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LOGIC

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Why Should You Study Logic?

What is "logic"? According to the American Heritage Dictionary, it is:

The study of the principles of reasoning, especially of the structure of propositions as distinguished from their content, and of method and validity in deductive reasoning.

Whew. Another definition:

The formal guiding principles of a discipline, school, or science.

Another definition:

The circuitry in a computer.

It is through all these definitions of "logic" that we discover why it's important to study Logic – no matter what you plan on doing. Whether you become a mathematician, an astrophysicist, a politician, or even an English professor, logical thinking is crucial.

By studying Logic you will accomplish several things at once:

You will study a new mathematical system, but which, like the integers with multiplication and addition, follows some rules you know, at least intuitively. And so we will venture into Abstract Mathematics, complete with variables, formulas, proofs. Consider this an introduction to Abstract Mathematics.

You will study a system of reasoning which is used in all aspects of life:

In Science, you need to set up a logical experiment, one whose outcome follows directly from your assumptions – as opposed to variables you didn't consider, or faulty reasoning along the way.

Or in Politics, to conduct a logical debate about the balanced budget.

Or right now, trying to convince your parents to let you go out on a Saturday night.

Which is the more convincing argument?

I did all my homework Friday.

So I'll have my Saturday night free.

Therefore, you should let me out on Saturday night.

or

I want to go out.

Everybody else gets to go.

Therefore, you should let me out on Saturday night.

While the first argument may not convince your parents, it does eliminate a possible homework argument on their part. As someone who has tried the second argument, let me assure you that it's full of holes – it's not a logical argument.

Another aspect of being able to think logically is that it empowers you to analyze and understand issues which directly or indirectly confront you in everyday life. It enables you to make decisions and render judgments about things, in a correct way – the logical way.

Think about the TV or radio commentators and newspaper editorial writers. Have you ever wondered what their roles are in society? Why is it that we need someone else, no matter how knowledgeable about a subject, to tell us what an article means, or how to make a decision about an issue? What if they are wrong and our decision, based on their interpretation of facts, could prove harmful to us? Don't misunderstand - we are not talking about acquiring additional, pertinent information for our decision making. What we are discussing are those instances when someone else's analysis or decision replaces our own, simply because we do not know how to reliably come to our own conclusion.

By studying logic, you can practice the art of presenting an argument in a coherent and meaningful way. You can participate in a discussion or debate or present a logical counter argument.

This summer you will study Logic to lay a foundation for you as an intelligent, educated individual. Studying logic is not just to improve our mathematics, but to improve our analytical reasoning skills, to enhance our problem solving skills, to exercise control over our destiny. And when you have control over your destiny, you are free in the fullest sense of the word.

LOGIC

"What's your name?"

"My name is Graciella."

The second sentence is a statement, which means it is either true or false. Is "what's your name?" a statement?

"My name is Graciella." is a simple statement. Simple statements combine to make more complicated ones, such as: "If the Chicago Bulls win at least three of their next five games or the Miami Heat loses at least two of their next five games, the Bulls will have home court advantage." Determining whether or not complicated statements are true from information on whether the component simple statements are true or false, is the subject of Logic.

When we study Logic we deal with statements, just as in arithmetic we deal with numbers. In the same way that being good in arithmetic helps you whenever numbers show up, being good in logic will help you whenever statements show up – which means it will help you with language in everyday life.

As in all sciences, Logic has its own jargon. So in Logic, when we ask "Is this statement true or false?" we say "What is the truth value for this statement?" The answer to which could be either "true" or "false." When we write it down, as you will see, we sometimes shorten this to T or F.

So let's start studying Logic.

Definition 1 *A* ***statement*** *is a sentence to which a truth value, either true or false but not both, can be assigned.*

We don't want to have to write "My name is Graciella" every time we refer to this statement, so we assign to it a letter, say p, and refer to p rather than to "My name is Graciella." Statements are usually denoted by (lower case) letters p, q, r, ... This is actually useful in a much more important way. Statements combine according to rules which do not depend on the content of the statements. Therefore, when we state the rules, we can state them for statements $p, q, r, \ldots$ and the rules will then apply for whatever statements these letters stand for.

You will soon see this device used in Algebra. In fact, it is one of the most important tools in all of Mathematics.

Example 2 *Let p be the statement, "Red is a color." Let q be the statement "Thanksgiving is in July." Hence, p is true and q is false.*

QUESTION: Can you give an example of a sentence which is *not* a statement?

The simplest operation on statements is forming their negation.

"Today is Tuesday." What do you think the negation of this statement is?

"My name is not Juan." What do you think the negation of this statement is?

Definition 3 *The* **negation** *of a statement p, denoted by $\sim p$ (read as "not p"), is the statement whose truth value is the opposite of the truth value of p.*

We want to be efficient in our description of the possible truth values of various statements. For this reason, we use **truth tables**. Here is an example:

TRUTH TABLE FOR $\sim p$

p	$\sim p$
T	F
F	T

You read this truth table in the following way: In the column headed by p are the possible truth values for p, and in the column headed by $\sim p$ are the **corresponding** truth values for $\sim p$. Therefore, the second line in the table says that when p is T, $\sim p$ is F.

Example 4 *Using statement p from Example 2, then $\sim p$ is the statement "Red is* **not** *a color," and $\sim q$ is the statement "Thanksgiving is* **not** *in July."*

A negated statement is again a statement, which means we can negate **it.**

"My name is Juan."

"My name is not Graciella."

"My name is not not Bill."

Can you find a simpler way of making the last statement?

Let us write a truth table for $\sim (\sim p)$:

p	$\sim p$	$\sim (\sim p)$
T	F	T
F	T	F

Notice that when p is T, so is $\sim (\sim p)$. And when p is F, $\sim (\sim p)$ is F. We say that p and $\sim (\sim p)$ have identical truth tables.

If someone says "My name is not not Bill" as opposed to just saying "My name is Bill" we think he is acting funny. We don't want to make a distinction between the two statements. And in general, we don't want to make a distinction between $\sim (\sim p)$ and p. More generally, we don't want to make distinctions between statements with the same truth tables.

Definition 5 *Two statements are said to be* ***logically equivalent*** *if and only if they have the same (identical) truth table.*

Notation: $p \equiv q$ (Read p is logically equivalent to q)

If p and q don't have the same truth tables, we write $p \not\equiv q$ (Read p is not logically equivalent to q).

COMBINING STATEMENTS

"My name is Madeleine and I live in Paris." This is a combination of two simpler statements: "My name is Madeleine" and "I live in Paris." Suppose the first statement is true and the second statement is false. Is "My name is Madeleine and I live in Paris" true or false?

"My name is Madeleine or I live in Paris." This is a combination of two simpler statements: "My name is Madeleine" and "I live in Paris." Suppose the first statement is true and the second statement is false. Is "My name is Madeleine or I live in Paris" true or false?

How do the truth values of the simpler statements combine to give a truth value for the combined statement? The key words are **and, or**. They tell us in what way the two component statements combine. In Logic, we call the combination of statements with the word "and" a conjunction.

Definition 6 ***Conjunction*** *of p and q, $p \bigwedge q$ (read as "p and q"), is the statement that is true if and only if both p and q are true.*

RULE FOR DETERMINING THE TRUTH VALUE OF A CONJUNCTION: Given statements p and q. The truth value for $p \bigwedge q$ is true whenever each of the statements is true; otherwise the truth value is false.

Note: The truth table now has to allow for q to be true or false for each truth value for p. We therefore have to add rows to the truth table for $p \bigwedge q$.

TRUTH TABLE FOR $p \bigwedge q$

p	q	$p \bigwedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 7 *Let p be the statement "Red is a color."*

Let q be the statement "Thanksgiving is in July."

Let r be the statement "Snow is white."

The conjunction of p and q is the statement "Red is a color and Thanksgiving is in July."

$p \bigwedge r$ is written as "Red is a color and snow is white."

$q \bigwedge r$ is written as "Thanksgiving is in July and snow is white."

So, $p \bigwedge q$ is false, $p \bigwedge r$ is true, and $q \bigwedge r$ is false.

QUESTION: Suppose we know the conjunction "The cookie jar is full and I didn't eat any cookies" is a false statement. What are all the different possible scenarios to explain why this is false?

QUESTION: How many rows will we need for the truth table for a statement which combines three statements? Four statement? Can you make a general rule?

QUESTION: What is the truth table for $p \bigwedge \sim p$?

A statement which is always false is called a **contradiction.**

Let's go back to the combination of statements with the word "or." "My name is Madeleine or I live in Paris." This way of combining statements is called a **disjunction.**

Definition 8 ***Disjunction*** *of p and q, $p \bigvee q$ (read as "p or q"), is the statement that is true if and only if at least one of p and q is true.*

RULE FOR DETERMINING THE TRUTH VALUE OF A DISJUNCTION: Given statements p and q. The truth value for $p \bigvee q$ is true whenever either component statements are true; otherwise the truth value is false.

