in VARs, e.g. Giannone and Reichlin (2006), Forni and Gambetti (2014), and Canova and Hamidi Sahneh (2017). A more recent literature has furthered our understanding of the identification problem when the system is not fully invertible but the shocks of interest can be revealed by linear combinations of current and past observations (i.e. are 'partially invertible'), can be revealed only to some degree of approximation (i.e. are 'approximately invertible', as in Sims and Zha, 2006), or can be recovered using future observables as well (the 'recoverability' concept proposed by Chahrour and Jurado, 2017). Our work slots into this effort by clarifying conditions for IV identification in SVAR models under partial invertibility (our results readily generalise to the case of approximate invertibility). In doing this, this paper is close in spirit to Forni et al. (2019) that study the conditions under which a SVAR is informative enough to estimate the dynamic effects of a shock, and to Plagborg-Møller and Wolf (2018b) that clarify the equivalence of SVAR and LP methods and address the validity of external instrument identification in the invertible and non-invertibile cases. While we share the emphasis on partial invertibility (referred to in Forni et al. 2019 as informational sufficiency), our paper focuses on the recent debate on the use of IV in empirical macro, and on its interaction with misspecifications in the modelling choices. Differently from Plagborg-Møller and Wolf (2018b), we focus on the conditions under which partially invertible shock are identifiable with SVAR with external instruments. However, we do not discuss the important issue of the inference on forecast variance decompositions with instrumental variables, for which bounds are provided by Plagborg-Møller and Wolf (2018a).

The paper is organised as follows. In Section 2 we introduce the notation and review the concepts of full invertibility and fundamentalness and some other useful results in the literature; a reader familiar with these concepts can skip the section. Sections 3 to 5 collect our main results. In Section 3 we discuss partial invertibility, and prove the existence of a semi-structural representation for covariance-stationary vector processes. We lay out the conditions for the identification of structural shocks in SVAR-IV under partial invertibility of the shock of interest in Section 4, while Section 5 compares the conditions for SVAR-IV with those required in LP-IV with controls. In Section 6 we discuss the challenges to identification and estimation of the IRFs in the case of misspecified systems. In Section 7 we provide a formal Hausman type test for violation of the limited lead-lag exogeneity condition, and for potential model misspecification. We apply the concepts discussed in this paper to artificial data from a NK-DSGE in Section 8, and in an empirical application in Section 9. Finally, Section 10 concludes.

2 Non-Fundamental Representations

To introduce the concept of non-fundamentalness, let us consider a covariance-stationary $n \times 1$ vector stochastic process Y_t, for $t \in \mathbb{Z}$, with rational spectral density and belonging to a Hilbert space $L^2(\Omega, \mathcal{F}, P)$ for some probability space $(\Omega, \mathcal{F}, P)$.[4] We define the Hilbert space generated by all the observations of Y_t up to time t as $\mathcal{H}_t^Y = \overline{\text{span}}\{Y_{t-j}, j \geq 0\}$. The process Y_t is a linear process and a VARMA(p,q) if it is stationary solution of the stochastic difference equation

$$\Phi(L)Y_t = \Psi(L)u_t \qquad u_t \sim \mathcal{WN}(0, \Sigma_u) \ , \tag{1}$$

[4] In the economic literature, the issue of non-fundamentalness (see Rozanov, 1967; Hannan, 1970) was first pointed out by Hansen and Sargent (1980, 1991) in a purely theoretical setting, while Lippi and Reichlin (1993, 1994) provided the first empirical application. Other more recent contributions on fundamentalness in macro models are in Chari et al. (2004), Christiano et al. (2007) and Fernandez-Villaverde et al. (2007). A useful review is in Alessi et al. (2011).

where $\Phi(L)$ and $\Psi(L)$ are generic autoregressive (AR) and moving average (MA) filters of order p and q respectively

$$\Phi(L) = \sum_{i=0}^{p} \Phi_i L^i \ , \qquad \Psi(L) = \sum_{i=0}^{q} \Psi_i L^i, \tag{2}$$

and u_t are the stochastic disturbances of the data generating process (i.e. the 'structural shocks' in the economic jargon), generally assumed to be orthogonal or orthonormal processes. If the process is causal – i.e., $det(\Phi(L))$ has all roots outside the unit circle, $det(\Phi(z)) \neq 0 \ \forall z = \zeta_i$ such that $|\zeta_i| < 1$ –, then it can be written as a (possibly infinite) MA in the structural shocks u_t

$$Y_t = \Theta(L) u_t, \qquad u_t \sim \mathcal{WN}(0, \Sigma_u). \tag{3}$$

Definition 1 (Invertibility and Fundamentalness). *Let Y_t be defined as in Eq. (1), and with structural MA representation as in Eq. (3).*

(i) If $det(\Psi(z))$ – and hence $det(\Theta(z))$ – has all roots outside the unit circle, i.e.

$$det(\Theta(z)) \neq 0, \qquad \forall z = \zeta_i \ s.t. \ |\zeta_i| < 1, \tag{4}$$

then the process in Eq. (1) is said to be invertible, and u_t are said to be Y_t-fundamental (i.e. $\mathcal{H}_t^Y = \mathcal{H}_t^u$ and the stochastic disturbances can be recovered from current and past realisation of the process Y_t). Y_t can be written in VAR form as

$$A(L) Y_t = \Theta_0 u_t \ , \tag{5}$$

where Θ_0 is an n-dimensional matrix.

(ii) If $det(\Theta(z))$ has at least one root inside the unit circle, then the process in Eq. (1) is 'non-invertible', and u_t is said to be Y_t–non-fundamental (i.e. $\mathcal{H}_t^Y \subset \mathcal{H}_t^u$).

(iii) If $det(\Theta(L))$ has at least one root on the unit circle, the process is said to be non-invertibile, but u_t are Y_t-fundamental ($\mathcal{H}_t^Y = \mathcal{H}_t^u$).

The Wold Representation Theorem guarantees that Y_t always admits a Wold decomposition of the form

$$Y_t = C(L)\nu_t \qquad \nu_t \sim \mathcal{WN}(0, \Sigma_\nu), \tag{6}$$

where $C(L) = \sum_j C_j L^j$ is a causal (i.e. no terms with $C_j \neq 0$ for $j < 0$), time-independent, square summable filter with $C_0 = \mathbb{I}_n$.[5] ν_t is the Wold innovation process – an uncorrelated sequence – to Y_t

$$\nu_t = Y_t - Proj(Y_t|Y_{t-1}, Y_{t-2}, \dots) \ , \tag{7}$$

that, by definition, belongs to the space generated by present and past values of Y_t (i.e. $\mathcal{H}_t^\nu = \mathcal{H}_t^Y$, since we are assuming Y_t to be a purely non-deterministic process). Given the invertibility of $C(L)$, we can rewrite Eq. (6) in VAR form

$$A(L)Y_t = \nu_t \qquad A_0 = \mathbb{I}_n \ . \tag{8}$$

If the Wold representation has absolute summable coefficients, then it admits a VAR representation with coefficient matrices that decay to zero rapidly; hence, it can be well approximated by a finite order VAR. This is always the case for causal finite-order ARMA processes.

If the structural shocks u_t are Y_t–fundamental, then u_t and ν_t generate the same space ($\mathcal{H}_t^u = \mathcal{H}_t^\nu, \ \forall t$). This implies that

$$\nu_t = \Theta_0 u_t \ , \tag{9}$$

where Θ_0 is non-singular. Hence, the structural disturbances u_t can be determined from current and lagged values of Y_t

$$u_t = Proj(u_t|Y_t, Y_{t-1}, \dots) \ . \tag{10}$$

If, however, the process is not invertible, and u_t is not Y_t–fundamental, the space generated by the VAR innovations does not coincide with that spanned by the structural shocks, i.e. $\mathcal{H}_t^\nu \subset \mathcal{H}_t^u$.[6] The following result guarantees that the Wold and the structural

[5] The Wold Theorem guarantees that any weakly stationary process Y_t can be written as $Y_t = \eta_t + C(L)\nu_t$, where η_t is a purely deterministic component. Without loss of generality, in what follows we disregard the possible presence of deterministic terms in order to focus on purely non-deterministic processes.

[6] Non-fundamentalness also naturally arises in systems in which the dimension of the vector Y_t (and hence of ν_t) is smaller than that of u_t. We provide a discussion of a related case when examining the

MA representations (Eq. 3) are connected by a class of transformations generated by means of Blaschke matrices.

Theorem 1 (Non-fundamental Representations). *Let Y_t be a covariance-stationary vector process with rational spectral density, i.e. an ARMA process. Let $Y_t = C(L)\nu_t$ be a fundamental representation of Y_t, i.e.*

(i) ν_t is a white noise vector;

(ii) $C(L)$ is a matrix of rational functions in L with no poles of modulus smaller or equal to unity (Causality);

(iii) $\det(C(L))$ has no roots of modulus smaller than unity (Invertibility).

Let $Y_t = \Theta(L)u_t$ be any other MA representation, i.e. one which fulfils (i), and (ii), but not necessarily (iii). Then

$$\Theta(L) = C(L)B(L) \ ,$$

where $B(L)$ is a Blaschke matrix.

Blaschke matrices are filters capable to flip the roots of a fundamental representation inside the unit circle (see Lippi and Reichlin, 1994). A complex-valued matrix $B(z)$ is a Blaschke matrix if: (i) It has no poles inside the unit circle; (ii) $B(z)^{-1} = \overline{B}'(z^{-1})$, where $\overline{B}$ indicates the complex conjugation.[7] The following result guarantees that any Blaschke matrix can be written as the product of orthogonal matrices, and matrices with a Blaschke factor as one of their entries.

Theorem 2 (Blaschke Factors). *Let $B(z)$ be an $n \times n$ Blaschke matrix, then $\exists\, m \in \mathbb{N}$ and $\exists\, \zeta_i \in \mathbb{C}$ for $i = 1, \dots, m$ such that*

$$B(z) = \prod_{i=1}^{m} K(\zeta_i, z) R_i \ , \tag{11}$$

where R_i are orthogonal matrices, i.e. $R_i R_i' = \mathbb{I}_n$, and

$$K(\zeta_i, z) = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & \dfrac{z - \zeta_i}{1 - \overline{\zeta}_i z} \end{pmatrix} \ , \tag{12}$$

implications of misspecifications in VAR models in Section 6.

[7] See Lippi and Reichlin (1994) for a proof of Theorems 1 and 2.

are matrices with a Blaschke factor as one of the entries.

The above results indicate that in general we can connect the structural and the Wold representation using a Blaschke matrix $B(L)$, that is

$$Y_t = \Theta(L)u_t = \Theta(L)B(L)^{-1}B(L)u_t = C(L)\nu_t, \tag{13}$$

where $B(L)$ flips the roots of the Wold fundamental representation inside the unit circle to obtain the structural MA. Hence,

$$\nu_t = B(L)u_t \ , \tag{14}$$

where we incorporate incorporate into $B(L)$ possibly also a constant scale matrix. In the case in which the structural representation is invertible, $B(L)$ is just Θ_0.

It is important to observe that, as it is clear from Eqs. (11-12), Blaschke factors may be acting only on a subset of the shocks. The remaining shocks can be recovered from current and past realisations of the variables, and are hence invertible. We discuss this relevant case in the next section.

3 Partial Invertibility

The property of invertibility guarantees identifiability of the dynamic effects of all the structural disturbances in a correctly specified VAR. In such a case, the problem of identification amounts to finding the correct matrix Θ_0 that connects the VAR residuals to the structural shocks as in Eq. (9). However, phenomena such as anticipation and foresight of economic shocks, which are often a feature of rational expectation models, can generate non-invertible representations (see e.g. Leeper et al., 2013). In such cases, the search for the correct Blaschke matrix can be a daunting problem (see Lippi and Reichlin, 1994).

In most empirical applications, however, often only one or a subset of the structural innovations is of interest. For example, one may want to identify only a monetary policy shock, or an oil price shock. This is the case of 'partial identification', when only a subset of the column entries of the matrix that maps the Wold residuals into the structural shocks has to be recovered. In such a setting, the relevant condition is that of partial invertibility in the subset of the shocks of interest (see also Sims and Zha, 2006, Stock and Watson, 2018 and Forni et al., 2019).

Definition 2 (Partial Invertibility). *Let Y_t be a covariance-stationary $n \times 1$ vector stochastic process, with rational spectral density, solution to the ARMA equation $\Phi(L)Y_t = \Psi(L)u_t$, where u_t is an $n \times 1$ vector of stochastic disturbances (structural shocks) with $u_t \sim \mathcal{WN}(0, \mathbb{I}_n)$. Y_t admits a Wold representation of the form $Y_t = C(L)\nu_t$ for a vector of innovations $\nu_t \sim \mathcal{WN}(0, \Sigma_\nu)$. Without loss of generality, let u_t^1 denote the first entry of u_t. The structural shock u_t^1 is invertible and Y_t–fundamental if*

$$u_t^1 = Proj(u_t^1 | Y_t, Y_{t-1}, \dots) \ . \tag{15}$$

Hence, u_t^1 is a linear combination of the innovations ν_t, that is, there exists an n-dimensional vector λ such that

$$u_t^1 = \lambda' \nu_t \ . \tag{16}$$

For a given VAR model, the property of partial invertibility guarantees that one or, more generally, some of the structural shocks, $u_t^{1:m} = (u_t^1, \dots, u_t^m)'$ for $m < n$, can be correctly recovered as a linear combination of the estimated innovations.[8] While seldom acknowledged, partial invertibility is always implicitly assumed in the empirical macroeconomic literature concerned with evaluating the effects of a specific type of shock – e.g. monetary policy shocks, spending shocks, etc.

The following proposition guarantees the existence of a representation for the covariance stationary vector process Y_t as an infinite moving average of the invertible shocks $u_t^{1:m}$, and of the $n - m$ linear combinations of the Wold innovations orthogonal to $u_t^{1:m}$. This is a key result that allows for the study of the propagation of structural shocks in reduced-form VAR models.

[8]The notion of partial invertibility – i.e. $u_t^i = Proj(u_t^i | \mathcal{H}_t^Y)$ – can be generalised by considering a continuous measure of the degree of invertibility – i.e. approximate invertibility – that is the case in which $0 \neq u_t^i - Proj(u_t^i | \mathcal{H}_t^Y) \neq u_t^i$ (see Sims and Zha, 2006 and Forni et al., 2019). In such a case, a measure of the degree of invertibility is provided by

$$\delta_i = \frac{Var(u_t^i) - Var(Proj(u_t^i | \mathcal{H}_t^Y))}{Var(u_t^i)}.$$

For values close to 1 (i.e. close to partial invertibility), the IRFs obtained from a VAR model can be close the the true ones. A even weaker condition than invertibility is that of recoverability $u_t^i = Proj(u_t^i | \mathcal{H}_\infty^Y)$, i.e. the shock of interest is recoverable from all leads and lags of the endogenous variables (see Chahrour and Jurado, 2017).

Proposition 1 (Semi-structural Moving Average Representation). *Let the n-dimensional covariance stationary vector process Y_t be solution to*

$$\Phi(L)Y_t = \Psi(L)u_t \qquad u_t \sim \mathcal{WN}(0, \mathbb{I}_n) \ , \tag{17}$$

where u_t is an n-dimensional vector of structural innovations, and let $\Psi(L)$ be a non-invertible moving average filter, i.e. $det(\Psi(z)) = 0$ for some ζ_i such that $|\zeta_i| < 1$. Let the Wold representation of Y_t be equal to

$$Y_t = C(L)\nu_t \qquad \nu_t \sim \mathcal{WN}(0, \Sigma_\nu). \tag{18}$$

where Σ_ν is the positive definite variance-covariance matrix of Wold innovations. If the system is partially invertible in the shocks u_t^i, for $i = 1, \ldots, m$, i.e. there exist m vectors λ_i such that $\lambda_i'\nu_t = u_t^i$, then Y_t admits a semi-structural moving average representation of the form

$$Y_t = C(L)\Sigma_\nu \sum_{i=1}^{m} \lambda_i u_t^i + C(L)\Sigma_\nu \tilde{\lambda}\xi_t \ , \tag{19}$$

where ξ_t is an $(n - m) \times 1$ vector of linear combinations of Wold innovations that is orthogonal to all u_t^i for $i = 1, \ldots, m$, i.e. $\mathbb{E}(u_t^i \xi_t') = 0$.