Let's take a look at the truth table for $p \bigvee q$:

TRUTH TABLE FOR $p \bigvee q$

p	q	$p \bigvee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 9 *Let $p, q,$ and r be the statements from Example 7. Then $p \bigvee q$ is the statement "Red is a color or Thanksgiving is in July." $p \bigvee r$ is written as "Red is a color or snow is white." $q \bigvee r$ is written as "Thanksgiving is in July or snow is white."*

$p \bigvee q$ is true, $p \bigvee r$ is true, and $q \bigvee r$ is true.

Example 10 *Suppose you know that the disjunction "Mozart was a sky diver or Gauss was not a mathematician" is false. What does that mean about the component statements "Mozart was a sky diver" and "Gauss was not a mathematician"? How could you change those statements to make our disjunction true?*

QUESTION: What is the truth table for $p \bigvee \sim p$?

A statement which is always true is called a **tautology**.

Now that we know three operations for statements (negation, conjunction, disjunction), we want to perform combinations of these operations. The order of operations is extremely important. We can open the door and then enter the room, but don't try doing this in the opposite order. When we combine operations we have to specify the order in which they're carried out. We do that with the help of parentheses.

In Logic (as in the rest of Mathematics), we **do what's in parentheses first**.

Let's see an example.

Example 11 *Consider the statements $(p \bigvee q) \bigwedge r$ and $p \bigvee (q \bigwedge r)$. Because they have different truth tables they are not logically equivalent.*

p	q	r	$(p \bigvee q) \bigwedge r$	$p \bigvee (q \bigwedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Observe that the last two columns are not the same. This means that the statements $(p \bigvee q) \bigwedge r$ and $p \bigvee (q \bigwedge r)$ do not have the same truth tables; they are not equivalent.

Note: To make the last statement the table did not need to be completed since their truth values differ in the second row. This means they are not logically equivalent.

Note: The expression $p \bigvee q \bigwedge r$ is not a statement since it is ambiguous. In other words, it isn't clear which operation, $\bigwedge$ or $\bigvee$ should be performed *first. Our table above shows that the order of applying operations changes the truth values.*

DeMorgan's Laws

Let us consider the conjunction "Red is a color and Thanksgiving is in July." What do you think the negation of this conjunction is? This requires some thought and where truth tables prove their value by helping us out.

p	q	$p \bigwedge q$	$\sim (p \bigwedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Observe that in the last column there is only one F, and the rest T. Have we seen such columns before?

Yes, we have: in truth tables for disjunctions. So now let's look for a disjunction which is false when both p and q are true, and true otherwise. Observe that when p and q are true, $\sim p$ and $\sim q$ are false. It is therefore reasonable to look at the truth table for the disjunction of $\sim p$ and $\sim q$.

p	q	$\sim p$	$\sim q$	$\sim p \bigvee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Look! The truth tables for $\sim p \bigvee \sim q$ and for $\sim (p \bigwedge q)$ are identical, and therefore the two statements are logically equivalent. So to answer our original question, the negation of the statement "Red is a color and Thanksgiving is in July" is the statement "Red is not a color or Thanksgiving is not in July."

$$\sim p \bigvee \sim q \equiv \sim (p \bigwedge q)$$

This is the first of **DeMorgan's Laws.**

The second DeMorgan Law is similar, and in fact follows from the first. We will show you how it follows, but before we do so, why don't you make your own conjecture and check the truth tables to see if you're right.

Done?

Now here's another proof:

Let us denote $\sim p$ by P, and $\sim q$ by Q. This implies that

$$\begin{aligned} p &\equiv \sim (\sim p) \\ &\equiv \sim P \end{aligned}$$

and similarly,

$$q \equiv \sim Q$$

Therefore, the first DeMorgan law reads

$$P \bigvee Q \equiv \sim \left(\sim P \bigwedge \sim Q\right)$$

Let us negate both sides. We get

$$\sim \left(P \bigvee Q\right) \equiv \sim P \bigwedge \sim Q$$

This is the second of **DeMorgan's Laws.**

So the negation of the statement "Cliff got an A or Rebecca got an A" is "Cliff did not get an A and Rebecca did not get an A."

Conditional Statements

We use other combinations of statements: "If it rains heavily then the ball game is cancelled." One component is the statement "It rains heavily" let us denote it by p; the other is "the ball game is cancelled" let us denote it by q. The combined statement is denoted by $p \rightarrow q$. This combination is called a conditional statement. In a moment we will give its formal definition which will include the rules for determining its truth value. But before we do that let's see another example.

"I'll give you \$5 when pigs fly"

To say this in the standard Logic form you'd say: "If pigs fly I'll give you \$5." This conditional statement is meant to be true, but in the sense that since pigs don't fly, I am not giving you the \$5. This is why we want the combination p (Pigs fly) is false and q (I'll give you \$5) either true or false to result in a conditional statement which is true. In fact the only time that the conditional statement is false is if pigs fly and I do not give you \$5; if p is true and q is false.

And now to the formal definition:

Definition 12 ***Conditional statement****, $p \to q$ (read as "If p, then q" or "p arrow q"), is the statement that is true unless p is true and q is false.*

p is called the ***antecedent*** *and q is called the* ***consequent****. Sometimes, p is called the* ***hypothesis*** *and q is called the* ***conclusion****.*

RULE FOR DETERMINING THE TRUTH VALUE OF A CONDITIONAL STATEMENT: Given statements p and q. The truth value for $p \to q$ is true unless p is true and q is false. If p is true and q is false, then $p \to q$ is false.

TRUTH TABLE FOR $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 13 *Let p be the statement "The temperature is greater than 0 degrees Celsius." Let q be the statement "Water will freeze." Then the conditional statement $p \to q$ is written as "If the temperature is greater than 0 degrees Celsius, then water will freeze." Is this conditional statement true?*

Example 14 *Let r be the statement "My dog barks too much." Let s be the statement "My neighbors complain." Then the conditional statement $r \to s$ is written as "If my dog barks too much, then my neighbors complain." Suppose we know that $r \to s$ is false. Does my dog bark too much? Do my neighbors complain?*

There are several ways of expressing verbally conditional statements. Here are some ways of saying $p \to q$:

"If p, then q."

"p is a sufficient condition for q."

"p only if q."

"q is a necessary condition for p."

"q if p."

Let us return to our first example of a conditional statement: "If it rains heavily then the ball game is cancelled." Consider the statement "If the ball game is not cancelled then it does not rain heavily." This is called the contrapositive of the first statement. In general:

Definition 15 *The* ***contrapositive*** *of $p \to q$ is $\sim q \to \sim p$.*

What is the contrapositive of the contrapositive?

"If the ball game is cancelled then it rains heavily." This is the converse of the first statement, and in general:

Definition 16 *The* ***converse*** *of $p \to q$ is $q \to p$.*

What is the converse of the converse?

"If it doesn't rain heavily then the ball game is not cancelled." This is the inverse of the statement, and in general:

Definition 17 *The* ***inverse*** *of $p \to q$ is $\sim p \to \sim q$.*

What is the inverse of the inverse?

Let us return to the original statement, "If it rains heavily then the ball game is cancelled." Suppose we know this is true. What can we say about its contrapositive, "If the ball game is not cancelled then it does not rain heavily?" Suppose we know the first statement is false. What can we say about its contrapositive? We summarize this by saying that a conditional statement and its contrapositive are logically equivalent. Now let's prove it.

Theorem 18 $p \to q \equiv \sim q \to \sim p$.

Proof.

p	q	$p \to q$	$\sim q \to \sim p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

We see that the truth tables for $p \to q$ and for $\sim q \to \sim p$ are the same, and so the two statements are logically equivalent. ∎

The converse of $p \to q$ is $q \to p$. The contrapositive of $q \to p$ is $\sim p \to \sim q$, which is the inverse of $p \to q$. So $q \to p \equiv \sim p \to \sim q$, or the converse is equivalent to the inverse.

Negations of Conditional Statements

What is the negation of "If it rains heavily then the ball game is cancelled"? Again, truth tables to the rescue.

We know the truth table for $\sim (p \to q)$:

p	q	$p \to q$	$\sim(p \to q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

We see that the truth table for $\sim (p \to q)$ is T in one case, and F otherwise. We are reminded of the truth table of a conjunction. Notice that our proposed conjunction is T only when p is T and q is F, alternatively when p and $\sim q$. Let us therefore check the truth table for $p \bigwedge \sim q$.

p	q	$p \to q$	$\sim(p \to q)$	$p \bigwedge \sim q$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

Therefore:

Theorem 19

$$\sim(p \to q) \equiv p \bigwedge \sim q$$

In fact, simply by taking the negation of both sides of this equation we have:

Theorem 20

$$p \to q \equiv \sim p \bigvee q$$

This means that **any conditional statement can be written as a disjunction**, and vice versa.

Remark 1 *We have learned quite a lot. Let's summarize what we have:*

1. $p \to q \equiv \sim q \to \sim p$ *(A conditional statement is equivalent to its contrapositive)*

2. $q \to p \equiv \sim p \to \sim q$ *(The converse is equivalent to the inverse)*

3. $\sim (\sim p) \equiv p$ ***(Double Negation)***

4. $p \to q \equiv \sim p \bigvee q$

5. $p \bigvee q \equiv q \bigvee p$; $p \bigwedge q \equiv q \bigwedge p$ ***(Commutative Properties)***

6. $p \to q \not\equiv q \to p$

7. $\sim (p \bigwedge q) \equiv \sim p \bigvee \sim q$ *(DeMorgan's Law)*

8. $\sim (p \bigvee q) \equiv \sim p \bigwedge \sim q$ *(DeMorgan's Law)*

9. $\sim (p \rightarrow q) \equiv p \bigwedge \sim q$

Let p be the statement "A triangle is isosceles." Let q be the statement "The triangle has at least two congruent sides." We see that $p \rightarrow q$ is true and $q \rightarrow p$ is true. We abbreviate the two statements to $p \leftrightarrow q$ is true. In our case this is written as "A triangle is isosceles if and only if it has at least two congruent sides." Alternatively, $p \leftrightarrow q$ may also be written as "If a triangle is isosceles, then the triangle has at least two congruent sides and if a triangle has at least two congruent sides, then the triangle is isosceles." Let's give a formal definition:

Definition 21 ***Biconditional statement:*** *$p \leftrightarrow q$ (read as "p if and only if q"). $p \leftrightarrow q$ means $(p \rightarrow q) \bigwedge (q \rightarrow p)$.*

TRUTH TABLE FOR $p \leftrightarrow q$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Let's go back to the statement "A triangle is isosceles if and only if it has at least two congruent sides." It is in fact a definition of the term "isosceles". All definitions in Mathematics are biconditional statements. They allow us to use a term, like "isosceles" instead of a longer condition "has at least two congruent sides."

PROPOSITIONAL CALCULUS

You may have heard of "Calculus" before, as a subject you will take in high school and in college. The word calculus comes from the Latin word for stone pebbles, and is the root of the word "calculate" – because people calculated by counting stones. We use the word here in the simplest sense, that of calculation. An example of a propositional calculation is DeMorgan's Law (the first or the second). It involves two operations on statements, $\sim p \bigvee \sim q$ and $\sim (p \bigwedge q)$ which turn out to be equivalent.

Since the calculations will all involve Logical Equivalence, let us summarize what we know about it:

PROPERTIES OF LOGICAL EQUIVALENCE:
Let p, q and r be any three statements. Then:

1. Reflexive Property: $p \equiv p$
2. Symmetric Property: If $p \equiv q$, then $q \equiv p$.
3. Transitive Property: If $p \equiv q$ and $q \equiv r$, then $p \equiv r$

4. Principle of Substitution. Given any statement. Then part of the statement can be substituted (replaced) by a logically equivalent statement without affecting the truth value of the original statement.

Note that these properties are the same ones held by = when dealing with numbers. It's useful to remember that we deal with operations or equivalences which have the similar properties to the ones we're already familiar with. (Much of mathematics is about abstracting specific results to general situations, and then proving they hold true.)

In fact, what we've constructed so far with our definitions is a whole area of Mathematics called **Propositional Calculus**.

Let us summarize what we know about the properties of the operations on statement:

Theorem 22 *The Algebra of Statements (Propositional Calculus). Let p, q and r be any three statements.*

1. *Closure: $p \bigwedge q$ is a statement and $p \bigvee q$ is a statement.*

2. *Commutative Properties: $p \bigvee q \equiv q \bigvee p$ and $p \bigwedge q \equiv q \bigwedge p$.*

3. *Associative Properties: $(p \bigvee q) \bigvee r \equiv p \bigvee (q \bigvee r)$ and $(p \bigwedge q) \bigwedge r \equiv p \bigwedge (q \bigwedge r)$.*

4. *Identity Properties: If c is a contradiction and t is a tautology, then $p \bigvee c \equiv p$ and $p \bigwedge t \equiv p$.*

5. *Inverse elements: $p \bigvee \sim p \equiv t$ and $p \bigwedge \sim p \equiv c$.*

6. *Distributive properties: $p \bigvee (q \bigwedge r) \equiv (p \bigvee q) \bigwedge (p \bigvee r)$ and $p \bigwedge (q \bigvee r) \equiv (p \bigwedge q) \bigvee (p \bigwedge r)$.*

7. *DeMorgan Properties: $\sim (p \bigwedge q) \equiv \sim p \bigvee \sim q$ and $\sim (p \bigvee q) \equiv \sim p \bigwedge \sim q$.*

8. *Idempotent Properties: $p \bigvee p \equiv p$ and $p \bigwedge p \equiv p$.*

9. *$(p \bigvee q) \bigwedge p \equiv p$ and $(p \bigwedge q) \bigvee p \equiv p$*

These are actually properties of **Boolean Algebra.** Boolean Algebras are used extensively in the design of electrical circuits, for example in the design of computers. So, again, what we are doing is useful in a broader context.

Theorem 23 *$p \bigvee t \equiv t$ and $p \bigwedge c \equiv c$.*

Proof. From Propositional Calculus (PC), #9 we know that

$$\left(t \bigwedge p\right) \bigvee t \equiv t$$

By PC4

$$t \bigwedge p \equiv p$$

so by substitution we have

$$p \bigvee t \equiv t$$

This holds for all statements p and so in particular for the statement $\sim p$. Therefore

$$\sim p \bigvee t \equiv t$$

By negation of the last conclusion we get

$$\sim \left(\sim p \bigvee t\right) \equiv \sim t$$

But the negation of a tautology is a contradiction, $\sim t \equiv c$, and

$$\sim \left(\sim p \bigvee t\right) \equiv c$$

By DeMorgan's law we have

$$\sim\sim p \bigwedge \sim t \equiv c$$

$$p \bigwedge \sim t \equiv c$$

$$p \bigwedge c \equiv c \blacksquare$$

We've examined Logic as a mathematical system. But remember that logic is made up of statements – words, ideas, sentences. At its core, Logic extends to Philosophy, Politics, Biology. Great philosophers from Ancient Greece established logical thinking as a method of proof for their arguments regarding the nature of Justice and Beauty. Scientists have used the logical method to construct their experiments. And of course, mathematicians use logic in constructing their proofs. Now that we have laid all this groundwork, it is time to reap the benefits and get to the meat of our subject: checking whether or not an argument is valid.

Definition 24 *An* ***argument*** *means the assertion that a certain statement (the conclusion) follows(is derived) from other statements (the premises).*

We've all encountered arguments that don't ring true, that are invalid:

"If it's sunny outside, then I'm happy. I'm happy. Therefore, it must be sunny outside."

It's pretty easy to see why this is an invalid argument: I never said I was *only* happy if it was sunny. We're going to lay a mathematical groundwork to show why this argument is invalid, and deal with more complicated ones as well. To do this we need a few more definitions.

An argument is typically a chain of implications. Let's see an example of an implication. Consider the statement p : "Les is Mrs. Brown's second husband." This implies that Mrs. Brown has been married at least twice. Let q be the statement "Mrs. Brown has been married at least twice." We say that the first statement implies the second.

Definition 25 *Let p and q be any two statements. p* ***implies*** *q if and only if $p \bigvee q \equiv q$. When this holds we write $p \Rightarrow q$ and say p implies q or that q follows from p, or that q is derived from p.*

Let's go back to our previous example that "Les is Mrs. Brown's second husband" implies that "Mrs. Brown has been married at least twice." Let's check to see if our mathematical definition matches our intuition that $p \Rightarrow q$.

Example 26

p	q	$p \bigvee q$
T	T	T
T^*	F^*	T^*
F	T	T
F	F	F

The truth table for $p \bigvee q$ does not depend on the statements p and q. For the particular statements p (Les is Mrs. Brown's second husband) and q (Mrs. Brown has been married at least twice) the starred line in the truth table is ruled out. If p is true q cannot be false. In order to have a second husband, you must have been married at least twice. Therefore only the 3 other rows count, and for these rows we see that the truth value for $p \bigvee q$ is indeed the same as that for q. Since all ***possible*** *cases show that the truth tables for $p \bigvee q$ and q are identical, we see that our mathematical definition agrees with our intuition.*

NOTE: **$p \Rightarrow q$ is a relation between statements p and q. This relation is called an implication and should not be confused with the conditional $p \to q$ which is a new statement combined from statements p and q.**

We say that an argument is valid if and only if its conclusion is true when **all** the hypotheses are true.