Proof. Let us consider a non singular $n \times n$ matrix $\Lambda = \begin{pmatrix} \lambda & \tilde{\lambda} \end{pmatrix}$, where λ is an $n \times m$ matrix, and $\tilde{\lambda}$ is an $n \times (n - m)$ matrix such that

$$\Lambda'\nu_t = \begin{pmatrix} \lambda' \\ \tilde{\lambda}' \end{pmatrix} \nu_t = \begin{pmatrix} u_t^{1:m} \\ \xi_t \end{pmatrix} \tag{20}$$

and

$$\Lambda'\Sigma_\nu\Lambda = \Lambda'\mathbb{E}\left[\nu_t\nu_t'\right]\Lambda = \mathbb{E}\left[\begin{pmatrix} u_t^{1:m} \\ \xi_t \end{pmatrix} \begin{pmatrix} u_t^{1:m\prime} & \xi_t' \end{pmatrix}\right] = \mathbb{I}_n \ . \tag{21}$$

Eq. (20) allows Λ to enforce the partial invertibility of ν_t in $u_t^{1:m}$, while Eq. (21) guarantees that Λ performs an orthogonalisation of the Wold innovations, such that $\mathbb{E}\left[u_t^i \xi_t'\right] = 0 \ \forall i = 1, \ldots, m$.

It is possible to constructively prove the existence of such a matrix Λ. Since Σ_ν is a symmetric positive-definite real matrix, we can use the Spectral Theorem to write (see, e.g., Sudipto and Roy, 2014)

$$\Sigma_\nu = QDQ' \ ,$$

where D is a diagonal matrix with the n distinct eigenvalues of Σ_ν along the main diagonal, and Q is an orthogonal matrix whose columns are the corresponding eigenvectors. Given that all the eigenvalues are positive, we can further decompose Σ_ν as

$$\Sigma_\nu = QD^{1/2}H(QD^{1/2}H)' \ ,$$

for a generic orthogonal matrix H, parametrised by $n(n-1)/2$ free parameters. Hence, a generic matrix Λ can be expressed as $\Lambda = QD^{-1/2}H$.

It is always possible to choose a column of Λ to be equal to λ_1, by fixing $n-1$ of the free parameters in H. The assumptions of partial invertibility and of unit variance of the structural shocks impose a constraint on λ_1, i.e. $\lambda_1'\Sigma_\nu\lambda_1 = \lambda_1'\mathbb{E}[\nu_t\nu_t']\lambda_1 = \mathbb{E}[u_t^1] = 1$. This leaves $(n-1)(n-2)/2$ free parameters of H spanning the residual group of rotations. It is possible to proceed in a similar manner to fix the remaining $m-1$ columns in the sub-matrix λ. In fact, the assumption of partial invertibility of the second shock u_t^2 imposes the constraint $\lambda_2'\Sigma_\nu\lambda_2 = 1$, while the assumption of orthogonality with the shock u_t^1 imposes an additional constraint $\lambda_2'\Sigma_\nu\lambda_1 = 0$, hence it is necessary to employ $n-2$ of the residual parameters of H, leaving a residual group of rotation with $(n-2)(n-3)/2$ parameters. Proceeding in similar steps for the remaining $m-2$ partially invertible shocks, one obtains the desired matrix Λ.[9] The remaining $(m-1)(m-2)/2$ free parameters of H span the $\mathcal{O}(m-1)$ residual group of rotations in the subspace of $\mathbb{R}^n$ formed by the vectors $\tilde{\lambda}_i$, $i = 1, \ldots, m-1$ conjugate to all the λ_i with respect to Σ_ν, i.e. such that $\lambda'\Sigma_\nu\tilde{\lambda}_i = 0$. Hence, while Λ always exists, it is not unique.

Since $\Sigma_\nu = (\Lambda\Lambda')^{-1}$, it follows that $\Sigma_\nu\Lambda\Lambda' = \mathbb{I}_n$. Using this identity, it is possible to write

$$Y_t = C(L)\nu_t = C(L)\Sigma_\nu\Lambda\Lambda'\nu_t = C(L)\Sigma_\nu(\lambda_1 u_t^1 + \lambda_2 u_t^2 + \ldots \lambda_m u_t^m) + C(L)\Sigma_\nu\tilde{\lambda}\xi_t \ ,$$

[9] More generally, any T such that

$$T'\Sigma_\nu T = \begin{pmatrix} D & 0 \\ 0 & M \end{pmatrix}$$

for D diagonal and $M = M'$ will produce a decomposition of the form of Eq. (19). It is possible to construct such a matrix from the matrix Λ defined above as $T = W\Lambda$, for any W such that $W'W = \begin{pmatrix} D & 0 \\ 0 & M \end{pmatrix}$.

that is the representation in Eq. (19). □

Proposition 1 guarantees that any covariance-stationary vector process Y_t that is solution to Eq. (17) admits the semi-structural MA representation in Eq. (19). In an independently derived result, Forni et al. (2019) propose a moving average equation (their Definition 4) similar to Eq. (19) for the $m = 1$ case, and provide some indication on how to obtain it. Proposition 1 provides a full characterisation for $m \geq 1$ invertible shocks and guarantees the existence of a semi-structural MA representation. Hence it can be thought of as a representation result.

The first term of Eq. (19) depends on the realisations of the invertible shocks u_t^i for $i = 1, \dots, m$. The second term is instead a function of $n - m$ linear combinations of the Wold innovations orthogonal to the invertible shocks, ξ_t. Due to the action of the Blaschke factors, ξ_t will be a convolution of past, current and future non-invertible shocks. It is worth stressing that, while the requirement that ξ_t and the invertible shocks $u_t^{1:m}$ are orthogonal is important, ξ_t does not need to span the space of all the non invertible structural shocks. Hence, while the representation in Eq. (19) always exists, it is not unique, since ξ_t can be redefined by selecting different values for the residuals $(m-1)(m-2)/2$ free parameters of H.

Importantly, Proposition 1 implies that if the VAR has a correctly specified lag order, under partial invertibility, the 'partially' identified SVAR impulse response functions $C(L)\Sigma_\nu \lambda_i u_t^i$ are the dynamic causal effects of the identified m invertible shocks.

Remark 1. *Under partial invertibility, the map between structural shocks and Wold innovations is of the form*

$$\nu_t = B(L)u_t = \begin{pmatrix} b_1 & b_2(L) \end{pmatrix} u_t \ , \tag{22}$$

where b_1 is a $n \times m$ matrix, and $b_2(L)$ is a matrix of dimensions $n \times (n-m)$ that contains Blaschke factors, and m is the number of partially invertible shocks.

Proof. This is a straightforward, since

$$\nu_t = B(L)u_t = \Sigma_\nu \Lambda \begin{pmatrix} u_t^{1:m} \\ \xi_t \end{pmatrix} = (b_1 \ \ b_2(L))\, u_t$$

where the first equality follows from Theorem 1, while the second from Proposition 1. Since b_1 has to be equal to the first m columns of $\Sigma_\nu \Lambda = (\Lambda')^{-1}$, it follows that it is a vector of constants, while $b_2(L)$ contains Blaschke factors mapping non-invertible shocks into the Wold residuals. □

4 IV Identification under Partial Invertibility

Let us consider a partially invertible VAR with reduced-form representation as in Eq. (8), repeated below for convenience

$$A(L)Y_t = \nu_t \qquad A_0 = \mathbb{I}_n \ . \tag{8}$$

Given an external instrument z_t, it is possible to identify u_t^1 and its effects on Y_{t+h}, $h = 0, \ldots, H$, under the conditions in the following proposition.

Proposition 2 (Identification in SVAR-IV under Partial Invertibility). *Let $u_t^{1:m}$ denote the m invertible structural shocks in the system, and $u_t^{m+1:n}$ the remaining $n-m$ non-invertible shocks. Let z_t be an instrument for the shock of interest u_t^1, and define $z_t^{\perp} = z_t - Proj\left(z_t|\mathcal{H}_{t-1}^Y\right)$. If z_t satisfies the following conditions:*

(i) $\mathbb{E}[u_t^1 z_t] = \alpha$ *(Relevance)*

(ii) $\mathbb{E}[u_t^{2:n} z_t^{\perp}] = 0$ *(Contemporaneous Exogeneity)*

(iii) $\mathbb{E}[u_{t-j}^{m+1:n} z_t^{\perp}] = 0$ *for all* $j \neq 0$ *for which* $\mathbb{E}[u_{t-j}^{m+1:n} \nu_t'] \neq 0$. *(Limited Lead-Lag Exogeneity)*

then, in a VAR that correctly captures the autocorrelation structure of the Wold representation of the process of interest, external instrumental variable estimation of the IRFs correctly estimates the causal effects of the shock of interest up to a constant, i.e.

$$IRF_{i1}^h \propto [C_h \Sigma_\nu \lambda_1]_i \ , \tag{23}$$

where C_h are the matrix coefficients of the Wold representation at lag h.

Proof. Recall that $\Sigma_\nu \Lambda \Lambda' = \mathbb{I}_n$, for Λ defined as in Proposition 1, and let λ_1 denote the first column of Λ. Using the property of partial invertibility in Eq. (20) we can write

$$\mathbb{E}[\nu_t z_t] = \mathbb{E}\left[\Sigma_\nu \Lambda \Lambda' \nu_t z_t\right] = \Sigma_\nu \Lambda \mathbb{E}\left[\begin{pmatrix} u_t^1 \\ u_t^{2:m} \\ \xi_t \end{pmatrix} z_t\right] = \Sigma_\nu \begin{pmatrix} \lambda_1 & \dots & \lambda_m & \tilde{\lambda} \end{pmatrix} \begin{pmatrix} \mathbb{E}[u_t^1 z_t] \\ \mathbb{E}[u_t^{2:m} z_t] \\ \mathbb{E}[\xi_t z_t] \end{pmatrix}.$$

Conditions (i) and (ii) set $\mathbb{E}[u_t^1 z_t] = \alpha$ and $\mathbb{E}[u_t^{2:m} z_t^\perp] = \mathbb{E}[u_t^{2:m} z_t] = 0$ respectively. We now need to prove that $\mathbb{E}[\xi_t z_t] = 0$. Recall first that $\mathbb{E}[\xi_t u_t^{1:m}] = 0$, which follows directly from the definition of Λ. Second, note that $\mathbb{E}[\xi_t u_{t-j}^i] = 0$ for $i = 1, \dots, m$ and $\forall\ j \neq 0$ since $\mathbb{E}[\xi_t u_{t-j}^i] = \tilde{\lambda}' \mathbb{E}[\nu_t \nu_{t-j}'] \lambda_i' = 0$. This follows from the Wold theorem that guarantees that the innovations ν_t are an uncorrelated white noise sequence. Hence, ξ_t and z_t do not correlate via past or future realisations of the invertible shocks. Finally, observe that Condition (ii) and (iii) together require that $\mathbb{E}[u_{t-j}^{m+1:n} z_t^\perp] = 0$ for all j for which $E[u_{t-j}^{m+1:n} \nu_t'] \neq 0$. Since $\xi_t^\perp = \xi_t - Proj(\tilde{\lambda}' \nu_t | \mathcal{H}_{t-1}^Y) = \xi_t$ by the definition of ν_t, it follows that $\mathbb{E}[\xi_t z_t] = \mathbb{E}[\xi_t^\perp z_t] = \mathbb{E}[\xi_t^\perp z_t^\perp] = 0$. Hence, ξ_t and z_t do not correlate via leads or lags of the non-invertible shocks either, leading to $\mathbb{E}[\xi_t z_t] = 0$. It follows that

$$\mathbb{E}[\nu_t z_t] = \alpha \Sigma_\nu \lambda_1.$$

Given the assumption of partial invertibility, the system can be written in the semi-structural representation of Eq. (19) in Proposition 1. Hence the SVAR-IV correctly estimates the relative dynamic causal effects of u_t^1 onto Y_t (i.e. up to a relative scale α). □

Conditions (i) and (ii) are the conventional validity conditions for instrumental variables (IV) that are standard in the micro and macro literatures (see Stock and Watson, 2018). Condition (iii) arises because of the dynamics, and requires that if there are any non-invertible shocks, they do not correlate with the component of the instrument that is orthogonal to past Y_t, at any leads and lags. Conversely, leads and lags (but not contemporaneous values) of other partial invertible shocks can contaminate the instrument without compromising the identification of the impact effects $\Sigma_\nu \lambda_1$, since they do not enter the VAR residuals.[10]

[10]Interestingly, leads, lags or even contemporaneous realisations of the non-invertible shocks can contaminate z_t, but only via their 'projectable' component $Proj(u_t^{m+1:n} | \mathcal{H}_{t-1}^Y) \neq u_t^{m+1:n}$ that lives in the space spanned by past realisations of Y_t.

If the system is invertible in all the structural shocks and the VAR correctly captures the data generating process of Y_t, then the third condition is trivially satisfied, since ν_t are a linear combination of the contemporaneous structural shocks u_t only. Conversely, when all the remaining shocks are non-invertible, Condition (iii) implies a stronger lead-lag exogeneity condition that applies to all the shocks but the partially invertible one. In the more general case in which only some of the remaining shocks are non-invertible, Proposition 2 ensures that identification with an external instrument is possible as long as the instrument is contaminated only by the past and future realisations of the invertible shocks. It is worth stressing that while Condition (iii) is a relatively stronger condition than that required for a fully invertible SVAR (where lead-lag exogeneity is not required), it is still much weaker than the strong lead-lag exogeneity condition required for identification in LP-IV without controls.

When Conditions (ii) or (iii) are violated, the instrument is contaminated by the contemporaneous realisations of any other shock or by leads and lags of some of the non-invertible shocks. This results in a bias in the estimated impulse response functions. We formalise this observation in the following remark.[11]

Remark 2 (Violation of the Exogeneity Conditions). *Let z_t be an instrument for the invertible shock u_t^1 that satisfies Condition (i) but possibly fails Condition (ii) and Condition (iii) of Proposition 2, due to contamination by lags, leads or contemporaneous realisations of a non-invertible shock $u_t^{\not{1}}$, i.e.*

$$z_t = \alpha u_t^1 + \sum_{k \in K} \beta_k u_{t-k}^{\not{1}} \ . \tag{24}$$

Given a VAR that correctly captures the autocorrelation structure of the Wold representation of the process of interest, IRFs estimated by external instrumental variable estimation are biased and, up to a constant, of the form

$$\widetilde{IRF}_{i1}^{h} \propto IRF_{i1}^{h} + \left[C_h \sum_{j \in J} \sum_{k \in K} b_{2,j,\not{1}} \frac{\beta_k}{\alpha} \delta_{jk} \right]_i \ , \tag{25}$$

where IRF_{i1}^h are the IRFs for variable i to the shock u_t^1 at horizon h, C_h are the matrix coefficients of the Wold representation at lag h, $b_{2,j,\not{1}}$ is the $\not{1}$ column of the matrix of

[11] A related result is in Plagborg-Møller and Wolf (2018a) that discuss the bias that arises in SVAR-IV methods when the shock of interest is non-invertible.

coefficients of the polynomial $b_2(L)$ at lag j and δ_{jk} is the Kronecker's delta.

Proof. Given a well specified VAR, the Wold representation is

$$Y_t = C(L)\nu_t \ ,$$

where

$$\nu_t = \Sigma_\nu \Lambda \begin{pmatrix} u_t^{1:m} \\ \xi_t \end{pmatrix} = \left(b_1 \ \ b_2(L)\right) u_t$$

from Proposition 1 and Remark 1. In this case

$$\mathbb{E}[\nu_t z_t] = \Sigma_\nu \Lambda \begin{pmatrix} \mathbb{E}[u_t^{1:m} z_t] \\ \mathbb{E}[\xi_t z_t] \end{pmatrix} = \alpha b_{1,1} + \sum_{j\in J}\sum_{k\in K} b_{2,j,\not{1}} \beta_k \delta_{jk} \ ,$$

where the Kronecker's delta singles out the common leads or lags of $u_t^{\not{1}}$ that appear both in the instrument and the column $\not{1}$ of the matrix $b_2(L)$. By normalising for the coefficient of correlation α and multiplying for the matrix C_h of lag h of the Wold representation one finds

$$\widetilde{IRF}_{i1}^{h} = \left[C_h b_{1,1} + C_h \sum_{j\in J}\sum_{k\in K} b_{2,j,\not{1}} \frac{\beta_k}{\alpha} \delta_{jk} \right]_i \ ,$$

which is the expression in Eq. (25). □

A few elements of Eq. (25) are worth highlighting. First, all else equal, the size of the bias in the estimated IRFs depends on how much the instrument correlates with the (leads, lags and contemporaneous realisations of the) contaminating shock as compared to the shock of interest – i.e. on the ratios $\frac{\beta_k}{\alpha}$. Second, the bias depends on the number of lags that are common to those contaminating the instrument (Eq. 24) and those that appear in the Blaschke matrix $b_2(L)$. Finally, and importantly, the bias depends on the relative order of magnitude of the coefficients $b_{2,j,\not{1}}$ as compared to b_1. These relate to the variance of variable i that is accounted for by the shock of interest and the contaminating shock. For example, very small values of $b_{2,j,\not{1}}$ relative to b_1 imply that shock $u_t^{\not{1}}$ explains a small share of the variance of the variable i, and hence the distortion is likely to be small.

For comparison, if the instrument correlated with leads, lags and contemporaneous realisations of another invertible shock, the equivalent of Eq. (25) would read

$$\widetilde{IRF}^{h}_{i1} = IRF^{h}_{i1} + \left[C_h b_{1,\not 1} \frac{\beta_0}{\alpha} \right]_i \ , \tag{26}$$

for $u_t^{\not 1}$ invertible. In fact, only a violation of Condition (ii) would matter, i.e. the contamination by contemporaneous realisations.

5 SVAR-IV under Partial Invertibility and LP-IV

In empirical applications, when doubts arise regarding the correct VAR specification, a direct estimation approach in the form of local projections (LP) is often suggested. However, as discussed in Stock and Watson (2018), in LP without control variables identification is achieved only under a strict lead-lag exogeneity condition (i.e. $\mathbb{E}[u^{2:n}_{t-j} z_t] = 0$ for all $j \neq 0$) that is potentially violated in practice.

In the empirically likely case in which the instrument satisfies the contemporaneous exogeneity condition but is not strictly lead-lag exogenous due to correlation with past shocks, the standard practice is to incorporate lagged macro variables in the LP regression, in order to control for these lagged shocks (LP-IV$^{\perp}$). In this case, Stock and Watson (2018) provide a 'no-free lunch' result by showing that the conditions for validity of IV identification are in general equivalent to assuming full invertibility of a VAR that incorporates the same information set. In other words, if the shock of interest is non-invertible then there exists some pathological IV that satisfies the contemporaneous exclusion restrictions but fails the dynamic exclusion restrictions, even conditional on controls. In this section we generalise this result to the case of partial invertibility, and show that LP and SVAR methods – with either external or internal instruments – identify shocks under the same conditions.

Consider the standard set up for LP-IV with controls

$$Y_{i,t+h} = \Theta_{h,i1} \widehat{Y}^1_t + \gamma'_h W_t + \zeta^h_{i,t+h} \ , \tag{27}$$

where W_t denotes a generic set of control variables, $\Theta_{h,i1}$ are the causal responses of $Y_{i,t+h}$ to u^1_t at horizon h, $\widehat{Y}^1_t$ is the fitted value of Y^1_t from the first-stage regression on the external instrument z_t, and $\zeta^h_{i,t+h}$ are serially correlated projection residuals.

The conditions for identification in the LP-IV$^{\perp}$ case are (see Stock and Watson, 2018)

(i) $\mathbb{E}[u_t^{1,\perp} z_t^{\perp}] = \alpha$ (Relevance)

(ii) $\mathbb{E}[u_t^{2:n,\perp} z_t^{\perp}] = 0$ (Contemporaneous Exogeneity)

(iii) $\mathbb{E}[u_{t-j}^{\perp} z_t^{\perp}] = 0$ for all $j \neq 0$ (Lead-Lag Exogeneity)

where $x_t^{\perp} = x_t - Proj(x_t|\mathcal{W}_t)$ for a given x_t, and $\mathcal{W}_t = \overline{\text{span}}\{W_t\}$.

The following proposition shows that an instrument that correctly identifies the shock of interest (up to a normalisation) in a SVAR-IV under partial invertibility, will also generally identify the same shock in LP-IV$^{\perp}$ when $\mathcal{W}_t \equiv \mathcal{H}_{t-1}^{Y}$, and vice versa. Conversely, an instrument that identifies a non-invertible shock in LP-IV$^{\perp}$ will also identify that same shock in a SVAR if used as an internal instrument, i.e. in a SVAR specified on $(z_t' \; Y_t')'$ (see also Plagborg-Møller and Wolf, 2018b). This specification is sometimes referred to as hybrid VAR (SVAR-H) in the empirical literature (see e.g. Ramey, 2016).

Proposition 3 (Relation between SVAR-IV under Partial Invertibility and LP-IV$^{\perp}$). *Let Z be the set of scalar stochastic processes z_t that satisfy LP-IV Conditions (i) and (ii) – i.e. $\mathbb{E}[u_t^1 z_t] = \alpha$ and $\mathbb{E}[u_t^{2:n} z_t] = 0$ –, but satisfy Condition LP-IV (iii) $\mathbb{E}[u_{t-j} z_t] = 0$ only for $j < 0$ and not for $j > 0$. Let $\widetilde{Z} \subseteq Z$ be such that any $z_t \in \widetilde{Z}$ satisfies the LP-IV$^{\perp}$ conditions for $\mathcal{W}_t \equiv \mathcal{H}_{t-1}^{Y}$. Assume also that $Proj(u_t|\mathcal{H}_{t-1}^{Y}) = 0$. z_t is an element of $\widetilde{Z}$ if and only if it either (a) satisfies the conditions for SVAR-IV identification under partial invertibility, or (b) satisfies the conditions for SVAR-H identification under non-invertibility.*

Proof. Let us consider the LP-IV$^{\perp}$ conditions for $\mathcal{W}_t \equiv \mathcal{H}_{t-1}^{Y}$. Condition LP-IV$^{\perp}$ (ii) is trivially equivalent to Condition SVAR-IV (ii), since $\mathbb{E}[u_t^{2:n,\perp} z_t^{\perp}] = \mathbb{E}[u_t^{2:n} z_t^{\perp}]$. Condition LP-IV$^{\perp}$ (iii) holds in its stronger LP-IV (iii) form, i.e. $\mathbb{E}[u_{t-j} z_t] = 0$, for $j < 0$, by assumption. For all the invertible shocks i in the system, LP-IV$^{\perp}$ (iii) is trivially satisfied, since $u_{t-j}^{i} = Proj(u_{t-j}^{i}|\mathcal{H}_{t-1}^{Y})$, and hence $u_{t-j}^{i,\perp} = 0$, for all $j > 0$. In this case, LP-IV$^{\perp}$ (iii) does not enforce any restriction on z_t, which can hence correlate with the lags of the invertible shocks. This corresponds to the case in which SVAR-IV (iii) is not active, since $\mathbb{E}[u_{t-j}^{i} \nu_t'] = 0$. In the case of the non-invertible shocks, $u_{t-j}^{i} \neq Proj(u_{t-j}^{i}|\mathcal{H}_{t-1}^{Y})$

for $j \geq 1$. Thus, $\mathbb{E}[u^{\perp}_{t-j} z^{\perp}_t] = 0$ implies that z_t can only correlate with the 'projectable component of the shock' i.e. $Proj(u^i_{t-j} | \mathcal{H}^Y_{t-1})$. In this case, LP-IV$^{\perp}$ (iii) implies $\mathbb{E}[u^{i,\perp}_{t-j} z^{\perp}_t] = \mathbb{E}[u^i_{t-j} z^{\perp}_t] = 0$, and coincides with SVAR-IV (iii) for $\mathbb{E}[u^i_{t-j} \nu'_t] \neq 0$. Hence, conditions (ii) and (iii) for LP-IV$^{\perp}$ and SVAR-IV are equivalent.[12] Condition LP-IV$^{\perp}$ (i) requires $\mathbb{E}[u^{1,\perp}_t z^{\perp}_t] = \mathbb{E}[(u^1_t - Proj(u^1_t | \mathcal{H}^Y_{t-1})) z_t] = \alpha$. If u^1_t is invertible, this is equivalent to $\mathbb{E}[u^1_t z_t] = \alpha$. Hence, under invertibility of the shock of interest, and with $\mathcal{W}_t \equiv \mathcal{H}^Y_{t-1}$, the conditions for LP-IV$^{\perp}$ and SVAR-IV are equivalent.

If u^1_t is non-invertible, $Proj(u^1_t | \mathcal{H}^Y_t) \neq u^1_t$ while $Proj(u^1_t | \mathcal{H}^Y_{t-1}) = 0$ by assumption.[13] In such a case, the conditions for identification in LP-IV$^{\perp}$ are satisfied, while those for SVAR-IV are violated. It is easy to realise that z_t correctly identifies the impact effects when used as an external instrument in a Structural VAR since $z^{\perp}_t$ correlates with u^1_t only at time t. However, a SVAR-IV would not correctly capture the dynamic effects of the non-invertible shock due to the presence of the Blashke factor $b(L)$. In this case, correct identification of the IRFs can still be obtained with a Cholesky identification in a VAR that includes the instrument as an endogenous variable ordered first (see discussion in Plagborg-Møller and Wolf, 2018b).[14] □

Table 1 summarises the content of the proposition and the previous discussion and compares SVARs and LP methods in terms of their ability to correctly estimate the relative impulse response functions of the shock of interest u^1_t on a given set of variables Y_t, given an instrumental variable z_t. The rows in the table consider different properties of z_t, while the columns of the table distinguish between the cases in which u^1_t is invertible or not. It is understood that invertibility of any u^i_t, $i = 1, \ldots, n$, is to be intended relative to $\mathcal{H}^Y_{t-1}$.

A few comments are in order. First, conditional on the same choice of the information set and instrument, generally SVAR (either as SVAR-IV or SVAR-H) and LP-IV with controls can identify a shock under the same set of conditions. Hence, the choice between

[12] This is almost trivial, since both methods require that $z^{\perp}_t$ does not correlate with the residuals of the projection of Y_t onto its past via any other shock except for the one of interest.

[13] Two cases are in principle possible: (a) $Proj(u^1_t | \mathcal{H}^Y_{t-1}) \neq 0$; and (b) $Proj(u^1_t | \mathcal{H}^Y_{t-1}) = 0$. Case (a) implies that u^1_t affects past realisations of Y_t or, equivalently, that Y_t depends on future realisations of u^1_t. This is not an econometrically interesting case – the shock at time t would have affected Y before time t –, and one could just redefine the t index to allow the shock to affect the system only from time t onwards.

[14] The intuition was first proposed in Ramey (2011) by observing that the inclusion of a measure of fiscal news shock in a standard VAR can make shocks that are non-invertible in a small information set, invertible in a larger one.

Table 1: Estimation of the Dynamic Causal Effects of u_t^1

	u_t^1 invertible	u_t^1 non-invertible
Strong Lead-Lag Exogeneity		
$\mathbb{E}[u_{t-j}^i z_t] = 0 \; \forall \; i \; \& \; j \neq 0$	LP-IV SVAR-IV SVAR-H	LP-IV SVAR-H
Limited Lead-Lag Exogeneity but Contamination by Past Shocks		
$\mathbb{E}[u_{t-j}^i z_t] \neq 0$ for some $j > 0$ $(= 0$ for $j < 0)$ but $\mathbb{E}[u_{t-j}^i z_t^\perp] = 0$ and $\mathbb{E}[u_{t-j}^i \nu_t'] = 0$	LP-IV$^\perp$ SVAR-IV SVAR-H	LP-IV$^\perp$ SVAR-H
Limited Lead-Lag Exogeneity but Contamination by Future Shocks		
$\mathbb{E}[u_{t-j}^i z_t] \neq 0$ for some $j < 0$ but $\mathbb{E}[u_{t-j}^i z_t^\perp] = 0$ and $\mathbb{E}[u_{t-j}^i \nu_t'] = 0$	SVAR-IV	–
Violation of Limited Lead-Lag Exogeneity		
$\mathbb{E}[u_{t-j}^i z_t^\perp] \neq 0$, $j > 0$ and i $s.t.$ $\mathbb{E}[u_{t-j}^i \nu_t'] \neq 0$	–	–

Note: The table reports the methods that are able to correctly estimate the dynamic effects of u_t^1 on a given vector Y_t depending on whether u_t^1 is invertible or not, and on the properties of the instrument z_t (in rows). $^\perp$ denotes orthogonality with respect to $\mathcal{H}_{t-1}^Y$. It is assumed that the conditions of Relevance $\big(\mathbb{E}[u_t^1 z_t] = \alpha\big)$ and Contemporaneous Exogeneity $\big(\mathbb{E}[u_t^{2:n} z_t] = 0\big)$ hold throughout.

LP and SVAR methods should not be based on considerations relative to the instrument, but rather on the specific empirical constraints dictated by the availability of the sample and variables of interests, and in light of the different finite-sample bias-variance properties of the two methods, as observed by Plagborg-Møller and Wolf (2018b).

Second, SVAR-IV allow for identification under partial invertibility also in those cases in which the instrument correlates with future invertible shocks, while this is never possible for LP-IV with or without controls (nor for SVARs that include the instrument ordered first). However, these cases are empirically unlikely.

Third, the three methods – LP-IV with controls, SVAR-IV and SVAR-H – deliver similar responses in most but not all of the relevant empirical cases. Hence, they could in principle be used to empirically gauge violations of the conditions for identification.

6 An Observation on VAR Misspecifications

In Section 4, we discussed how the contamination of the instrument biases both the impact and the dynamic responses. In this section, we show that as long as partial invertibility and the conditions for identification of Proposition 2 hold, model misspecification biases the dynamic responses but does not prevent the correct identification of the impact effects of the shock of interest. Canova and Ferroni (2019) provide a background to and complement our discussion by analysing how VAR misspecification challenges the identification of structural shocks, and provide detailed examples using DSGE models.

Let us consider a purely nondeterministic, stationary VARMA(p,q) process $Y_t = (y'_{1,t} \;\; y'_{2,t})'$

$$\begin{pmatrix} \Phi_{11}(L) & \Phi_{12}(L) \\ \Phi_{21}(L) & \Phi_{22}(L) \end{pmatrix} \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \Psi_{11}(L) & \Psi_{12}(L) \\ \Psi_{21}(L) & \Psi_{22}(L) \end{pmatrix} \begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} . \tag{28}$$

Fitting a VAR(k) to $y_{1,t}$ corresponds to imposing some or all of the following restrictions

$$\Phi_{11,i} = 0, \qquad i = k+1, k+2, \dots p, \tag{29}$$

$$\Phi_{12,i} = 0, \qquad i = 1, 2, \dots p, \tag{30}$$

$$\Psi_{11,i} = 0, \qquad i = 1, 2, \dots q, \tag{31}$$

$$\Psi_{12,i} = 0, \qquad i = 1, 2, \dots q. \tag{32}$$

Let us consider the case in which only some of these restrictions are not reflected in the data generating process. The first restriction (conditional on the others being true) corresponds to understating the VAR lag order, with $k < p$. The second restriction instead implies the exclusion of relevant variables $y_{2,t}$. This is also a trivial case of non-invertibility due to the number of variables being smaller than the number of shocks. Finally, the last two restrictions correspond to disregarding the MA structure of the process. Braun and Mittnik (1993) discuss and quantify the asymptotic biases resulting from these misspecifications.

We now examine what these misspecifications imply for the identification of the shock of interest u_t^1, under the assumption of partial invertibility. Let us assume that a condition of partial invertibility for u_t^1 on the subvector $y_{1,t}$ holds, i.e.

$$u_t^1 = Proj(u_t^1 | y_{1,t}, y_{1,t-1}, \dots) . \tag{33}$$

This condition guarantees that u_t^1 can be obtained from the linear projection of $y_{1,t}$ onto its lags (potentially infinitely many).

Let us now consider the case of a too short lag order. In this case, the autoregressive coefficients are biased and inconsistent. However, if the system contains sufficiently many lags to fulfil the partial invertibility condition in Eq. (33), then identification of the impact effects is still obtained. Hence, while impact responses of the variables to the shocks of interest are correctly estimated, their dynamics are distorted even asymptotically. Exactly the same logic applies to the case of a misspecified moving average component, that can always be mapped into a VAR with infinitely many lags. It is worth observing that while in the first case (Eq. 29) more lags trivially resolve the issue, in the second case (Eqs. 31-32) longer lags only asymptotically approximate the correct Wold representation.