Definition 27 *An argument is* ***valid*** *if and only if the conjunction of the premises (the hypotheses) implies the conclusion.*

To check the validity of arguments, we need to check implications. It's useful therefore to review properties of implications:

Theorem 28 *Reflexive Property: For any statement p, $p \Rightarrow p$*

Proof. Recall that $p \Rightarrow q$ means $p \bigvee q \equiv q$, so that $p \Rightarrow p$ means $p \bigvee p \equiv p$ which is true by the idempotent property of disjunction, PC 8. ■

Theorem 29 *Antisymmetric Property: If $p \Rightarrow q$ and $q \Rightarrow p$, then $p \equiv q$.*

Proof. Recall that $p \Rightarrow q$ means $p \bigvee q \equiv q$, so that $q \Rightarrow p$ means $p \bigvee q \equiv p$. Since we are given that both hold we write:

$$p \equiv p \bigvee q \equiv q$$

and by the transitive property of logical equivalence we have $p \equiv q$. ■

Theorem 30 *Transitive Property. If $p \Rightarrow q$ and $q \Rightarrow r$, then $p \Rightarrow r$.*

Proof. This is going to be more fun than the previous two proofs. To indicate why each step in the proof is valid, we will organize the information in a more formal way. We have two "givens", we call these **premises.** The first premise is $p \Rightarrow q$, which means $p \bigvee q \equiv q$. Let us write

$$P_1 : \; p \bigvee q \equiv q$$

We are also given $q \Rightarrow r$, which means $q \bigvee r \equiv r$. This is the second premise and we write

$$P_2 : \; q \bigvee r \equiv r$$

In the proof below we also refer to PC3 which stands for the 3rd assertion of Propositional Calculus, the associative law of conjunctions. From P_2 follows that when we consider $p \bigvee r$ we can replace r by $q \bigvee r$:

$$\begin{aligned} p \bigvee r &\overset{P_2}{\equiv} p \bigvee \left(q \bigvee r\right) \\ &\overset{\text{Associative Law, PC3}}{\equiv} \left(p \bigvee q\right) \bigvee r \\ &\overset{P_1}{\equiv} q \bigvee r \\ &\overset{P_2}{\equiv} r. \end{aligned}$$ ■

Let us consider a simple valid argument: If Trina gets a perfect 800 on her math SAT's, then Trina's parents will treat her with tickets to go see the ballet. Trina scored an 800 on her math SAT's. Therefore, Trina's parents will treat her with tickets to go see the ballet.

It makes perfect sense, doesn't it?

The rule which we applied without invoking it formally is called Modus Ponens. While simple, it makes for a powerful tool when you're proving more complicated implications.

Theorem 31 *Rule of Modus Ponens*

$$[(p \to q) \bigwedge p] \Rightarrow q$$

Proof. This theorem, like all those involving logical equivalences can be proved by examining the truth tables.

p	q	$[(p \to q) \bigwedge p] \bigvee q$
T	T	T
T	F	F
F	T	T
F	F	F

■

You can also prove it using our Propositional Calculus.

Again, we are trying to show $[(p \to q) \bigwedge p] \bigvee q \equiv q$.

Remember that we can write each conditional statement as a disjunction. By Remark 1 #4,

$$(p \to q) \equiv \sim p \bigvee q$$

$$[(p \to q) \bigwedge p] \bigvee q \overset{\text{Substitution}}{\equiv} [\left(\sim p \bigvee q\right) \bigwedge p] \bigvee q$$

$$\overset{\text{Distributive Law (PC 6)}}{\equiv} \left(\left(\sim p \bigwedge p\right) \bigvee \left(q \bigwedge p\right)\right) \bigvee q$$

Recall that $\sim p \bigwedge p \equiv c$, where c is a contradiction, by Inverse Elements (PC 5)

$$\overset{\text{Substitution}}{\equiv} \left(c \bigvee \left(q \bigwedge p\right)\right) \bigvee q$$

$$\overset{\text{Identity (PC 4)}}{\equiv} \left(q \bigwedge p\right) \bigvee q$$

$$\overset{\text{(PC 9)}}{\equiv} q$$

Therefore, $[(p \to q) \bigwedge p] \bigvee q \equiv q$.

Note that by proving these theorems using Propositional Calculus, we have in fact proved them for all Boolean Algebras. You're becoming real mathematicians.

Let us consider another simple argument; it's a less happy story for Trina this time:

If Trina gets a perfect score on her math SAT's then Trina's parents will treat her with tickets to go see the ballet. Trina's parents did not give Trina tickets to go see the ballet. Therefore, Trina did not score an 800 on her math SAT's.

The rule which we applied without invoking it formally is called Modus Tollens.

Theorem 32 *Rule of* ***Modus Tollens***.

$$[(p \to q) \bigwedge \sim q] \Rightarrow \sim p$$

Modus Tollens is, in fact, Modus Ponens, with substitution of the contrapositive for the conditional statement which we know we are allowed to do because they are equivalent.

We're doing great, lets see another example:

If it's a cool day then Carlos drinks hot chocolate. If Carlos drinks hot chocolate then he gets an upset stomach. Therefore, if it's a cool day, Carlos will get an upset stomach. This is an application of the rule of Hypothetical Syllogism. Let us state it formally:

Theorem 33 *Rule of* ***Hypothetical Syllogism***

$$[(p \to q) \bigwedge (q \to r)] \Rightarrow p \to r$$

You can prove this on your own checking truth tables.

Here are two easy theorems. If both p and q are true then p is true. Formally:

Theorem 34 $p \bigwedge q \Rightarrow p$.

Proof. We know that $r \Rightarrow q$ if and only if $r \bigvee q \equiv q$. We take r to be $p \bigwedge q$ and we get that $p \bigwedge q \Rightarrow p$ if and only if $(p \bigwedge q) \bigvee q = q$. But this is just PC9. ■

If p is true then of course the statement "either p or q" is true. Formally:

Theorem 35 $p \Rightarrow p \bigvee q$.

Proof. $p \Rightarrow p \bigvee q$ is true if and only if $p \bigvee (p \bigvee q) \equiv p \bigvee q$. But, by the associative property, $p \bigvee (p \bigvee q) \equiv (p \bigvee p) \bigvee q$. From the idempotent property, $p \bigvee p \equiv p$ so that

$$\begin{aligned} p \bigvee \left(p \bigvee q\right) &\equiv \left(p \bigvee p\right) \bigvee q \\ &\equiv p \bigvee q \end{aligned}$$

which means that $p \Rightarrow p \bigvee q$. ■

The relation of an implication has a close relationship to the conditional statement. The connection is asserted in the following theorem.

Theorem 36 $p \Rightarrow q$ *if and only if* $p \to q$ *is a tautology.*

Proof. Let us assume $p \Rightarrow q$, that is to say $p \bigvee q \equiv q$. Therefore:

$$\begin{aligned} p \to q &\equiv \sim p \bigvee q && \text{By Remark 1.4} \\ &\equiv \sim p \bigvee (p \bigvee q) && \text{By the hypothesis, } p \bigvee q \equiv q \\ &\equiv ((\sim p) \bigvee p) \bigvee q && \text{Associative law} \\ &\equiv t \bigvee q && \text{PC5} \\ &\equiv t && \text{By Theorem 23} \end{aligned}$$

In the other direction, suppose that $p \to q$ is a tautology, t, so that $\sim p \bigvee q$ is a tautology. We need to show then that $p \bigvee q \equiv q$.

$$\begin{aligned} p \bigvee q &\equiv (p \bigwedge t) \bigvee q && \text{By the definition of tautology, } p \bigwedge t \equiv p \\ &\equiv (p \bigwedge (\sim p \bigvee q)) \bigvee q && \text{By the hypothesis, } \sim p \bigvee q \equiv t \\ &\equiv ((p \bigwedge (\sim p)) \bigvee (p \bigwedge q)) \bigvee q && \text{Distributive law} \\ &\equiv (c \bigvee (p \bigwedge q)) \bigvee q && \text{PC5} \\ &\equiv (p \bigwedge q) \bigvee q && \text{By PC4} \\ &\equiv q && \text{By PC9. ■} \end{aligned}$$

We have seen some valid arguments; the following arguments are commonly used but are **not valid**:

1. AFFIRMING THE CONSEQUENT. Here's an example: If it's sunny, then I am happy. I am happy. Therefore it is sunny. (Please note why it's called affirming the consequent.) Let p denote the statement "it's sunny" and q the statement "I am happy". The symbolic representation of this argument is $[(p \to q) \bigwedge q] \Rightarrow p$. Let us see that it is **not valid**:

	p	q	$[(p \to q) \bigwedge q] \bigvee p$
	T	T	T
	T	F	T
$\longrightarrow$	F	T	T
	F	F	F

Notice that when p is false (and q is true), then $[(p \to q) \bigwedge q] \bigvee p$ is true. The two statements are therefore not logically equivalent, that is $[(p \to q) \bigwedge q] \bigvee p \not\equiv p$. By the definition of "imply" then $[(p \to q) \bigwedge q] \not\Rightarrow p$. Therefore, the argument is not valid.

2. DENYING THE ANTECEDENT. Here's an example: If I find a lucky penny, then the teacher will not check my homework. I did not find a lucky penny. Therefore the teacher will check my homework.

If we denote "I find a lucky penny" by p, "The teacher will not check my homework" by q, then the symbolic representation of the argument is $[(p \to q) \bigwedge \sim p] \Rightarrow \sim q$. Let us see that it is **not valid**:

	p	q	$\sim q$	$[(p \to q) \bigwedge \sim p] \bigvee \sim q$
	T	T	F	F
	T	F	T	T
$\longrightarrow$	F	T	F	T
	F	F	T	T

We see that the truth tables for $\sim q$ and for $[(p \to q) \bigwedge \sim p] \bigvee \sim q$ are not the same, and so $[(p \to q) \bigwedge \sim p] \not\Rightarrow \sim q$.