Consider now the case of omitted variables (Eq. 30). If Eq. (33) holds for the subset of variables $y_{1,t}$, then also in this case the impact effect are correctly retrieved, while the IRFs at longer horizons are distorted. However, interestingly, in this case too longer lags would asymptotically capture the correct dynamics of the system, and hence asymptotically recover the true IRFs. To see this, note that the Wold Representation Theorem implies that also $y_{1,t}$ has an invertible MA representation. For the n_1-dimensional subprocess $y_{1,t} = JY_t$, where $J_t = (\mathbb{I}_{n_1} \ \ 0_{n-n_1})$ is a selector matrix, we can write

$$\Phi_{11}(L)y_{1,t} = -\Phi_{12}(L)y_{2,t} + \Psi_1(L)u_t \ . \tag{34}$$

If Y_t is covariance-stationary, $y_{1,t}$ is also covariance stationary, with first and second moments respectively equal to $\mathbb{E}(y_{1,t}) = J\mathbb{E}(Y_t)$, and $\Gamma_{y_1}(h) = J\Gamma_Y(h)J'$, where $\Gamma(h)$ is the autocovariance of Y_t at lag h. The Wold Representation Theorem also guarantees the existence of an ARMA representation of the form

$$\widetilde{\Phi}_1(L)y_{1,t} = \widetilde{\Psi}_1(L)\nu_{1,t} \ . \tag{35}$$

The structural innovations u_t are trivially non-invertible in $y_{1,t}$. In fact, the n innovations u_t are compounded and reduced to the $n_1 < n$ innovations $\nu_{1,t}$, which do not have a meaningful structural interpretation. If, however, the system is partially invertible and Eq. (33) holds, then the impact effects of the shock of interest u_t^1 are correctly estimated; moreover, the existence of a Wold representation guarantees that the dynamics of the system are asymptotically approximated by infinitely many lags of $y_{1,t}$ only. It is worth noting that direct methods with controls (Jordà, 2005) can in principle be used to improve over VAR estimates in all these cases in which VARs can only asymptotically approximate the true dynamics of the system.

Interestingly, these observations provide a simple way to gauge the contamination of an instrument versus the misspecification of the chosen model – two dimensions along which structural identification may be problematic and deliver unstable results. In fact, if one can assume partial invertibility across different specifications of an empirical model, an instrument that fulfils the conditions for identification of Proposition 2 delivers stable impact responses but unstable IRFs across models. In this case, increasing the number of lags and/or selectively adding variables that may be of importance for the transmission of the shock should help stabilising the dynamics responses. The intuition for this is that additional controls may be important for the transmission of the structural shocks. Conversely, an instrument that violates the lead-lag exogeneity condition is likely to also deliver unstable impact responses across different models.[15] In the following section, we discuss a formal statistical test that formalises these remarks.

7 A Test for (Lagged) Exogeneity

The informal discussion provided in the previous section can be generalised and formalised with a Hausman (1978)'s type test procedure.[16] Let us consider an expression for the i-th equation of the structural model in Eq. (1) in terms of a set of observables W_t, e.g. lagged values of components of Y_t, and of the shock of interest u_t^1

$$Y_{i,t} = b_{i1}u_t^1 + \gamma_i' W_t + U_{i,t} \qquad i = 1, \dots, n \;, \tag{36}$$

where U_t absorbs all the structural components of $Y_{i,t}$ not captured by W_t and not due to u_t^1.

All the methods discussed – SVAR-IV, SVAR-H, LP-IV$^{\perp}$ – estimate the structural relationships, at different horizons, between $Y_{i,t}$ and u_t^1 by means of linear regressions of

[15]In this case, the use of a much larger information set can help resolving the issue. The intuition is that structural shocks are likely to be fundamental and invertible in larger models, hence improving the performance of contaminated instruments (see Giannone and Reichlin, 2006).

[16]See Godfrey (1991) for a thorough discussion on misspecification tests in econometrics and for an introduction to Hausman type tests.

the variables of interest onto a set of controls and an instrument that is assumed exogenous conditional on the set of controls, i.e.

$$Y_{i,t} = b_{i1}^* z_t + \gamma_i^{*'} W_t + \zeta_{i,t} \qquad i = 1, \dots, n \ . \tag{37}$$

For example, in a VAR(p), identification with external instruments amounts to a two-stage procedure: (i) in the first stage, each endogenous variable is regressed on its own lags and on the lags of the other variables, equation by equation; (ii) in the second stage, the structural impact coefficients are identified (up to normalisation) from the coefficients of the linear projection of the residuals of the first-stage regression on the instrument. This is equivalent to instrumenting a relevant variable in the VAR and then regressing each variable onto the past realisations of the variables in the system, i.e. $W_t = \{Y_{t-1}, Y_{t-2}, \dots Y_{t-p}\}$, and the contemporaneous realisation of the instrumented variable. Alternatively, one can consider a single-stage case in which the instrument and its lags are included among the endogenous variables in (i); (this would also correspond to a SVAR-H). In the LP-IV framework, the same would apply horizon by horizon as in Eq. (27).

The conditional exogeneity condition

$$\mathbb{E}[z_t^{\perp} U_{i,t}] = 0 \ , \tag{38}$$

is a necessary condition to have structural interpretation of the coefficient of the 'predictive regressions' of $Y_{i,t}$ onto z_t in Eq. (37), that is $b_{i1}^* \propto b_{i1}$. Together with the relevance condition, the exogeneity condition requires that the residuals of the projection of the instrument on W_t – i.e. $z_t^{\perp} = z_t - Proj(z_t|\mathcal{W}_t)$ –, correlate with $Y_{i,t}^{\perp} = z_t - Proj(Y_{i,t}|\mathcal{W}_t)$ only via the shock of interest at time t, u_t^1. In the previous section, this conditions was formalised in relation to different models. An important implication of this condition is that, when satisfied, estimates of b_{i1}^* are robust to the inclusion of additional controls.

Given an instrument z_t, the smallest information set W_t for which the conditional exogeneity condition is satisfied is sometimes defined as the 'minimum relevant information set' (Heckman and Navarro-Lozano, 2004) or as the 'core covariates' (Lu and White, 2014). This provides a different angle to understand the conditions for identification discussed in the previous sections, that can be seen as the necessary and sufficient properties for z_t, conditional on a W_t of choice. The 'dual' problem to this, is to define a minimum relevant information set such that the instrument z_t satisfies the conditional exogeneity condition in Eq. (38). For a perfect instrument (or an instrument with classic measurement error only), the core information set has dimension zero. That is, any regression of a variable of interest onto the instrument with or without controls delivers the correct impact coefficients. Conversely, for instruments contaminated by other shocks, the minimum relevant information set can be impossible to obtain, given the available observables.

Therefore, a necessary condition for valid instrumental inference on the dynamic responses to a structural shock of interest is that the model correctly incorporates the core information set. Hence, once that set has been correctly identified, robustness is necessary for valid causal inference, and the regression coefficient of $Y_{i,t}$ onto z_t has to be insensitive to adding or dropping variables beyond those in the core set.

This idea can be formalised in a Hausman type test, following White and Chalak (2010) and Lu and White (2014). Let us consider a general specification, as in Eq. (36), for which we can define a core regression model

$$Y_{i,t} = b_{i1}^{(1)*} z_t + \gamma_i^{(1)*'} W_t^{(c)} + \zeta_{i,t} \qquad i = 1, \ldots, n \ , \tag{39}$$

along with a set of 'robustness check regressions'

$$Y_{i,t} = b_{i1}^{(j)*} z_t + \gamma_i^{(j)*'} W_t^{(c)} + \kappa_i^{(j)*'} W_t^{(j)} + \zeta_{i,t}^{(j)} \qquad i = 1, \ldots, n, j = 2, \ldots J \ , \tag{40}$$

where $W_t^{(c)}$ is the assumed core information set – the set of variables needed to achieve the conditional exogeneity condition – and $W^{(j)}$ is a set of non-core variables. For example in a VAR, $W_t^{(c)}$ would be given by the lags of the core variables, while $W^{(j)}$ may contain lags of other variables of interest.

Let the ordinary least squares estimates of the coefficients for the core and the robustness check regressions be, respectively, $\hat{\delta}_{i,T}^{(1)} = (\hat{b}_{i,T}^{(1)} \ \hat{\gamma}_{i,T}^{(1)'})'$ and $\hat{\delta}_{i,T}^{(j)} = (\hat{b}_{i,T}^{(j)} \ \hat{\gamma}_{i,T}^{(j)'} \ \hat{\kappa}_{i,T}^{(j)'})'$, where T denotes the sample size. Defining the joint vector of the estimators $\hat{\delta}_{i,T} \equiv (\hat{\delta}_{i,T}^{(1)'} \ \hat{\delta}_{i,T}^{(2)'} \ldots \hat{\delta}_{i,T}^{(J)'})'$, under the mild conditions discussed in Chalak and White (2011), and without assuming correct specification, it follows that

$$\sqrt{T}(\hat{\delta}_{i,T} - \delta_i^*) \xrightarrow{\mathcal{D}} \mathcal{N}(0, M_i^{*-1} V_i^* M_i^{*-1}) \ , \tag{41}$$

where, accordingly, $\delta_i^* \equiv (\delta_i^{(1)*'} \ \delta_i^{(2)*'} \ldots \delta_i^{(J)*'})'$, and $\delta_i^{(j)*}$ is the vector of coefficients of each model. M_i^* and V_i^* are given by

$$M_i^* \equiv diag(M_i^{(1)*}, \ldots M_i^{(J)*}) \ , \qquad \text{and} \qquad V_i^* \equiv [V_i^{(kj)*}] \ , \tag{42}$$

where for $j = 1, \ldots J$

$$M_i^{(j)*} \equiv \begin{pmatrix} \mathbb{E}[z_t^2] & \mathbb{E}[z_t X_t^{(j)'}] \\ \mathbb{E}[X_t^{(j)} z_t] & \mathbb{E}[X_t^{(j)} X_t^{(j)'}] \end{pmatrix} \tag{43}$$

$$V_i^{(kj)*} \equiv \begin{pmatrix} \mathbb{E}[\zeta_{i,t}^{(k)} \zeta_{i,t}^{(j)} z_t^2] & \mathbb{E}[\zeta_{i,t}^{(k)} \zeta_{i,t}^{(j)} z_t X_t^{(j)'}] \\ \mathbb{E}[\zeta_{i,t}^{(k)} \zeta_{i,t}^{(j)} X_t^{(k)} z_t] & \mathbb{E}[\zeta_{i,t}^{(k)} \zeta_{i,t}^{(j)} X_t^{(k)} X_t^{(j)'}] \end{pmatrix} \tag{44}$$

and $X_t^{(j)} = (W_t^{(c)'} \ W_t^{(j)'})'$ is the full set of regressors in specification j. Given the asymptotic results in Eq. (41), it is possible to devise a test on the robustness of $b_{i1}^{(1)*}$, i.e.

$$\text{H}_0 : \Delta b_{i1}^* = 0 \ , \tag{45}$$

or an alternative test on a more general hypothesis involving some of the other coefficients

$$\text{H}_0' : \Delta S \delta_i^* = 0 \ , \tag{46}$$

where S is a selection matrix that extract the J subvectors $\delta_{i,T}^{(j)} = (\hat{b}_{i,T}^{(j)} \ \hat{\gamma}_{i,T}^{(j)'})'$, and Δ is the $(J-1)j_0 - \times Jj_0$ differencing matrix (for j_0 the number of regressors considered), defined as

$$S = \begin{pmatrix} I & -I & 0 & \ldots & 0 \\ I & 0 & -I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I & 0 & 0 & \ldots & -I \end{pmatrix} \ . \tag{47}$$

Defining $R \equiv \Delta S$ the Hausman (1978)'s type test statistic is

$$\mathcal{R}_{J,T} = T\hat{\delta}_{i,T} R' [R \hat{M}_{i,T}^{-1} \hat{V}_{i,T} \hat{M}_{i,T}^{-1} R']^{-1} R \hat{\delta}_{i,T} \ , \tag{48}$$

with $\hat{V}_{i,T}$ and $\hat{M}_{i,T}$ respectively consistent estimators of M_i^* and V_i^* (and $RM_i^{*-1} V_i^* M_i^{*-1} R'$

assumed to be nonsingular). Under H_0

$$\mathcal{R}_{J,T} \xrightarrow{\mathcal{D}} \chi^2_{(J-1)j_0} \ . \tag{49}$$

As it is standard, the robustness of the impact coefficients (or of some, or all of the core model coefficients) is rejected at the α level if $\mathcal{R}_{J,T}$ exceeds the $1-\alpha$ percentile of the $\chi^2_{(J-1)j_0}$ distribution. As observed in Lu and White (2014) since the test is a completely standard parametric test, it has power against local alternatives at rate $T^{-1/2}$.

A few observations are important. First, in general, the form of robustness assessed by this test is not sufficient, but it is nevertheless necessary for valid causal inference. In fact, for example, while the use of lagged variables does not rule out that the instrument violates the contemporaneous exogeneity conditions, rejection of robustness indicates a failure of the conditional exogeneity assumption. Second, it is worth noting that in the case of a VAR, the proposed test can be helpful in detecting both violations of the conditional exogeneity condition and lack of robustness of the VAR coefficients to the inclusion of additional endogenous regressors (Eq. 46).[17] Finally, from an applied perspective, the instability of the impact responses across specifications detected by the test is a signal that should alert the researcher, since a contamination of the instrument at any lags may well also imply a contamination by other contemporaneous shocks that the test would not detect.[18] It is important to stress that the robustness of the coefficients is a necessary but not sufficient condition for the exogeneity of the instrument, and hence this test does not exonerate the researcher from providing explicit arguments in support of the maintained assumptions.

8 Partial Invertibility in a Simulated System

We use a stylised New Keynesian DSGE model that features (i) a representative infinitely-lived household that chooses between consumption and leisure; (ii) firms that produce a continuum of goods using a Cobb-Douglas technology to aggregate capital and labour; (iii) a government that consumes a share of output for wasteful public spending; and

[17] In a more general setting also a violation of linearity would be signalled by a rejection of H_0, see Lu and White (2014) for a discussion.

[18] A possible way around this issue is to introduce in the test, as controls, instruments for other shocks that may be contaminating the instrument at test.

Table 2: Degree of Invertibility

	u_t^r	u_t^a	u_t^g	u_t^π
δ_i	0.069	0.799	0.494	0.343

Note: Degree of invertibility of the structural shocks in the model. $\delta_i = 0$ denotes invertibility; $\delta_u = 1$ denotes insufficient information for shocks invertibility. VAR(4).

(*iv*) a central bank that sets the interest rate using a Taylor rule with smoothing. There are four stochastic disturbances that generate fluctuations in the economy, namely, a monetary policy shock u_t^r, a government spending shock u_t^g, a technology shock u_t^a, and an inflation-specific shock u_t^π.

The processes for technology, spending, inflation, and the policy rate are defined as follows. Log technology a_t evolves with a news component as

$$a_t = \rho_a \, a_{t-1} + \sigma_a u^a_{t|t-4} \ , \tag{50}$$

where u_t^a is an i.i.d. normally distributed technology news shock. Similarly, an element of fiscal foresight characterises the spending process g_t, that evolves according to

$$g_t = \rho_g \, g_{t-1} + u^g_{t-4} \ , \tag{51}$$

where u_t^g is an i.i.d. normally distributed spending shock. The monetary authority sets the nominal interest rate using a Taylor rule with smoothing

$$r_t = \rho_r \, r_{t-1} + (1 - \rho_r) \left(\phi_\pi \overline{\pi_t} + \phi_y \overline{\Delta y_t} \right) + \sigma_r u_t^r \ , \tag{52}$$

where $\overline{\pi_t}$ is the average inflation rate over the last four periods, $\overline{\Delta y_t}$ is the average growth rate of output, and u_t^r is a white noise i.i.d. normally distributed monetary policy shock. Finally, price dynamics are governed by a New Keynesian Phillips Curve, as follows

$$\pi_t = \gamma_\pi \pi_{t-1} + \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\theta_\pi)(1-\theta_\pi \beta)}{\theta_\pi} mc_t + u_t^\pi, \tag{53}$$

where mc_t are marginal costs, and u_t^π is an i.i.d. normally distributed inflation-specific shock. All the model details, including the calibrated parameters, are reported in Appendix A.