You should be careful to distinguish between the validity of an argument and the truth values of its premises and conclusions.

Let us check the following argument:

If $2+2=5$, then 12 is a negative integer. This is the first premise. Let us denote it by P_1.

$2+2=5$. This is the second premise. Let us denote it by P_2.

Therefore, 12 is a negative integer. This is the conclusion. Let us denote it by C.

Let p denote the statement "$2+2=5$" and q denote the statement "12 is a negative integer." The symbolic form of the argument is

$$\begin{array}{l} P_1 : p \rightarrow q \\ P_2 : p \\ \hline C : q \end{array}$$

The argument is valid by the rule of Modus Ponens. Remember, we are checking the validity of the argument – not whether or not each statement is true. Because our hypothesis was false, our conclusion is false. But our argument was valid. This kind of proof – attaining a conclusion you know to be false through an argument which is valid – serves to show us that at least one of our hypotheses must be false. This is called **proof by contradiction** and is used a great deal in Mathematics.

This is so important that we want to show you a more interesting example of a proof by contradiction. The example requires some knowledge of Algebra, so that if you haven't had any Algebra yet, you can skip it. Or, you can look at it and be motivated to learn Algebra.

Example 37 *Let us show that if the square of an integer is an even number, then the original number itself is even.*

Let us denote the number by n. We know that its square, n^2, is even, and we want to show that n is even.

Let the premise be that n is an odd number. This means that when we divide it by 2 we get a remainder of 1. Therefore $n = 2k+1$ for some integer

k. We square both sides of the equation and get:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= (2k+1)(2k+1) \\ &= 4k^2+4k+1 \end{aligned}$$

But $4k^2+4k$ is an even number, when we divide it by 2 we get $2k^2+2k$ and no remainder. This means that $n^2 = 4k^2+4k+1$ is an odd number. We have shown that the premise "n is odd" implies the conclusion "n^2 is odd", which is false for our integer n which was chosen so that n^2 is even. The argument is valid and so the premise must be false, and n is not odd, that is to say n is even.

We have already seen an example which showed us that a correct argument can lead to a false conclusion. Don't let a true conclusion make you assume that the argument is valid. Consider the following argument:

If $2 \cdot 6 = 12$, then 23 is a positive integer.

23 is a positive integer.

Therefore, $2 \cdot 6 = 12$.

This argument has the form of Affirming the Consequent, and is therefore not valid. As we wrote earlier, just because the conclusion is true we cannot conclude the argument is valid!

Proving the Validity of Arguments.

We've done pretty well with simple arguments. It's time to flex our Logic muscles and determine whether or not more complex arguments are valid. Consider the following argument:

If Craig is smart, then he can do mathematics. If he is not smart, then he cannot write poetry. But, he cannot do mathematics. Therefore, he cannot write poetry.

Is this argument valid?

To determine this, let's write out what we have symbolically.

Let s denote "Craig is smart."

Let m denote "Craig can do mathematics."

Let p denote "Craig can write poetry."

Once again, let P_1 denote our first premise, P_2 our second premise, etc. Then we can write our argument symbolically as:

$$\begin{array}{l} P_1 : s \to m \\ P_2 : \sim s \to \sim p \\ P_3 : \sim m \\ \hline C : \sim p \end{array}$$

Let's take a look at it horizontally:

$$P_1 \bigwedge P_2 \bigwedge P_3 \Rightarrow C$$

Which comes to:

$$[(s \to m) \bigwedge (\sim s \to \sim p) \bigwedge \sim m] \Rightarrow \sim p$$

Take a good look at this. By the tools we have at hand, how can we simplify the left side? Well, by the Commutative property (PC 2) we can rearrange our terms on the left hand side of the implication to get:

$$\Big[(s \to m) \bigwedge (\sim m) \bigwedge (\sim s \to \sim p)\Big]$$

Note that we don't have to worry about parentheses here because of the Associative property for Conjunctions (PC 3).

Well, by Modus Tollens we know

$$(s \to m) \bigwedge \sim m \Rightarrow \sim s$$

Returning to our implication we have

$$\Big[(s \to m) \bigwedge (\sim m) \bigwedge (\sim s \to \sim p)\Big] \Rightarrow \sim s \bigwedge (\sim s \to \sim p)$$

By Commutativity (PC 2) we have

$$(\sim s \bigwedge (\sim s \to \sim p)) \equiv ((\sim s \to \sim p) \bigwedge \sim s)$$

Which promptly reminds us of Modus Ponens, which we use to get:

$$((\sim s \to \sim p) \bigwedge \sim s) \Rightarrow \sim p$$

Which means that our argument is valid.

In fact, when you will be asked to prove an argument is valid, your final proof will have a more structured format. Here is the proof in the format you will be following to show that our argument is valid:

$$\begin{array}{l} P_1 : s \rightarrow m \\ P_2 : \sim s \rightarrow \sim p \\ P_3 : \sim m \\ \hline C : \sim p \end{array}$$

Proof:

1. $s \rightarrow m$	Premise (Given)
2. $\sim s \rightarrow \sim p$	Premise (Given)
3. $\sim m$	Premise (Given)
4. $(s \rightarrow m) \bigwedge \sim m$	Conjunction $(1, 3)$
5. $\sim s$	Modus Tollens (4)
6. $(\sim s \rightarrow \sim p) \bigwedge \sim s$	Conjunction $(2, 5)$
7. $\sim p$	Modus Ponens (6)

Therefore, the argument is valid.

As we said, this is the form your final proof should take. It might be helpful for you to map out informally what steps you will take first, like we did above.

Example 38 *Check the validity of the following argument:*

If this car is made in Italy (c)*, then parts are difficult to obtain* (d)*. This car is very expensive* (e) *or it is not difficult to obtain parts. But, this car is not very expensive. Therefore, it is not made in Italy.*

The symbolic form of the argument is:

$$P1 : c \to d$$
$$P2 : e \bigvee \sim d$$
$$\underline{P3 :\sim e}$$
$$C :\sim c$$

Proof:

1. $c \to d$	*Premise*
2. $e \bigvee \sim d$	*Premise*
3. $\sim e$	*Premise*
4. $\sim d \bigvee e$	*Commutative Property (PC 2)*
5. $d \to e$	*Remark* 1.4
6. $(d \to e) \bigwedge \sim e$	*Conjunction (*3, 5*)*
7. $\sim d$	*Modus Tollens (*6*)*
8. $(c \to d) \bigwedge \sim d$	*Conjunction (*1, 7*)*
9. $\sim c$	*Modus Tollens (*8*)*

Therefore, the argument is valid.

If an argument is not valid, you can prove this by using Truth Tables. Sometimes we can show that an argument is invalid by reducing it to a known false argument, such as Affirming the Consequent or Denying the Antecedent.

QUANTIFIERS AND QUANTIFIED STATEMENTS

So far the types of statements we've been dealing with have been pretty specific:

"My name is Jose" or "Thanksgiving is in July."

The negations of such statements are straightforward:

"My name is **not** Jose" and "Thanksgiving is **not** in July."

But consider the statement "All dogs are Dalmatians." We know that's false – there are St. Bernards, Chihuahuas, Doberman Pinschers, all sorts of different dog breeds. But what's the negation? It can't be "All dogs are **not** Dalmatians" because that statement is false as well – we know that some dogs are indeed Dalmatians. Remember, in order to be the negation of "All dogs are Dalmatians," a false statement, we need a true statement. In fact, our negation of "All dogs are Dalmatians" is the statement "**Some** dogs are **not** Dalmatians."

"All politicians are men." What is the negation?

Let's return to the first example. The negation of "All dogs are Dalmatians" is "Some dogs are not Dalmatians." What do you think is the negation of "Some dogs are not Dalmatians"?

Recall that $\sim (\sim p) \equiv p$. So, we know that the negation of "Some dogs are not Dalmatians" is "All dogs are Dalmatians."

"Some musicians play piano." What do you think is the negation of this statement?

What makes these statements different are the adjectives, "all" and "some."

The words "some" and "all" tell us, in some sense, **how many** dogs we are considering in the statements "All dogs are Dalmatians" and "Some dogs are not Dalmatians." This is why we call these types of statements **quantified** statements: the word "quantify" is related to the word "quantity" which deals with how many.

The word "all" says that the statement holds true universally – for all dogs. That is to say if something is a dog then the statement holds true for it. This is why we call "all" a universal quantifier.

Definition 39 *The adjective "All" is called a **universal quantifier**.*

Notice that when we say "All dogs are Dalmatians" we are making a statement about the set of dogs. Another way of phrasing it is to say "For all x, if x is a dog, then x is a Dalmatian."