We consider a VAR(4) in the policy rate, inflation, output, and government spending. Under the chosen set of parameters, the model fails the 'poor man's invertibility condition' of Fernandez-Villaverde et al. (2007), hence, the four structural shocks cannot all be recovered from a VAR in the observables. However, the specification of the Taylor rule ensures that the monetary policy shock is partially invertible from a VAR(4) in $[r_t, \pi_t, y_t]'$. Table 2 reports the degree of invertibility δ_i of each of the structural shocks in the model, as defined in Sims and Zha (2006) and calculated following Forni et al. (2019) as

$$\delta_i = \text{var}[u_t^i - Proj(u_t^i|\mathcal{H}_t^Y)]/\sigma_{u_i}^2, \tag{54}$$

where $\sigma_{u_i}^2$ denotes the variance of the shock u_t^i, and $\mathcal{H}_t^Y$ denotes the space spanned by the vector of observables Y_t and its lags. δ_i is a deterministic function of the model's deep parameters, and measures the unexplained variance of the orthogonal projection of each of the structural shocks onto the VAR residuals. A value of 0 implies that the shock is invertible from the VAR, whereas increasing values of δ_i imply non-fundamentalness and an increasing degree of non-invertibility.

Table 2 shows that the value of δ_i for technology is very close to 1, confirming the inability of the VAR to recover this structural shock. The inflation and spending shocks are also non-invertible, but with a higher degree of invertibility. The monetary policy shock is the only invertible shock in the system. The four shocks play a different role in driving economic fluctuations in the model. Table 3 reports the share of variance of the four observables that is accounted for by each of the four shocks in the model. We note that the government spending shock plays a negligible role.

In Figure 1 we report the distribution of δ_i for each of the shocks across simulations from the model, and compare it against the model implied ones (green dashed lines). Specifically, we simulate from the model 5,000 economies each of sample size $T = 300$ periods. For each set of simulated data, we then estimate a VAR(4) in the four observables – output, inflation, spending and the policy interest rate –, and we calculate δ_i by projecting on the residuals of each VAR. In all cases, the distribution of the simulated δ_i has most of its mass concentrated around the true, model-implied value.

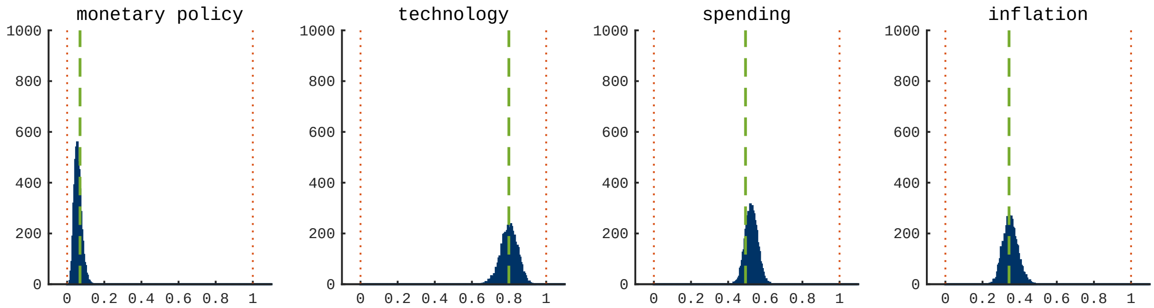
We use the same set of simulated data to identify the monetary policy shock from VARs using the following four external instruments:

Table 3: Variance Decomposition

		u_t^r	u_t^a	u_t^g	u_t^π
output	y_t	16.45	77.01	0.98	12.95
spending	g_t	0.00	0.00	61.91	0.00
inflation	π_t	9.07	51.34	0.01	67.03
policy rate	r_t	25.32	19.81	0.15	14.47

Note: Share of variance accounted for by each shock. Numbers may not add up to 100 due to non-zero correlation of simulated shocks in small samples.

Figure 1: Degree of Invertibility of the Structural Shocks



Note: Distribution of δ_i across 5000 simulated economies. $\delta_i = 0$ denotes invertibility; $\delta_i = 1$ denotes insufficient information for shocks invertibility. VAR(4). Green dashed lines are the model-implied values of δ_i.

$$z_{0,t} = u_t^r \ , \tag{55}$$

$$z_{1,t} = 0.7u_t^r - 0.5u_{t-2}^r + \varsigma_t \ , \tag{56}$$

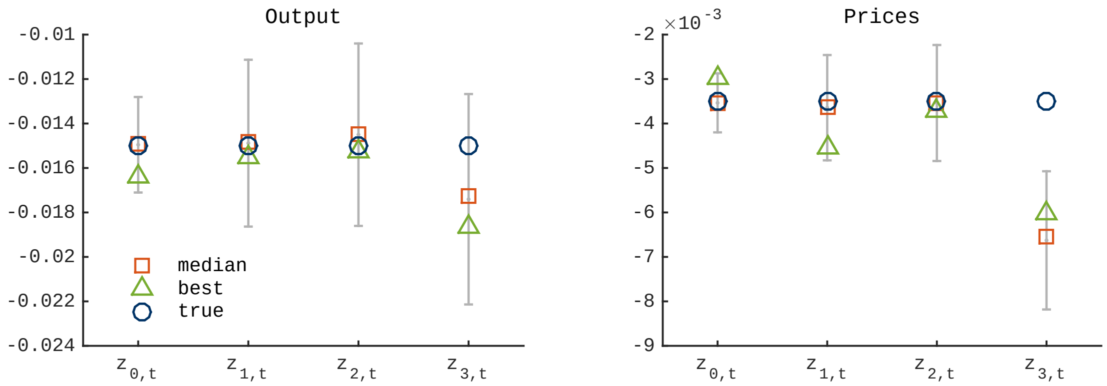
$$z_{2,t} = 0.7u_t^r - 0.5\left(u_{t-1}^g + u_{t-2}^g + u_{t-3}^g\right) + \varsigma_t \ , \tag{57}$$

$$z_{3,t} = 0.7u_t^r + 0.5\left(u_{t-1}^a + u_{t-2}^a + u_{t-3}^a\right) + \varsigma_t \ . \tag{58}$$

In Eq. (55) the shock is perfectly observable. This is the case discussed in Stock and Watson (2018). The instrument in Eq. (56) is contaminated by classic white noise measurement error, and the second lag of the monetary policy shock. The instruments in Eqs. (57-58) both fail the limited lead-lag exogeneity condition of Proposition 1. In fact, while $z_{2,t}$ is contaminated by lagged spending shocks, $z_{3,t}$ correlates with lagged technology shocks. In all cases, ς_t is a normally distributed random measurement error with zero mean and variance equal to that of the structural shocks. A VAR(4) is partially invertible in the monetary policy shock and also captures the model's dynamics sufficiently well. Hence, we use $p = 4$ as the benchmark case.[19]

[19] In the Appendix we also report the extreme cases of $p = 1$ and $p = 2$ where the model is more

FIGURE 2: IMPACT RESPONSES TO MONETARY POLICY SHOCK



Note: Impact responses to monetary policy shock from partially-invertible DSGE identified with external instruments and estimated with a VAR(4) in four observables. $z_{0,t}$: observed shock case; $z_{1,t}$: instrument correlates with monetary policy shock only; $z_{2,t}$: instrument also correlates with past spending shocks; $z_{3,t}$ instrument correlates also with past technology shocks. Grey vertical lines are 2 standard deviations error bars from the distribution of impact responses across 5,000 simulated economies of sample size $T = 300$ periods. True impact (blue circle), median across simulations (orange square), minimum distance from median (best) simulation (green triangle).

Impact responses for output and inflation recovered from the four instruments and a VAR(4) are in Figure 2.[20] In each subplot, we use blue circles for the model's responses (true), orange squares for the median across simulations, and green triangles for the simulation which is the closest to the median (best).[21] The error bars are two standard deviations intervals constructed from the distribution across simulations. A few elements are worth highlighting. As also noted in Stock and Watson (2018), when the shock is observable ($z_{0,t}$), the assumption of full invertibility can be dispensed with for the validity of SVAR-IV. However, the shock is correctly recovered also under the milder conditions introduced in Proposition 2. In fact, correct impact responses are recovered also with $z_{1,t}$. The introduction of a measurement error in $z_{1,t}$ widens the distribution of impact responses across simulations, but recovers the correct impact effects. The picture changes substantially when we consider the case of $z_{3,t}$. In this case, the instrument correlates with lagged non-invertible technology shocks which the data in the VAR cannot provide sufficient information for by construction. This results in severely biased impact responses. An interesting case arises when the instrument also correlates with lagged

severely misspecified and the identification becomes more challenging.

[20] IRFs are normalised such that the impact response of the policy rate to a monetary policy shock equals that of the model.

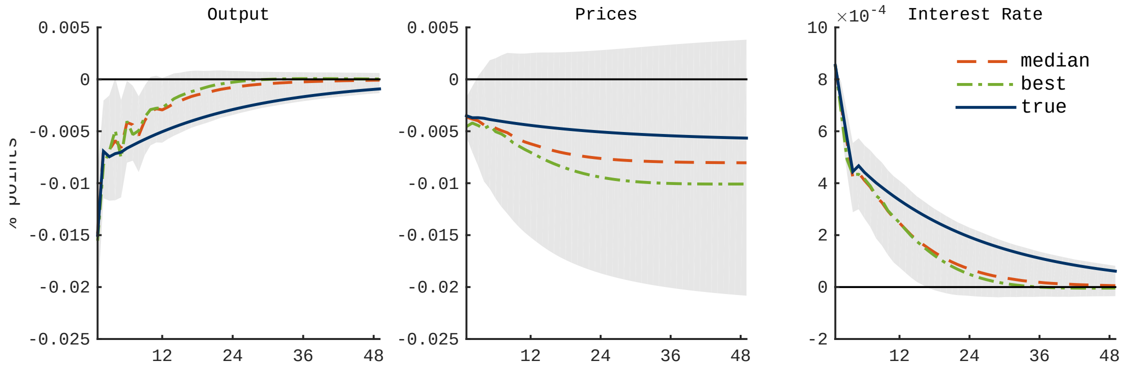
[21] We select the simulation whose IRFs minimise the sum of square deviations from median IRFs over the first 12 periods. The choice allows to put more weight at shorter horizons where responses display richer dynamics. Changing the truncation horizon yields qualitatively similar results.

spending shocks ($z_{2,t}$). The spending shock is not invertible in the system; however, as noted, it is responsible for a negligible share of the variance of the simulated variables. In this case the impact responses recovered are close to the true ones, consistently with what observed in Remark 2.

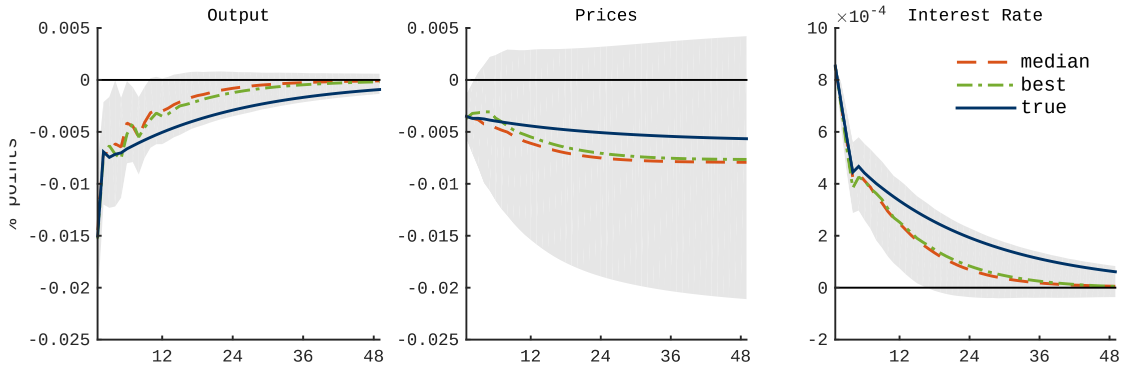
The argument extends in an equivalent way to responses at farther away horizons. Figure 3 reports IRFs over 48 periods estimated using $z_{1,t}$ (Panel A, top), $z_{2,t}$ (Panel B, centre), and $z_{3,t}$ (Panel C, bottom). In the first two cases the model responses lie within the bands generated across the simulations. Conversely, the responses of all variables lie outside of the simulation confidence regions when the shock is identified using $z_{3,t}$.

In this exercise we have used data simulated from a NK-DSGE to show that if the conditions in Proposition 2 are satisfied, full invertibility is not necessary for the identification of invertible shocks in SVAR-IVs. Furthermore, even when the instrument violates the limited lead-lag exogeneity condition, the extent to which the estimated IRFs are distorted depends on the share of variance that is accounted for by the non-invertible shocks that contaminates the instrument.

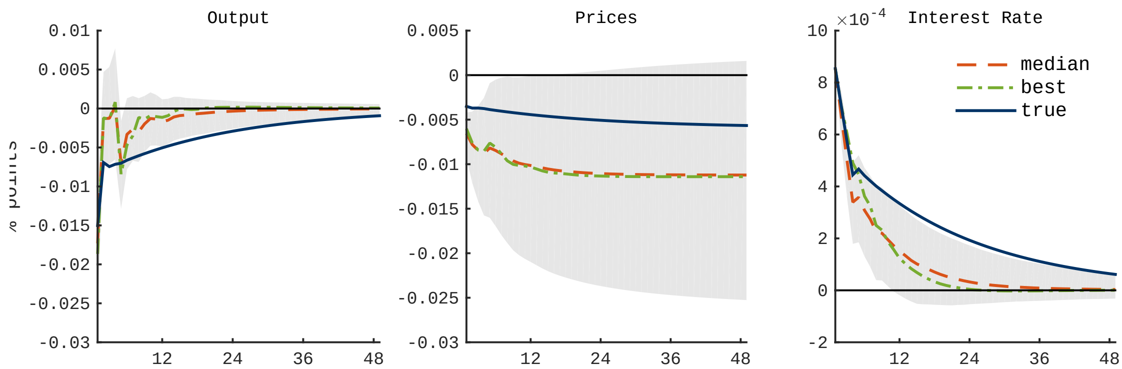
FIGURE 3: RESPONSES TO MONETARY POLICY SHOCK – SIMULATION



(A) $z_{1,t}$: external instrument correlates with monetary policy shock only



(B) $z_{2,t}$: external instrument also correlates with lagged spending shocks



(C) $z_{3,t}$: external instrument also correlates with lagged technology shocks

Notes: Impulse responses to monetary policy shock from partially-invertible DSGE identified with external instruments and estimated with a VAR(4) in four observables. Instrument correlates with monetary policy shock only (Panel A). Instrument correlates with monetary policy shock and lagged spending shocks (Panel B). Instrument correlates with monetary policy shock and lagged technology shocks (Panel C). Grey shaded areas denote 90th quantiles of the distribution of IRFs across 5,000 simulated economies of sample size $T = 300$ periods. Model responses (true, blue solid), median across simulations (orange dashed), minimum distance from median (best) simulation (green dash-dotted).

9 IV Identification of Monetary Policy Shocks

In this section, we look at the empirical identification of monetary policy shocks and use the results in the previous sections to shed light on the distortions to both the impact effects and the dynamic responses that arise from either the contamination of the instrument, or the misspecification of the chosen VAR. In particular, we consider different instruments for monetary policy shocks, some of which may be contaminated, and different VARs, some of which are likely to be misspecified.

We consider three external instruments, all constructed from the high-frequency surprises of Gürkaynak et al. (2005), that measure monetary policy innovations through the surprise reactions of federal funds futures markets around FOMC announcements, following the insight of Kuttner (2001). The first of these instruments is constructed by measuring high-frequency surprises around all the scheduled FOMC meetings between 1990 and 2012. This is equivalent to the instrument used in e.g. Stock and Watson (2018) and Caldara and Herbst (2018), and we denote it by $z_{\mathrm{A},t}$. The second instrument is a monthly moving average of high-frequency surprises around all FOMC announcements from 1990 to 2012. This is the instrument originally proposed in Gertler and Karadi (2015), denoted $z_{\mathrm{B},t}$. The third external instrument is the residual of a projection of high-frequency surprises around all FOMC meetings onto their lags and Fed Greenbook forecasts from 1990 to 2009. This is the instrument in Miranda-Agrippino and Ricco (2017), denoted $z_{\mathrm{C},t}$. This projection can be seen as a pre-whitening step that removes the contamination by other past and contemporaneous shocks related to the state of the economy induced by the presence of a signalling channel of monetary policy (see e.g. Melosi, 2017).