The following statements say the same thing – they are logically equivalent:

1. All chalkboards are green.

2. For all chalkboards x, x is green.

3. For every chalkboard x, x is green.

4. For each chalkboard x, x is green.

5. For all x, if x is a chalkboard, then x is green.

You can replace the universal quantifier *"All"* with *"For all,"* *"For every,"* or *"For each"* without changing the meaning of the statement.

The word "some" in "Some dogs are not Dalmatians" says that the statement holds true for some dog. In other words, there **exists** some dog which is not a Dalmatian. That's why we call "some" an existential quantifier – existential is related to the word "exist."

Definition 40 *The adjective "Some" is called an **existential quantifier.***

Notice that in the statement "Some cats are calico" we are saying that there is at least one cat which is calico. In other words, in the set of cats there exists one which is calico. Notice that we are not making a statement that there is **only** one, but that there is at least one.

The following statements say the same thing – they are logically equivalent:

1. Some cats are calico.

2. There is a cat that is calico.

3. There exists a cat that is calico.

4. There is at least one cat that is calico.

5. There is an x such that x is a cat and x is calico.

Notice that you can replace the existential quantifier *"Some"* with *"There is," "There exists,"* or *"There is at least one."* Mathematically, we **don't** assume that *"Some"* is plural – even though Statement (1) might read like that. To say "Some cats are calico" means "there **exists** a cat that is calico" – can you see why we call *"Some"* an **exist**ential quantifier?

Terms and Predicates

In English grammar, "predicate" is defined as the part of the sentence which expresses something about the subject. So, in the sentence "The

Toyota was stranded by the side of the road" the predicate is "was stranded by the side of the road."

Similarly, in Logic consider the statement: 3 is a whole number. "is a whole number" is called a **predicate.** 3 is called a **term.**

"x is an odd number." Here x is the **term**, and "is an odd number" is the **predicate.**

Just like statements, we can use shorthand for predicates. For instance, let W denote the predicate "is a whole number." Then we'd read $W3$ as "3 is a whole number." How do you think we'd read $W17$? How would you read Wy?

Using this same notation, we could write the statement "7.5 is not a whole number" as $\sim W7.5$.

The statement "z is not a whole number" is written as $\sim Wz$.

Let's keep translating into symbolic form. We know how to symbolically denote statements without quantifiers; now let's do the same for statements with quantifiers. There are many advantages to using symbolic notation. Like shorthand, it makes it faster. More importantly, it allows us to analyze arguments without being influenced by the content of the statements. And finally, it makes it easier for us to communicate with mathematicians all over the world.

Symbolic Representations of Quantified Statements

Let's learn the symbolic notation for the universal quantifier "*For all x*" and the existential quantifier "*There is an x*":

$\forall x$ will replace and be read as "*For all x*." Notice that the symbol is an upside-down A, as in "All."

$\exists x$ will replace and be read as "*There is an x*." Notice that the symbol is a backwards E, as in "Exists."

So let's consider the statement: All Frigidaires are refrigerators..

Since we know the symbol for "For all x" let's write this statement as "For all x, if x is a Frigidaire, then x is a refrigerator.."

So we have "$\forall x$, if x is a Frigidaire, then x is a refrigerator.."

Let F be the predicate "is a Frigidaire."

Let R be the predicate "is a refrigerator.."

Now we have "$(\forall x)\,(F_x \rightarrow R_x)$" This is the **symbolic form** of our original statement.

Let's do another.

"Some fire trucks are red."

This statement is logically equivalent to "There exists an x such that x is a fire truck and x is red."

Which can be written as "$\exists x$ such that x is a fire truck and x is red."

Let T be the predicate "is a fire truck."

Let R be the predicate "is red."

So now our statement can be written as "$(\exists x)\,(T_x \bigwedge R_x)$."

Example 41 *Rewrite symbolically the statement "Some cereals contain vitamins."*

Because of the existential quantifier "some" we know we want to rewrite this statement as "There exists an x such that x is a cereal and x contains vitamins."

Let C be the predicate "is a cereal."

Let V be the predicate "contains vitamins."

Then symbolically we have $(\exists x)\ (C_x \bigwedge V_x)$

NEGATION OF QUANTIFIED STATEMENTS

Remember that when we first considered quantified statements, we found it tricky to determine their negations. So let's think about it some more.

Consider the statement "All dogs have long hair." If this statement is true then there are many statements which are false: "German Shepherds have short hair", "Beagles have short hair", "Some dogs have short hair." However if "All dogs have long hair" is false, it **does not follow** that "German Shepherds have short hair", or "Beagles have short hair" are true statements. It does follow, however that "Some dogs have short hair" is true.

Recall: A negation of a statement is true if the statement is false, and the negation is false if the statement is true. Therefore the negation of the statement "All dogs have long hair" is "Some dogs have short hair".

We'd like to examine this result in a more mathematical way.

We state "All dogs have long hair" as "For all x, if x is a dog, then x has long hair."

We state the negation "Some dogs have short hair" as "There exists x such that x is a dog and x does not have long hair."

Let's write these statements symbolically.

Let D be the predicate "is a dog," and let H be the predicate "has long hair."

Then "All dogs have long hair" is written as:

$$(\forall x)\,(D_x \rightarrow H_x)$$

And the negation, "Some dogs have short hair" is written as:

$$(\exists x)\left(D_x \bigwedge \sim H_x\right)$$

So we have

$$\sim ((\forall x)\,(D_x \rightarrow H_x)) \equiv (\exists x)\left(D_x \bigwedge \sim H_x\right)$$

This is in fact the rule for negating quantified statements. Recall, by Theorem 19, that for statements p, q

$$\sim (p \rightarrow q) \equiv p \bigwedge \sim q$$

which in our case is

$$\sim (D_x \rightarrow H_x) \equiv D_x \bigwedge \sim H_x$$

So that

$$\sim ((\forall x)\,(D_x \rightarrow H_x)) \equiv (\exists x) \sim (D_x \rightarrow H_x)$$

This gives us a convenient rule to create the negations of statements:

$$\sim (\forall x) \equiv (\exists x) \sim$$

This should be understood in this way: whatever follows $(\forall x)$ should follow $(\exists x) \sim$. In words, the negation of "For all (term) (statement)" is "There exists (term) (negation of statement)"

The last formula enables us to negate statements which contain the quantifier "some", the existentially quantified statements. We will take the negation on the left and on the right of both sides in

$$\sim (\forall x) \equiv (\exists x) \sim$$

and get

$$\sim\sim (\forall x) \sim \equiv \sim (\exists x) \sim\sim$$

The double negations cancel, leaving us with

$$(\forall x) \sim \equiv \sim (\exists x)$$

Let us see some examples: The statement

$$(\exists x)\left(D_x \bigwedge \sim H_x\right)$$

reads: "There exists x such that x is a dog and x does not have long hair." The rule we just developed says that the negation of

$$(\exists x)\left(D_x \bigwedge \sim H_x\right)$$

is

$$(\forall x) \sim \left(D_x \bigwedge \sim H_x\right)$$

and Theorem 19 teaches us that

$$\left(D_x \bigwedge \sim H_x\right) \equiv \sim (D_x \to H_x)$$

so that

$$\begin{aligned} &\sim \left(D_x \bigwedge \sim H_x\right) \equiv \sim\sim (D_x \to H_x) \\ &\equiv (D_x \to H_x) \end{aligned}$$

and so

$$(\forall x) \sim \left(D_x \bigwedge \sim H_x\right) \equiv (\forall x)\,(D_x \to H_x)$$

To summarize, we have shown

$$\sim (\exists x)\left(D_x \bigwedge \sim H_x\right) \equiv (\forall x)\,(D_x \to H_x)$$

In words: The negation of "There exists x so that x is a dog and x does not have long hair" is "for all x if x is a dog then x has long hair."

SUMMARY OF NEGATIONS OF QUANTIFIED STATEMENTS

p	$\sim p$
All x are y	Some x are not y
Some x are y	All x are not y

Rule of Instantiation or Term Substitution

Consider the following argument:

P_1 :All cats have whiskers.

P_2 :Garfield is a cat.

C :Therefore, Garfield has whiskers.

It makes sense that this argument is valid. To prove it is valid, we use the **Rule of Instantiation** or **Term Substitution.** Namely, in any statement preceded by a **universal** quantifier, the variable indicated by the quantifier may be replaced throughout the statement with any term.

Let's look at our argument. Let T be the predicate "is a cat" and W be the predicate "has whiskers."

The first premise P_1 is symbolized by $(\forall x)(T_x \rightarrow W_x)$

The second premise P_2 is symbolized by $T_{Garfield}$

The conclusion is symbolized by $W_{Garfield}$

Symbolically our argument looks like:

$$P_1 : (\forall x)(T_x \rightarrow W_x)$$

$$P_2 : T_{Garfield}$$

$$C : W_{Garfield}$$

Here is the proof for the argument:

Proof:

1. $(\forall x)(T_x \rightarrow W_x)$	1. Given
2. $T_{Garfield}$	2. Given
3. $T_{Garfield} \rightarrow W_{Garfield}$	3. Instantiation (1)
4. $(T_{Garfield} \rightarrow W_{Garfield}) \bigwedge T_{Garfield}$	4. Conjunction (3,2)
5. $W_{Garfield}$	5. Modus Ponens (4)

Therefore, the argument is valid.