Table 4 reports Granger causality tests for the three instruments on the first ten macroeconomic and financial factors estimated from the monthly dataset in McCracken and Ng (2015). For each instrument we estimate the following regression

$$z_t = \theta_0 + \theta_1 z_{t-1} + \sum_{j=1}^{10} \theta_{f_j} f_{j,t-1} + v_t \tag{59}$$

at monthly frequency and over the sample 1990-1:2009-12. The numbers in the table are Wald test statistics for the null that the factors' coefficients are jointly equal to zero, i.e. H_0: $\theta_{f_1} = \ldots = \theta_{f_{10}} = 0$. Test results suggest a possible contamination of the instruments $z_{\mathrm{A},t}$ and $z_{\mathrm{B},t}$ by lagged macroeconomic shocks, with p-values well beyond the rejection

Table 4: Contamination of Monetary Policy Instruments

	$z_{\mathrm{A},t}$	$z_{\mathrm{B},t}$	$z_{\mathrm{C},t}$
$F_{(10,227)}$	2.12 (0.0240)		
$F_{(10,226)}$		3.52 (0.0002)	
$F_{(10,215)}$			1.77 (0.0669)
N	239	238	227

Note: Wald test statistics. Regressions include a constant and one lag of the dependent variable. Sample 1990:2009. p-values in parentheses.

region. This serves as motivation for our next exercise.

We evaluate the effect of the instruments' contamination on the estimation of the IRFs in an empirical setup that encompasses standard monetary VARs such as those in Coibion (2012) and Gertler and Karadi (2015). Our benchmark VAR is monthly and estimated with 12 lags from 1979-1 to 2012-12. The variables included are the one-year government bond rate as the policy variable, an index of industrial production, the unemployment rate, the consumer price index, a commodity price index, and the excess bond premium (EBP) of Gilchrist and Zakrajšek (2012).[22] Stock and Watson (2018) show that in this system there is no statistically significant evidence against the null hypothesis of invertibility.[23]

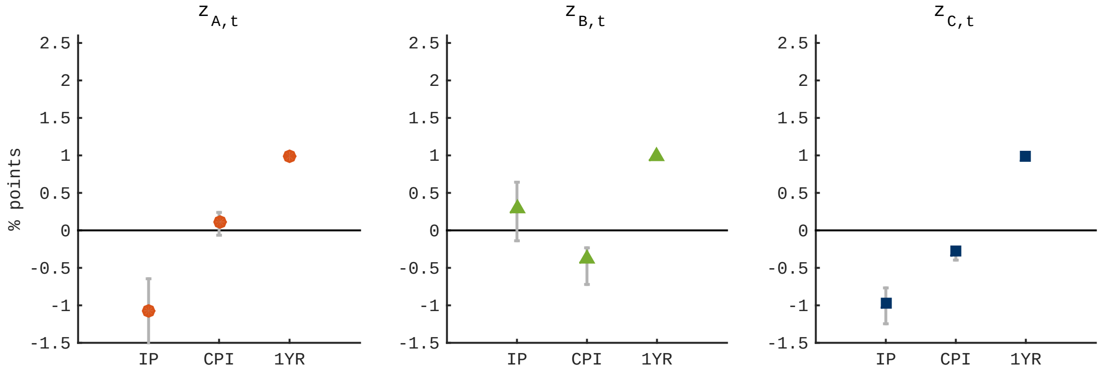
We also consider a VAR estimated over the same sample that omits the unemployment rate, the EBP variable, and the commodity price index, and includes only 2 lags. This VAR is likely to be misspecified, but is compatible with a central bank setting the interest rate using a simple Taylor rule, hence conventional monetary policy shocks are potentially invertible in this smaller VAR. In all cases, we estimate the impact responses from a regression of the VAR innovations onto one of the above instruments, while IRFs are retrieved from the coefficients of the VAR. Responses are normalised such that the policy rate increases by 1% on impact.

We start by looking at the impact responses retrieved by the three instruments in the two VARs, reported in Figure 4. The top row collects results for the baseline VAR, while

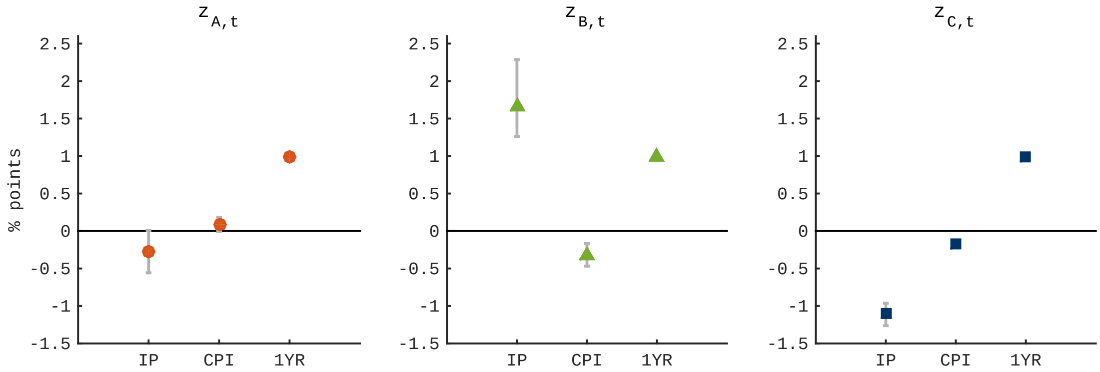
[22]Data for bond yields, industrial production, and the consumer price index are from the St Louis FRED Database, the commodity price index is from the Commodity Research Bureau, the EBP data are from the Federal Reserve Board.

[23]Stock and Watson (2018) do not reject the null of invertibility in a system that includes industrial production, the index of consumer prices, the one year interest rate and the excess bond premium variable. The test is however sensitive to the number of lags included (Plagborg-Møller and Wolf, 2018b).

Figure 4: Impact Responses to Monetary Policy Shocks – 1979:2012



(A) Baseline VAR



(B) Misspecified VAR

Notes: Baseline VAR(12) in all variables, top panel (A). Misspecified VAR(2) in three variables, bottom panel (B). VARs estimated with standard macroeconomic priors. Identification in all cases uses the full length of the instruments. $z_{\text{A},t}$: high-frequency surprises at scheduled FOMC meetings; $z_{\text{B},t}$: moving average of high-frequency surprises within the month; $z_{\text{C},t}$: residuals of $z_{\text{A},t}$ on Fed Greenbook forecasts. Shaded areas denote 90% posterior coverage bands.

the misspecified VAR is in the bottom row. Comparing the impact responses for each given instrument across VARs we note that while those estimated with $z_{\text{C},t}$ are stable across models, those recovered under either $z_{\text{A},t}$ or $z_{\text{B},t}$ vary and are statistically different. Modal impact responses of production to a contractionary monetary policy shock go from being not significant to strongly positive at almost 2% under $z_{\text{B},t}$, and from -1% to essentially zero under $z_{\text{A},t}$. The impact response under $z_{\text{C},t}$ is largely unchanged.

We formalise the discussion in Table 5 where we report the results of the test for conditional exogeneity of the instrument introduced in Section 7. Results are reported for the two main variables of output and prices, and we compare a simple specification in which the set of controls is equal to that in our baseline VAR with one which only includes lags of the dependent variable. We report two specifications for the test. One in which the instrument is included among the endogenous variables (denoted (1) in the

Table 5: Test for Conditional Exogeneity of the Instuments

	$z_{A,t}$		$z_{B,t}$		$z_{C,t}$	
	(1)	(2)	(1)	(2)	(1)	(2)
output	4.55	2.87	6.38	4.56	0.01	0.02
	(0.033)	(0.090)	(0.012)	(0.033)	(0.928)	(0.879)
prices	0.50	3.01	1.79	15.63	0.28	1.24
	(0.478)	(0.083)	(0.181)	(0.000)	(0.599)	(0.266)
N	270	270	270	270	228	228

Note: Test $\sim \chi_{(1)}$, p-values in parentheses. Simple model: $y_t = bz_t + \gamma y_{t-1} + \nu_t$, richer model: $y_t = bz_t + \kappa w_{t-1} + \nu_t$, where y_t is either output or prices, and w_t includes IP, UNRATE, CPI, CRBPI, GS1, EBP. (1): the instrument is included as one of the endogenous variables. (2) : residuals are estimated in a first stage, and the regression on the instrument run in a second stage. $z_{A,t}$: sum of high-frequency surprises within the month; $z_{B,t}$: moving average of high-frequency surprises within the month; $z_{C,t}$: residuals of $z_{A,t}$ on Fed Greenbook forecasts.

table), and one in which the test is run on first-stage residuals, as in the SVAR-IV here presented (denoted (2) in the table). This latter specification does not take into account parameters uncertainty and hence tends to over-reject the null. Test results show that the null of conditional exogeneity (i.e. equal impact coefficients across specifications) is rejected for both $z_{A,t}$ and $z_{B,t}$.

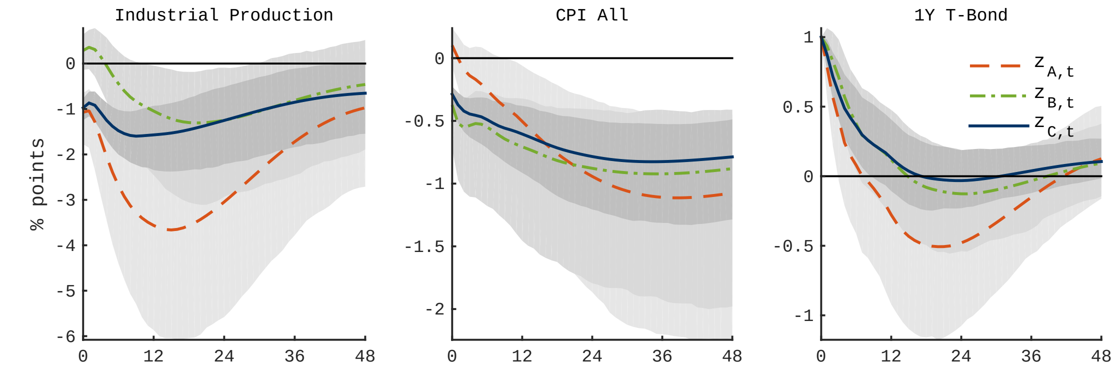
We then turn to the full dynamic responses reported in Figure 5. Despite the differences in the estimated impact effects, the responses in the baseline VAR are qualitatively coherent; all instruments identify a monetary policy shock that eventually triggers an economic recession and lowers prices. However, the picture changes quite materially when the misspecified VAR is used (bottom row of Figure 5).[24]

Using the simple heuristic developed in Section 6 and the test results in Table 5, we can postulate that the dependence of the impact effects on the model specification is likely due to both $z_{A,t}$ and $z_{B,t}$ violating the limited lead-lag exogeneity condition, i.e. they correlate with other shocks, likely related to developments in financial markets and the real economy, that the trivariate VAR(2) is not able to control for.[25] In fact, a possible interpretation for these results is that the instruments $z_{A,t}$ and $z_{B,t}$ may be contaminated by structural shocks that are non-invertible in the trivariate VAR, but which become invertible in the larger system. In such a case, the IRFs obtained in the smaller system

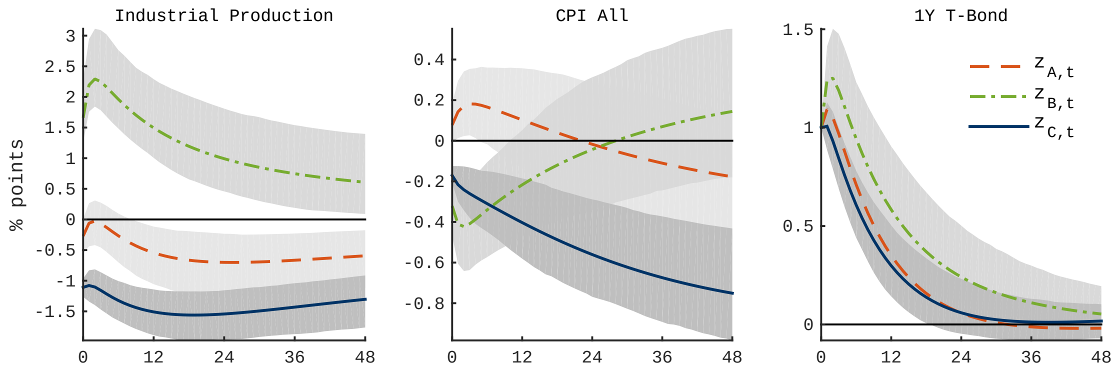
[24]These findings hold across different samples. Figure B.4 in the Appendix reports IRFs from VARs estimated from 1990-1, date that coincides with the start date of the three instruments.

[25]The first factor used in Table 4 is typically regarded as a synthetic measure of economic activity, see e.g. McCracken and Ng (2015). Other than a barometer for financial markets' health levels, the EBP has strong predictive power for an array of measures of economic activity, and is hence likely to account also for other omitted variables (see e.g. Gilchrist and Zakrajšek, 2012; Gertler and Karadi, 2015).

Figure 5: Responses to Monetary Policy Shocks – 1979:2012



(A) Baseline VAR



(B) Misspecified VAR

Notes: Baseline VAR(12) in all variables, top panel (A). Misspecified VAR(2) in three variables, bottom panel (B). VARs estimated with standard macroeconomic priors. Identification in all cases uses the full length of the instruments. $z_{\mathrm{A},t}$: sum of high-frequency surprises within the month; $z_{\mathrm{B},t}$: moving average of high-frequency surprises within the month; $z_{\mathrm{C},t}$: residuals of $z_{\mathrm{A},t}$ on Fed Greenbook forecasts. Shaded areas denote 90% posterior coverage bands.

are distorted due to the bias induced by the violation of both the limited lead-lag and the contemporaneous exogeneity conditions, as in Eq. (25). By adding financial variables to the system, some of the non-invertible shocks become invertible. Hence, the extent of the bias is much reduced and only due to the violation of the contemporaneous exogeneity conditions, as in Eq. (26). Interestingly, the dynamic responses obtained with $z_{\mathrm{C},t}$ are largely similar across the two VARs, which, using the same heuristic, indicates an overall small degree of model misspecification.

10 Conclusions

This paper provides conditions for identification with external instruments in Structural VARs under the assumption of partial invertibility. This property requires that only one or a subset of the structural shocks in the system are invertible, and hence recoverable from the residuals of the chosen empirical model.

We show that, under partial invertibility, correct identification of the dynamic causal effects of interest is obtained in SVAR-IV methods (and LP-IV with controls) if the instrument satisfies a limited lead-lag exogeneity condition, on top of the standard IV validity conditions of relevance and contemporaneous exogeneity. This limited lead-lag exogeneity condition allows to achieve correct identification even when the instrument correlates with other invertible shocks in the system. Overall, the conditions for identification used in this paper are weaker than both the standard full invertibility condition typically required for SVAR-IV, and also the strong lead-lag exogeneity condition needed for LP-IV without controls. Importantly, they allow to extend the range of empirical settings in which SVAR-IV and LP-IV with controls can be used.

Lastly, we show that the identification of impact effects is possible even in the presence of model misspecification of different nature. In this case, an empirical trade-off between efficiency and accuracy of the impulse response functions arises, and the use of larger information sets, or of direct methods, can help producing more robust inference.

References

Alessi, Lucia, Matteo Barigozzi, and Marco Capasso, "Non-Fundamentalness in Structural Econometric Models: A Review," *International Statistical Review*, 04 2011, *79* (1), 16–47.

Altavilla, Carlo, Luca Brugnolini, Refet S. Gürkaynak, Roberto Motto, and Giuseppe Ragusa, "Measuring euro area monetary policy," Working Paper Series 2281, European Central Bank May 2019.

Arezki, Rabah, Valerie A. Ramey, and Liugang Sheng, "News Shocks in Open Economies: Evidence from Giant Oil Discoveries," *The Quarterly Journal of Economics*, 2017, *132* (1), 103–155.

Arias, Jonas E., Juan F. Rubio-Ramirez, and Daniel F. Waggoner, "Inference in Bayesian Proxy-SVARs," Working Papers 18-25, Federal Reserve Bank of Philadelphia November 2018.