Example 42 *Prove that the following argument is valid:*

P_1 : *All camellias are white*

P_2 : The bluebonnet is not white.
C : Therefore, the bluebonnet is not a camellia.
Let M be the predicate "is a camellia" and W be the predicate "is white."
The symbolic form for our argument is:

$$P_1 : (\forall x)\,(M_x \rightarrow W_x)$$

$$P_2 :\sim W_{bluebonnet}$$

$$C :\sim M_{bluebonnet}$$

The proof goes as follows:

1. $(\forall x)\,(M_x \rightarrow W_x)$	1. *Given*
2. $\sim W_{bluebonnet}$	2. *Given*
3. $M_{bluebonnet} \rightarrow W_{bluebonnet}$	3. *Instantiation (1)*
4. $(M_{bluebonnet} \rightarrow W_{bluebonnet}) \bigwedge \sim W_{bluebonnet}$	4. *Conjunction (3,2)*
5. $\sim M_{bluebonnet}$	5. *Modus Tollens (4)*

Therefore, the argument is valid.

ELEMENTARY SET THEORY

A basket full of different colored yarn. A litter of kittens. A jar full of pennies dating 1973 and earlier. The even numbers. What do these all have in common?

They are all examples of **sets.**

Definition 43 *A* ***set*** *is a collection of objects together with a rule which tells you whether a given object belongs to the set or not. An* ***element*** *of a set is an object which belongs to the set.*

Sets can be a collection of anything: boys named Jaime, blue-eyed Presidents, an infinite number of numbers, even other sets!

We use $\{...\}$ to denote a set. For example, $\{2,4,6,8,...\}$ denotes the set of even numbers greater than zero. We could also have written this set as

$$\{x \mid x \text{ is an even number greater than zero}\}$$

or

$$\{x \mid x \text{ is even and } x \geq 2\}$$

Is 3 an element of this set? Is 252 an element of this set?

Let $A = \{-1, 5, 17, 42\}$. 5 is an element of this set, so we write $5 \in A$. Since 2 is not an element of this set we write $2 \notin A$.

Let B be the set of all numbers divisible by zero. There are no elements in this set. B is called an **empty set.**

Definition 44 *An* ***empty set*** *is a set with no elements. You denote the empty set by* $\emptyset$.

Let $X = \{2, 4, 5, 17\}$. Let $Y = \{2, 4\}$. Notice that every element in Y is an element of X.

Let Z be the set of integers and let R be the set of real numbers. Notice that every element in Z is an element of R.

Definition 45 *A set A is said to be a* ***subset*** *of B if every element of A is an element of B. We write this as $A \subset B$.*

So in our previous example, $Y \subset X$ and $Z \subset R$.

Let A be a set. Is the following true: $A \subset A$?

To answer this question, we have to check the definition. The statement is true if and only if the conditions in the definition are satisfied. Is it true that "every element of A is an element of A"? Yes! Therefore for all sets A, $A \subset A$.

Can you give an example of two sets A and B so that A is **not** a subset of B?

If I say that a certain set A is not a subset of B, what do I have to show?

Theorem 46 *An empty set $\emptyset$ is a subset of any set A.*

Proof. Suppose $\emptyset$ is not a subset of some set A. Then by the definition of subset, this means $\emptyset$ contains some element which is not an element of A. But $\emptyset$ has no elements by definition of the empty set. Therefore, $\emptyset$ is a subset of any set A. ■

Theorem 47 *Properties of Inclusions*

1. *Reflexive Property: For any set A, $A \subset A$.*
2. *Transitive Property: Let A, B, and C be sets. If $A \subset B$ and $B \subset C$, then $A \subset C$.*

Proof. We have already argued the reflexive property. Let's see the transitive property: Take an element a in A. It is an element of B since $A \subset B$. But, since $B \subset C$, every element of B is an element of C, and so a is an element of C. We have shown that any element of A is an element of C, which says that $A \subset C$. ■

Consider $A = \{1, 2, 5\}$ and $B = \{5, 1, 2\}$. We don't want to differentiate between two sets which have exactly the same elements. Let's define what it means for two sets to be equal:

Definition 48 *Two sets A and B are said to be* ***equal*** *if and only if A is a subset of B and B is a subset of A. We write this as $A = B$.*

We leave it for you to show that this definition means that A and B contain exactly the same elements.

Theorem 49 *Properties of Set Equality*

1. *Reflexive Property: For any set A, $A = A$.*

2. *Symmetric Property: If $A = B$, then $B = A$.*

3. *Transitive Property: If $A = B$ and $B = C$, then $A = C$.*

We leave the proof of this theorem to you.

Since any two empty sets are equal, we will refer to an empty set as **the** empty set.

Definition 50 *A set A is a **proper subset** of B if and only if:*

1. $A \subset B$

2. $A \neq B$

3. $A \neq \emptyset$.

Let $X = \{2, 4, 5, 17\}$. Let $Y = \{2, 4\}$ and $T = \{5, 17, 23\}$. Then $Y \subset X$. In fact, Y is a proper subset of X. Notice that $23 \in T$, but $23 \notin X$. So T is not a subset of X and we write $T \not\subset X$. But we do notice that 5 and 17 are elements of X **and** of T.

Definition 51 *The **intersection** of two sets A and B is the set of all elements common to both A and B. We write the intersection of A and B as $A \bigcap B$. Thus*

$$A \bigcap B = \{x \mid x \in A \text{ and } x \in B\}$$

So in our earlier example, $T \bigcap X = \{5, 17\}$ and $Y \bigcap X = \{2, 4\} = Y$. In fact, more generally:

Theorem 52 *Let A and B be any sets. If $A \subset B$ then $A \bigcap B = A$.*
Proof. *To prove two sets are equal we need to show that they are subsets of one another.*

Let $x \in A$. Since $A \subset B$, this means $x \in B$. So x is a common element of A and B, and $x \in A \bigcap B$. Thus $A \subset A \bigcap B$.

Let $x \in A \bigcap B$. By definition of $A \bigcap B$, then $x \in A$. So $A \bigcap B \subset A$.

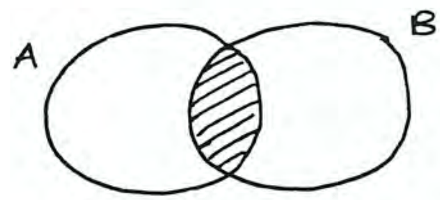
Therefore, $A \bigcap B = A$. ■

Notice that in this proof we also showed the result:

Theorem 53 *Let A and B be any sets. $A \bigcap B \subset A$.*

It can be very helpful to use diagrams when working with sets:

The shaded area represents $A \bigcap B$



A ∩ B

Let $A = \{-43, 0, 8\}$, $B = \left\{-3\frac{5}{8}, 0, 7, 8\right\}$, and $C = \left\{-3\frac{5}{8}, 2, 13\right\}$.
Then $A \bigcap B = \{0, 8\}$ and $B \bigcap C = \left\{-3\frac{5}{8}\right\}$. Notice $A \bigcap C = \emptyset$. We say that A and C are **disjoint.**

The notion of intersection reminds us of conjunctions; in fact, there is a conjunction in the definition of intersection. Remember,

$$A \bigcap B = \left\{x \mid x \in A \bigwedge x \in B\right\}$$

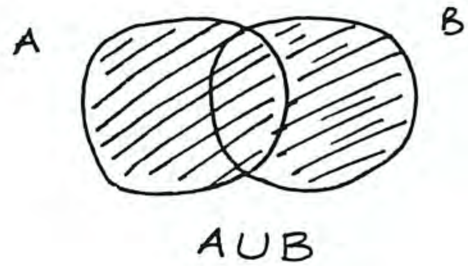
It is natural to examine the set

$$\left\{x \mid x \in A \bigvee x \in B\right\}$$

This is called the union of A and B.

Definition 54 *The* ***union*** *of two sets A and B is the set of all elements belonging to A or to B. We denote "A union B" by $A \bigcup B$.*

The shaded area represents $A \bigcup B$



Let $A = \{-43, 0, 8\}$, $B = \left\{-3\frac{5}{8}, 0, 7, 8\right\}$, and $C = \left\{-3\frac{5}{8}, 2, 13\right\}$.

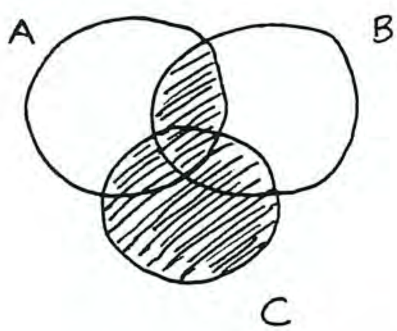
Then $A \bigcup B = \left\{-43, -3\frac{5}{8}, 0, 7, 8\right\}$ and $A \bigcup C = \left\{-43, -3\frac{5}{8}, 0, 2, 8, 13\right\}$. What is $B \bigcup C$?