Barsky, Robert B. and Eric R. Sims, "News shocks and business cycles," *Journal of Monetary Economics*, 2011, *58* (3), 273–289.

Beaudry, Paul and Franck Portier, "Stock Prices, News, and Economic Fluctuations," *American Economic Review*, September 2006, *96* (4), 1293–1307.

Braun, Phillip A. and Stefan Mittnik, "Misspecifications in vector autoregressions and their effects on impulse responses and variance decompositions," *Journal of Econometrics*, October 1993, *59* (3), 319–341.

Caldara, Dario and Edward Herbst, "Monetary Policy, Real Activity, and Credit Spreads : Evidence from Bayesian Proxy SVARs," *American Economic Journal: Macroeconomics, forthcoming*, 2018.

Calvo, Guillermo A., "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, September 1983, *12* (3), 383–398.

Canova, Fabio and Filippo Ferroni, "Mind the gap! Stylized dynamic facts and structural models.," June 2019. mimeo.

— **and Mehdi Hamidi Sahneh**, "Are Small-Scale SVARs Useful for Business Cycle Analysis? Revisiting Nonfundamentalness," *Journal of the European Economic Association*, 2017, *16* (4), 1069–1093.

Cascaldi-Garcia, Danilo and Marija Vukotic, "Patent-Based News Shocks," Technical Report 2019.

Chahrour, Ryan and Kyle Jurado, "Recoverability," Boston College Working Papers in Economics 935, Boston College Department of Economics November 2017.

Chalak, Karim and Halbert White, "Viewpoint: An extended class of instrumental variables for the estimation of causal effects," *Canadian Journal of Economics/Revue canadienne d'économique*, 2011, *44* (1), 1–51.

Chari, V. V., Patrick J. Kehoe, and Ellen R. McGrattan, "A Critique of Structural VARs Using Real Business Cycle Theory," Levine's Bibliography, UCLA Department of Economics November 2004.

Christiano, Lawrence J., Martin Eichenbaum, and Robert Vigfusson, "Assessing Structural VARs," in "NBER Macroeconomics Annual 2006, Volume 21" NBER Chapters, National Bureau of Economic Research, Inc, 2007, pp. 1–106.

Cloyne, James, "Discretionary Tax Changes and the Macroeconomy: New Narrative Evidence from the United Kingdom," *American Economic Review*, June 2013, *103* (4), 1507–1528.

— **and Patrick Hürtgen**, "The Macroeconomic Effects of Monetary Policy: A New Measure for the United Kingdom," *American Economic Journal: Macroeconomics*, October 2016, *8* (4), 75–102.

Coibion, Olivier, "Are the Effects of Monetary Policy Shocks Big or Small?," *American Economic Journal: Macroeconomics*, April 2012, *4* (2), 1–32.

Fernandez-Villaverde, Jesus, Juan F. Rubio-Ramirez, Thomas J. Sargent, and Mark W. Watson, "ABCs (and Ds) of Understanding VARs," *American Economic Review*, June 2007, *97* (3), 1021–1026.

Fieldhouse, Andrew J. and Karel Mertens, "A Narrative Analysis of Mortgage Asset Purchases by Federal Agencies," NBER Working Papers 23165, National Bureau of Economic Research, Inc February 2017.

Fieldhouse, Andrew J, Karel Mertens, and Morten O Ravn, "The Macroeconomic Effects of Government Asset Purchases: Evidence from Postwar U.S. Housing Credit Policy," *The Quarterly Journal of Economics*, 2018, *133* (3), 1503–1560.

Forni, Mario and Luca Gambetti, "Sufficient information in structural VARs," *Journal of Monetary Economics*, 2014, *66* (C), 124–136.

_ , _ , and Luca Sala, "Structural VARs and noninvertible macroeconomic models," *Journal of Applied Econometrics*, 2019, *34* (2), 221–246.

Galì, Jordi and Mark Gertler, "Inflation dynamics: A structural econometric analysis," *Journal of Monetary Economics*, 1999, *44* (2), 195 – 222.

Gertler, Mark and Peter Karadi, "Monetary Policy Surprises, Credit Costs, and Economic Activity," *American Economic Journal: Macroeconomics*, 2015, *7* (1), 44–76.

Giannone, Domenico and Lucrezia Reichlin, "Does information help recovering structural shocks from past observations?," *Journal of the European Economic Association*, 04-05 2006, *4* (2-3), 455–465.

Gilchrist, Simon and Egon Zakrajšek, "Credit Spreads and Business Cycle Fluctuations," *American Economic Review*, 2012, *102* (4), 1692–1720.

Godfrey, L. G., *Misspecification Tests in Econometrics* number 9780521424592. In 'Cambridge Books.', Cambridge University Press, December 1991.

Gürkaynak, Refet S, Brian Sack, and Eric Swanson, "Do Actions Speak Louder Than Words? The Response of Asset Prices to Monetary Policy Actions and Statements," *International Journal of Central Banking*, May 2005, *1* (1).

Hamilton, James D., "What is an oil shock?," *Journal of Econometrics*, 2003, *113* (2), 363 – 398.

Hannan, Edward J., *Multiple Time Series* A Wiley publication in applied statistics, Wiley, 1970.

Hansen, Lars P. and Thomas J. Sargent, "Formulating and estimating dynamic linear rational expectations models," *Journal of Economic Dynamics and Control*, May 1980, *2* (1), 7–46.

_ **and** _ , "Two problems in interpreting vector autoregressions," in "Rational Expectations Econometrics" Westview, Boulder, 1991, pp. 77–119.

Hansen, Stephen, Michael McMahon, and Matthew Tong, "The long-run information effect of central bank communication," Bank of England working papers 777, Bank of England January 2019.

Hausman, Jerry A, "Specification Tests in Econometrics," *Econometrica*, November 1978, *46* (6), 1251–1271.

Heckman, James and Salvador Navarro-Lozano, "Using Matching, Instrumental Variables, and Control Functions to Estimate Economic Choice Models," *The Review of Economics and Statistics*, 2004, *86* (1), 30–57.

Jordà, Òscar, "Estimation and Inference of Impulse Responses by Local Projections," *American Economic Review*, March 2005, *95* (1), 161–182.

_ **, Moritz Schularick, and Alan M. Taylor**, "Betting the house," *Journal of International Economics*, 2015, *96* (S1), 2–18.

Känzig, Diego R., "The Macroeconomic Effects of Oil Supply News: Evidence from OPEC Announcements," 2019. SSRN: https://ssrn.com/abstract=3185839.

Kilian, Lutz, "Exogenous Oil Supply Shocks: How Big Are They and How Much Do They Matter for the U.S. Economy?," *The Review of Economics and Statistics*, May 2008, *90* (2), 216–240.

Kuttner, Kenneth N., "Monetary policy surprises and interest rates: Evidence from the Fed funds futures market," *Journal of Monetary Economics*, June 2001, *47* (3), 523–544.

Leeper, Eric M., Todd B. Walker, and Shu-Chun Susan Yang, "Fiscal Foresight and Information Flows," *Econometrica*, May 2013, *81* (3), 1115–1145.

Lippi, Marco and Lucrezia Reichlin, "The Dynamic Effects of Aggregate Demand and Supply Disturbances: Comment," *American Economic Review*, June 1993, *83* (3), 644–652.

_ **and** _ , "VAR analysis, nonfundamental representations, Blaschke matrices," *Journal of Econometrics*, July 1994, *63* (1), 307–325.

Lu, Xun and Halbert White, "Robustness checks and robustness tests in applied economics," *Journal of Econometrics*, 2014, *178*, 194 – 206. Annals Issue: Misspecification Test Methods in Econometrics.

McCracken, Michael W. and Serena Ng, "FRED-MD: A Monthly Database for Macroeconomic Research," Working Papers 2015-12, Federal Reserve Bank of St. Louis June 2015.

Melosi, Leonardo, "Signaling Effects of Monetary Policy," *Review of Economic Studies*, 2017, *forthcoming.*

Mertens, Karel, "Advances in Empirical Macroeconomics, Lecture 2," Bonn Summer School 2014.

_ **and José Luis Montiel-Olea**, "Marginal Tax Rates and Income: New Time Series Evidence," *The Quarterly Journal of Economics*, 2018, *133* (4), 1803–1884.

_ **and Morten O. Ravn**, "Empirical Evidence on the Aggregate Effects of Anticipated and Unanticipated US Tax Policy Shocks," *American Economic Journal: Economic Policy*, May 2012, *4* (2), 145–81.

_ **and** _ , "The Dynamic Effects of Personal and Corporate Income Tax Changes in the United States," *American Economic Review*, June 2013, *103* (4), 1212–47.

Miranda-Agrippino, Silvia and Giovanni Ricco, "The Transmission of Monetary Policy Shocks," The Warwick Economics Research Paper Series (TWERPS) 1136, University of Warwick, Department of Economics 2017.

_ **, Sinem Hacioglu-Hoke, and Kristina Bluwstein**, "When Creativity Strikes: News Shocks and Business Cycle Fluctuations," Discussion Papers 1823, Centre for Macroeconomics (CFM) August 2018.

Paul, Pascal, "The Time-Varying Effect of Monetary Policy on Asset Prices," Working Paper Series 2017-9, Federal Reserve Bank of San Francisco May 2017.

Plagborg-Møller, Mikkel and Christian K. Wolf, "Instrumental Variable Identification of Dynamic Variance Decompositions," Technical Report, Princeton University 2018.

_ and _ , "Local Projections and VARs Estimate the Same Impulse Responses," Working Paper 2018.

Ramey, Valerie A., "Identifying Government Spending Shocks: It's all in the Timing," *The Quarterly Journal of Economics*, 2011, *126* (1), 1–50.

_ , "Macroeconomic Shocks and Their Propagation," in John B. Taylor and Harald Uhlig, eds., *Handbook of Macroeconomics*, Vol. 2 of *Handbook of Macroeconomics*, Elsevier, 2016, chapter 2, pp. 71 – 162.

_ and Sarah Zubairy, "Government Spending Multipliers in Good Times and in Bad: Evidence from US Historical Data," *Journal of Political Economy*, 2018, *126* (2), 850–901.

Ricco, Giovanni, Giovanni Callegari, and Jacopo Cimadomo, "Signals from the government: Policy disagreement and the transmission of fiscal shocks," *Journal of Monetary Economics*, 2016, *82* (C), 107–118.

Romer, Christina D. and David H. Romer, "A New Measure of Monetary Shocks: Derivation and Implications," *American Economic Review*, September 2004, *94* (4), 1055–1084.

_ and _ , "The Macroeconomic Effects of Tax Changes: Estimates Based on a New Measure of Fiscal Shocks," *American Economic Review*, June 2010, *100* (3), 763–801.

Rozanov, Yu. A., *Stationary random processes* Holden-Day series in time series analysis, Holden-Day, 1967.

Sims, Christopher A, "Macroeconomics and Reality," *Econometrica*, January 1980, *48* (1), 1–48.

Sims, Christopher A. and Tao Zha, "Does Monetary Policy Generate Recessions?," *Macroeconomic Dynamics*, April 2006, *10* (02), 231–272.

Stock, James H., "What's New in Econometrics: Time Series, Lecture 7," NBER Summer Institute, Short course lectures 2008.

_ **and Mark W. Watson**, "Disentangling the Channels of the 2007-09 Recession," *Brookings Papers on Economic Activity*, 2012, *44* (1 (Spring), 81–156.

_ **and** _ , "Identification and Estimation of Dynamic Causal Effects in Macroeconomics Using External Instruments," *The Economic Journal*, 2018, *128* (610), 917–948.

Sudipto, Banerjee and Anindya Roy, *Linear Algebra and Matrix Analysis for Statistics* Chapman & Hall/CRC Texts in Statistical Science, Chapman and Hall/CRC, 2014.

White, Halbert and Karim Chalak, "Testing a conditional form of exogeneity," *Economics Letters*, November 2010, *109* (2), 88–90.

Appendix

A Model

The economy is populated by a representative infinitely-lived household seeking to maximise

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, H_t) \ , \tag{A.1}$$

with a period utility

$$U(C_t, H_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{H_t^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}} \ , \tag{A.2}$$

where σ is the risk aversion parameter, φ is the Frisch elasticity, and H_t are hours worked. C_t is a consumption bundle defined as

$$C_t \equiv \left(\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} \right)^{\frac{\varepsilon}{1-\varepsilon}} \ , \tag{A.3}$$

where $C_t(i)$ is the quantity of good i consumed by the household in period t. A continuum of goods $i \in [0, 1]$ exists. The log-linearised households optimality conditions are given by the Euler equation

$$c_t = \mathbb{E}[c_{t+1}] - \frac{1}{\sigma} \left(r_t - \mathbb{E}[\pi_{t+1}] \right) \ , \tag{A.4}$$

and by the labour supply schedule

$$w_t = \frac{1}{\varphi} h_t + \sigma \, c_t \ , \tag{A.5}$$

where w_t is the labour wage on a competitive labour market. Agents maximise their intertemporal utility subject to a flow budget constraint. Agents can hold bonds or firms capital, and a no arbitrage condition between bonds and capital holds

$$\frac{1}{\beta} \left(r_t - \mathbb{E}[\pi_{t+1}] \right) = \frac{1}{\beta - (1-\delta)} \mathbb{E}[z_{t+1}] \ , \tag{A.6}$$

where δ is the rate of depreciation of capital. Firms produce differentiated goods $j \in [0, 1]$ by using a Cobb-Douglas technology to aggregate capital and labour

$$Y_t(j) = A_t K_{t-1}(j)^{\alpha} H_t(j)^{1-\alpha} \tag{A.7}$$

where, importantly, log technology $a_t \equiv log(A_t)$ has a news component

$$a_t = \rho_a \, a_{t-1} + \sigma_a u^a_{t-4} \ , \tag{A.8}$$

where u^a_t is an i.i.d. normally distributed technology shock. The static optimality condition on the production inputs delivers the linearised relation

$$w_t + h_t = k_{t-1} + z_t \ . \tag{A.9}$$

The log-linearised production function of the firms is

$$y_t = a_t + \alpha k_{t-1} + (1-\alpha)\, h_t \ . \tag{A.10}$$

Firms set prices in a staggered way à la Calvo (1983) with an indexation mechanisms of the type proposed by Galì and Gertler (1999). Thus, each period, a measure $1-\theta$ of firms reset their prices, while prices for a fraction θ of the firms are $P_t(j) = P_{t-1}\pi^{\gamma}_{t-1}$. θ is an index of price stickiness. The firms that can reset their prices maximise the expected sum of profits

$$max_{P^*_t(j)} \sum_{\tau=0}^{\infty} (\beta\theta)^{\tau} \left(P^*_t(j) \left(\frac{P_t - 1 + \tau}{P_{t-1}} \right)^{\gamma} - MC_{t+\tau} \right) Y_{t+\tau}(j) \ , \tag{A.11}$$

where MC_t are the real marginal costs in period t. The first order conditions from this problem, combined with the aggregate price equation, form a hybrid New Keynesian Phillips Curve

$$\pi_t = \gamma \, \pi_{t-1} + \beta \mathbb{E}[\pi_{t+1}] + \lambda mc_t \ , \qquad \lambda \equiv \frac{(1-\theta)\ (1-\beta\,\theta)}{\theta} + u^{\pi}_t, \tag{A.12}$$

where u^{π}_t is an i.i.d. normally distributed inflation-specific shock, and marginal costs evolve as

$$mc_t = \alpha \, z_t + (1-\alpha)\, w_t - a_t \ . \tag{A.13}$$

The linearised law of motion for firms capital is

$$I_t = K_{t+1} - (1-\delta) K_t \ , \tag{A.14}$$

where K_t is physical capital and I_t is investment. The log-linearisation of this equation yields[26]

$$i_t = k_t - (1-\delta)\; k_{t-1}\ . \tag{A.15}$$

A fiscal authority absorbs a share of output into wasteful government spending

$$G_t = (1-\rho_g)G + \rho_g G_{t-1} e^{u^g_{t-4}} \tag{A.16}$$

the log-linearised equation for government spending is

$$g_t = \rho_g\, g_{t-1} + u^g_{t-4}\ , \tag{A.17}$$

where u^g_t is an i.i.d. normally distributed government demand shock. At the steady state $G = gY$. A monetary authority sets the nominal interest rate using a monetary rule with a smoothing term

$$r_t = \rho_r\, r_{t-1} + (1-\rho_r)\left(\phi_\pi \overline{\pi_t} + \phi_y \overline{\Delta y_t}\right) + \sigma_r u^r_t\ , \tag{A.18}$$

where $\overline{\pi_t}$ and $\overline{\Delta y_t}$ are, respectively, average inflation and the average rate of output growth over the last four periods, and u^r_t is a white noise i.i.d. normally distributed monetary policy shock. The monetary policy innovation can be recovered from current and past values of the policy rate, inflation and output. Finally, the aggregate economy clears

$$Y y_t = C c_t + I i_t + G g_t\ . \tag{A.19}$$

Table A.1 reports the calibration for this benchmark NK model. For this set of parameters the model fails the 'poor man's invertibility condition' of Fernandez-Villaverde et al. (2007).

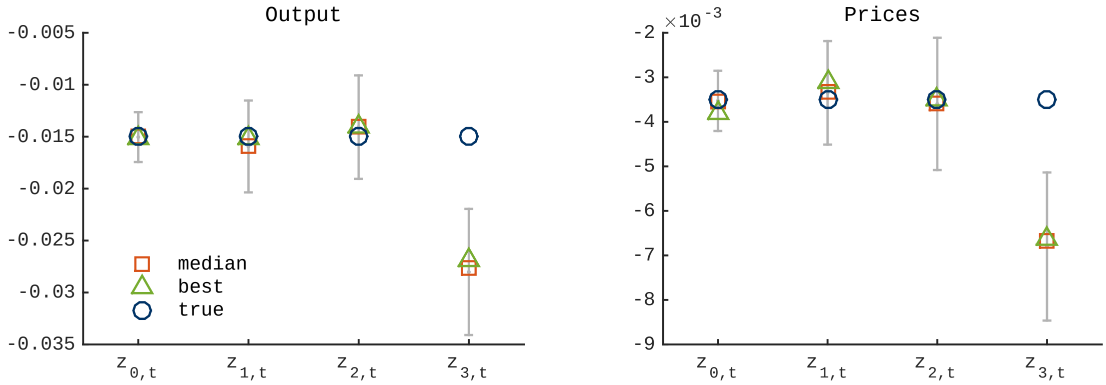
[26] In order to have smoother impulse response functions, without introducing autocorrelation in the shock processes, we added an ad hoc quadratic adjustment of the form $i_t = k_t - (1-\delta)\; k_{t-1} + (k_t - (1-\delta)\; k_{t-1})^2$.

Table A.1: Calibrated Parameters

Parameter	Value	Description
α	0.4	share of capital in output
β	0.99	discount factor
δ	0.025	depreciation of capital
σ	1	risk aversion consumption
φ	2	labor disutility
g	0.2	share of public spending in output
θ	0.75	price stickiness
γ	0.2	indexation parameter (NK Phillips curve backward term)
ϵ	10	substitutability goods
ρ_r	0.95	monetary policy smoothing
ϕ_y	0.5	monetary policy output growth
ϕ_r	1.2	monetary policy inflation
ρ_a	0.5	productivity autocorrelation
ρ_g	0.95	public spending autocorrelation
σ_a	10	
σ_r	0.1	

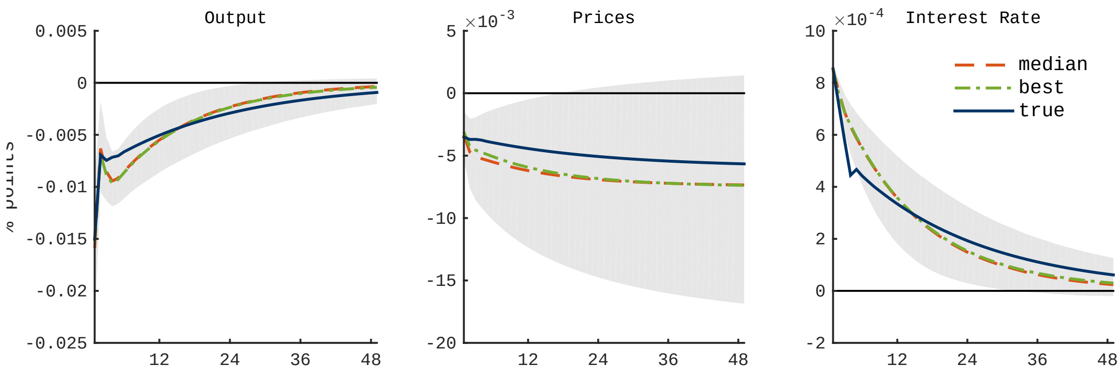
B Additional Charts

Figure B.1: Responses to MP Shock – Simulation & VAR(1)



(A) Impact Responses: All Instruments

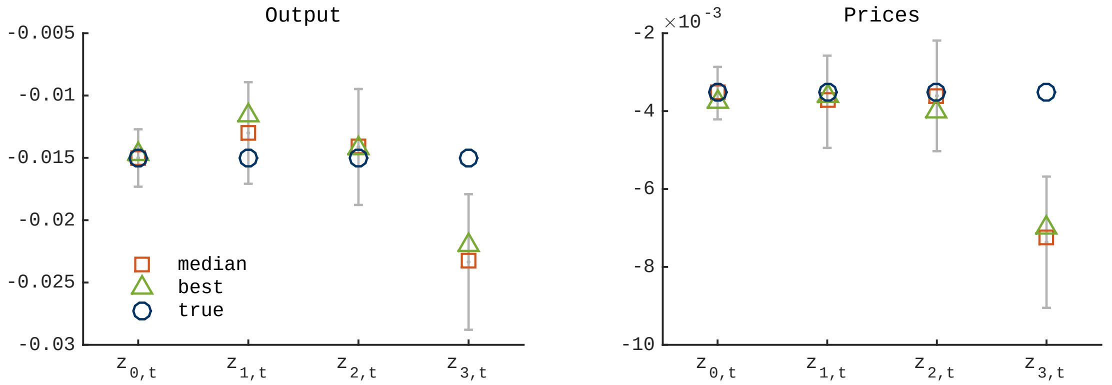
Note: Impact responses to monetary policy shock from partially-invertible DSGE identified with external instruments and estimated with a VAR(1) in four observables. $z_{0,t}$: observed shock case; $z_{1,t}$: instrument correlates with monetary policy shock only; $z_{2,t}$: instrument also correlates with past spending shocks; $z_{3,t}$ instrument correlates also with past technology shocks. Grey vertical lines are 2 standard deviations error bars from the distribution of impact responses across 5,000 simulated economies of sample size $T = 300$ periods. True impact (blue circle), median across simulations (orange square), minimum distance from median (best) simulation (green triangle).



(B) $z_{1,t}$: external instrument correlates with monetary policy shock only

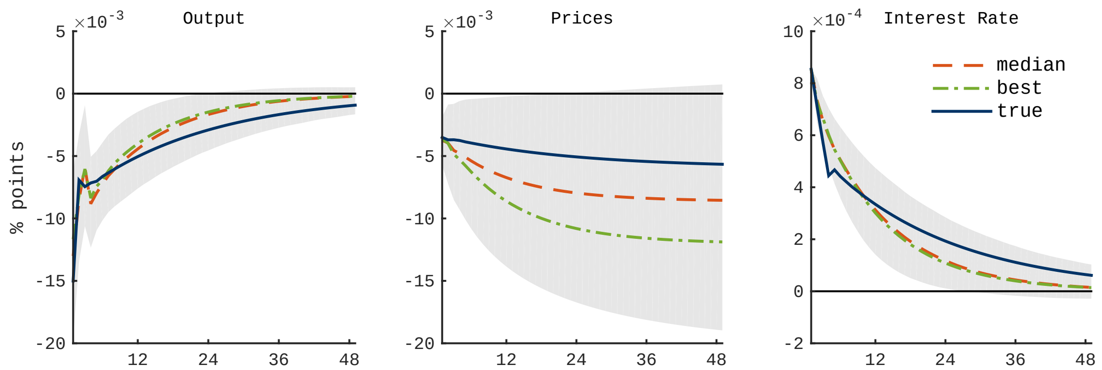
Notes: Impulse responses to monetary policy shock from partially-invertible DSGE identified with external instruments and estimated with a VAR(1) in four observables. Instrument correlates with monetary policy shocks only. Grey shaded areas denote 90th quantiles of the distribution of IRFs across 5,000 simulated economies of sample size $T = 300$ periods. Model responses (true, blue solid), median across simulations (orange dashed), minimum distance from median (best) simulation (green dash-dotted).

Figure B.2: Responses to MP Shock – Simulation & VAR(2)



(A) Impact Responses: All Instruments

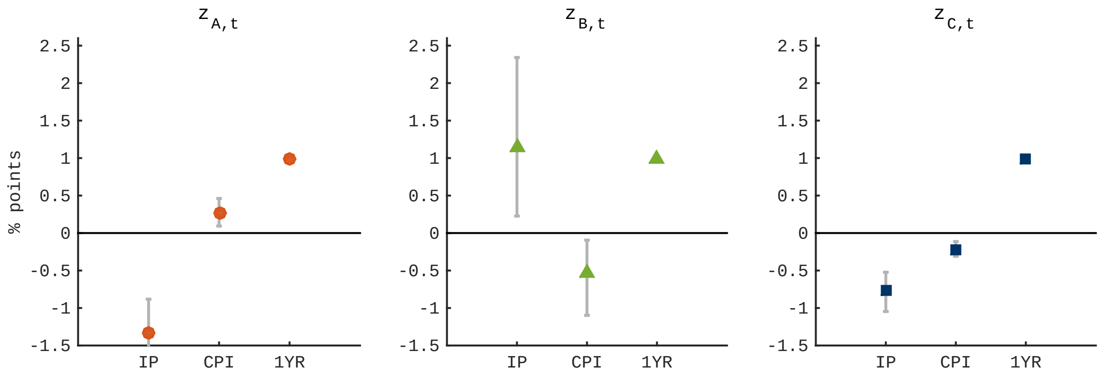
Note: Impact responses to monetary policy shock from partially-invertible DSGE identified with external instruments and estimated with a VAR(2) in four observables. $z_{0,t}$: observed shock case; $z_{1,t}$: instrument correlates with monetary policy shock only; $z_{2,t}$: instrument also correlates with past spending shocks; $z_{3,t}$ instrument correlates also with past technology shocks. Grey vertical lines are 2 standard deviations error bars from the distribution of impact responses across 5,000 simulated economies of sample size $T = 300$ periods. True impact (blue circle), median across simulations (orange square), minimum distance from median (best) simulation (green triangle).



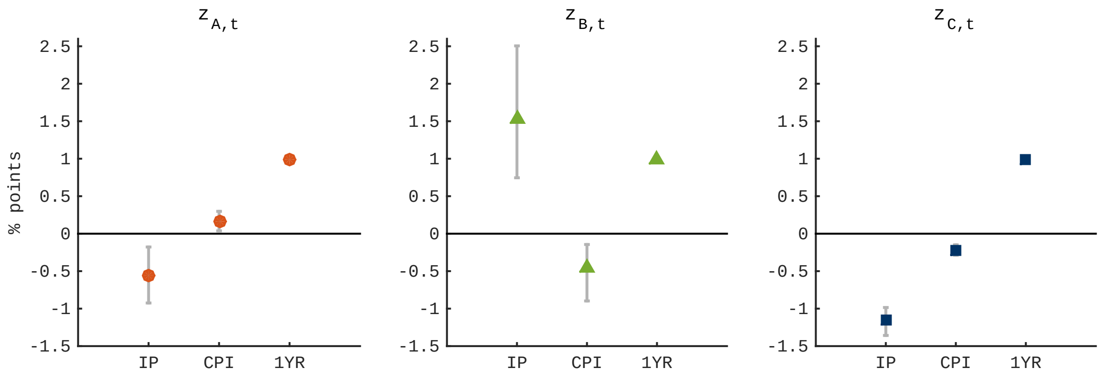
(B) $z_{1,t}$: external instrument correlates with monetary policy shock only

Notes: Impulse responses to monetary policy shock from partially-invertible DSGE identified with external instruments and estimated with a VAR(2) in four observables. Instrument correlates with monetary policy shocks only. Grey shaded areas denote 90th quantiles of the distribution of IRFs across 5,000 simulated economies of sample size $T = 300$ periods. Model responses (true, blue solid), median across simulations (orange dashed), minimum distance from median (best) simulation (green dash-dotted).

Figure B.3: Impact Responses to Monetary Policy Shocks – 1990:2012



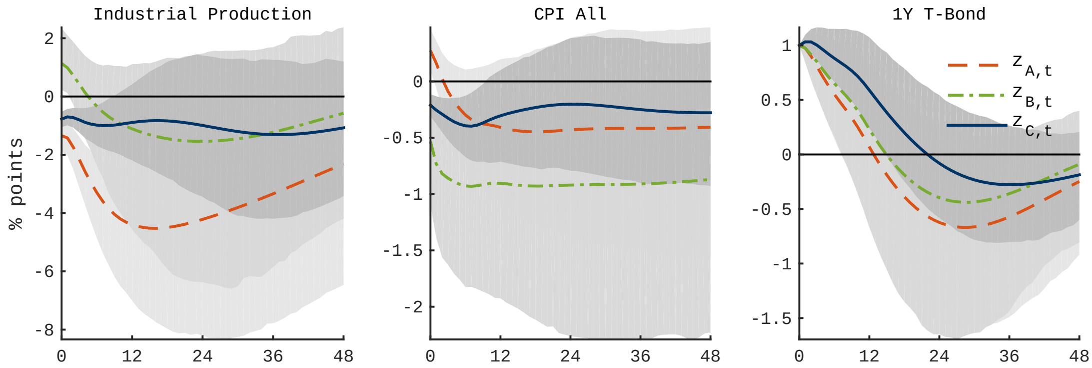
(A) Baseline VAR



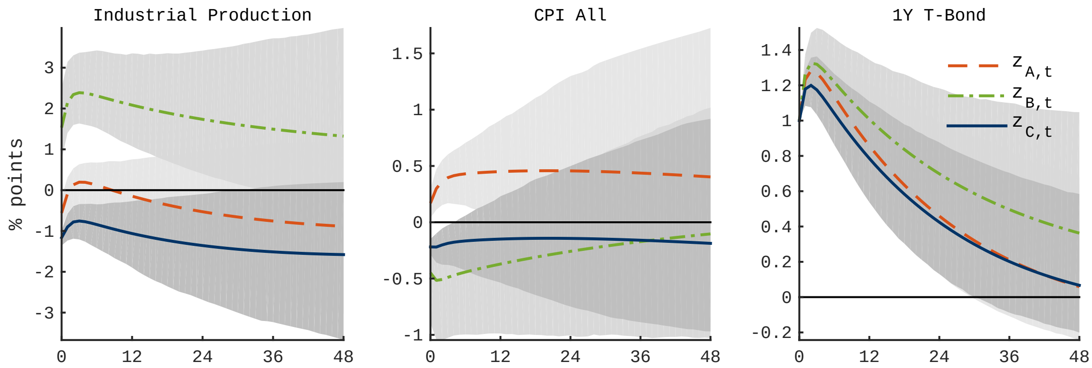
(B) Misspecified VAR

Notes: Baseline VAR(12) in all variables, top panel (A). Misspecified VAR(2) in three variables, bottom panel (B). VARs estimated with standard macroeconomic priors. Identification in all cases uses the full length of the instruments. $z_{\mathrm{A},t}$: high-frequency surprises at scheduled FOMC meetings; $z_{\mathrm{B},t}$: moving average of high-frequency surprises within the month; $z_{\mathrm{C},t}$: residuals of $z_{\mathrm{A},t}$ on Fed Greenbook forecasts. Shaded areas denote 90% posterior coverage bands.

Figure B.4: Responses to Monetary Policy Shocks – 1990:2012



(A) Baseline VAR



(B) Misspecified VAR

Notes: Baseline: VAR(12) in all variables. Misspecified: VAR(2) in three variables. VARs estimated with standard macroeconomic priors. Identification in all cases uses the full length of the instruments. $z_{\mathrm{A},t}$: sum of high-frequency surprises within the month; $z_{\mathrm{B},t}$: moving average of high-frequency surprises within the month; $z_{\mathrm{C},t}$: residuals of $z_{\mathrm{A},t}$ on Fed Greenbook forecasts. Shaded areas denote 90% posterior coverage bands.