Once again, as in Logic, we have two operations and we need to be careful about the order of operations.

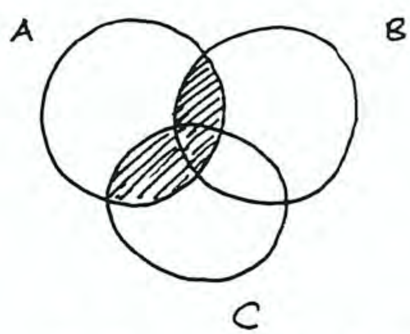
We do what's in parentheses first. For instance, in the above example $(A \bigcap B) \bigcup C = \left\{-3\frac{5}{8}, 0, 2, 8, 13\right\}$. But $A \bigcap (B \bigcup C) = \{0, 8\}$. So $(A \bigcap B) \bigcup C \neq A \bigcap (B \bigcup C)$.

Take a look at the diagrams:

$(A \bigcap B) \bigcup C$



$A \bigcap (B \bigcup C)$

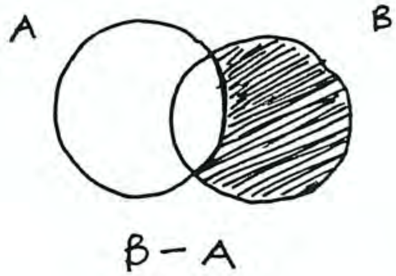


Definition 55 *Let A and B be subsets of a set X. The set $B - A$, called the **difference** of B and A, is the set of all elements which are in B and not in A. Thus,*

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}$$

Let us see the diagram representing $B - A$:

The shaded area represents $B - A$



Let $A = \{-43, 0, 8\}$, $B = \left\{-3\frac{5}{8}, 0, 7, 8\right\}$, and $C = \left\{-3\frac{5}{8}, 2, 13\right\}$.
Notice that $B - A \neq A - B$ since $B - A = \left\{-3\frac{5}{8}, 7\right\}$ and $A - B = \{-43\}$.
What is $C - B$?
What is $C - A$?
Can you make any general conclusions about $C - A$ for any two disjoint sets A and C?

Definition 56 *If $A \subset X$, then $X - A$ is sometimes called the* **complement** *of A with respect to X. Also used to denote set complements are the symbols*

$$C_X A, \ CA, \ \sim A, \text{ and } A'.$$

Thus

$$C_X A = \{x \in X \mid x \notin A\}$$

Theorem 57 *Let A and B be subsets of X. Then $A - B = A \bigcap C_X B$.*

Proof. Let $x \in A - B$. Then by the definition of $A - B$, $x \in A$ and $x \notin B$, so $x \in C_X B$. Thus $x \in A \bigcap C_X B$ and $A - B \subset (A \bigcap C_X B)$.

Let $x \in A \bigcap C_X B$. So $x \in A$ and $x \in C_X B$, which means $x \notin B$. Thus $x \in A - B$, and $A \bigcap C_X B \subset (A - B)$.

Therefore $A - B = A \bigcap C_X B$. ■

Definition 58 *Let A be a finite set. Then $|A|$ is the number of elements in A.*

Is it true that $|A \bigcup B| = |A| + |B|$?

Lets see a couple of examples. If $A = \{1, 2, 4\}$ and $B = \{3, 5, 6, 7\}$ then $A \bigcup B = \{1, 2, 3, 4, 5, 6, 7\}$. $|A| = 3$, $|B| = 4$ and $|A \bigcup B| = 7 = |A| + |B|$.

But: if $A = \{1, 2, 4\}$ and $B = \{1, 2, 4, 5\}$ then $|A| = 3$, $|B| = 4$. What is $A \bigcup B$?

$$A \bigcup B = \{1, 2, 4, 5\}$$

so that $|A \bigcup B| = 4$, which is certainly not $3 + 4$. What happened?

The difference between the two examples is that in the first example the sets A an B are disjoint. When we add the number of elements in A and in B we do not double count any element. In the second example, the elements $1, 2, 4$ are in both A and in B. These elements are counted once when we consider $|A \bigcup B|$ and are counted twice when we counted $|A| + |B|$. In general, when we count $|A \bigcup B|$ in terms of $|A|$ and $|B|$ we need to subtract the number of elements in $A \bigcap B$. Therefore

$$\left|A \bigcup B\right| = |A| + |B| - \left|A \bigcap B\right|$$

This can be used in some neat problems:

Example 59 *A survey was done of* 100 *teenagers. The report indicated that of these* 100, 78 *ate potato chips,* 71 *ate pretzels, and* 48 *ate both. Can you see why this was an invalid study?*

Let $A = \{\textit{potato chip eaters}\}$, $B = \{\textit{pretzel eaters}\}$. *This means that* $A \bigcap B = \{\textit{potato chip and pretzel eaters}\}$. $|A| = 78$, $|B| = 71$ *and* $|A \bigcap B| = 48$. *Since* 100 *teenagers were interviewed,* $|A \bigcup B| \leq 100$. *(We do not necessarily have* $|A \bigcup B| = 100$ *since some teenagers may eat neither potato chips nor pretzels). But*

$$\begin{aligned}\left|A \bigcup B\right| &= |A| + |B| - \left|A \bigcap B\right| \\ &= 78 + 71 - 48 \\ &= 101\end{aligned}$$

which is too many. The person collecting the data was not counting right.

Properties of Union, Intersection and Complementation

Theorem 60 *Let X be an arbitrary set and let $P(X)$ be the set of all subsets of X. $P(X)$ is called the* **power set** *of X. Let $A, B,$ and C be arbitrary elements of $P(X)$.*

1. $A \bigcap B = B \bigcap A$ and $A \bigcup B = B \bigcup A$ (Commutative Law for Intersection and Union)
2. $A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C$ and $A \bigcup (B \bigcup C) = (A \bigcup B) \bigcup C$ (Associative Law for Intersection and Union)
3. $A \bigcap X = A$
4. $A \bigcap \emptyset = \emptyset$
5. $A \bigcap B \subset A$
6. $A \bigcup X = X$
7. $A \bigcup \emptyset = A$
8. $A \subset A \bigcup B$
9. $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$ (Distributive Law of Union with respect to Intersection)
10. $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$ (Distributive Law of Intersection with respect to Union)
11. $C_X (A \bigcup B) = C_X A \bigcap C_X B$
12. $C_X (A \bigcap B) = C_X A \bigcup C_X B$
13. $A \bigcap C_X A = \emptyset$
14. $A \bigcup C_X A = X$.

BOOLEAN ALGEBRA

When we studied Propositional Calculus, recall that we said that statements form a Boolean Algebra. In fact, sets also form a Boolean Algebra. Let's review our definition of a Boolean Algebra.

Definition 61 *A* ***Boolean Algebra,*** B, *is a set of elements with two binary operations* $\bigwedge$ *and* $\bigvee$ *satisfying the following for* $a, b, c \in B$

B1. CLOSURE LAWS: $a \bigwedge b \in B$ *and* $a \bigvee b \in B$.

B2. COMMUTATIVE LAWS: $a \bigwedge b = b \bigwedge a$ *and* $a \bigvee b = b \bigvee a$.

B3. ASSOCIATIVE LAWS: $(a \bigvee b) \bigvee c = a \bigvee (b \bigvee c)$ *and* $(a \bigwedge b) \bigwedge c = a \bigwedge (b \bigwedge c)$.

B4. IDENTITY ELEMENTS: There exist elements z, $u \in B$ *such that for each* $a \in B$, $a \bigvee z = a$ *and* $a \bigwedge u = a$.

B5. INVERSE ELEMENTS: For each $a \in B$, *there exists* $a' \in B$ *such that* $a \bigvee a' = u$ *and* $a \bigwedge a' = z$. a' *is called the* ***inverse*** *or* ***complement*** *of* a.

B6. DISTRIBUTIVE LAWS: $a \bigwedge (b \bigvee c) = (a \bigwedge b) \bigvee (a \bigwedge c)$ *and* $a \bigvee (b \bigwedge c) = (a \bigvee b) \bigwedge (a \bigvee c)$.

Do the Real Numbers with the operations $+$ and $\cdot$ form a Boolean Algebra? Which properties do they fulfill? Which properties of a Boolean Algebra do they fail?

Theorem 62 *IDEMPOTENT LAWS: For each* $a \in B$, $a \bigvee a = a$ *and* $a \bigwedge a = a$.

Proof. *Consider first* $a \bigvee a = a$.

Recall that we denote by z *is the identity element for* $\bigvee$.

$$\begin{aligned}
&By\ B4: & a \bigvee z &= a \\
&By\ B5: & a \bigvee (a \bigwedge a') &= a \\
&By\ B6: & (a \bigvee a) \bigwedge (a \bigvee a') &= a \\
&By\ B5: & (a \bigvee a) \bigwedge u &= a \\
&By\ B4: & a \bigvee a &= a
\end{aligned}$$

We leave it to you to show $a \bigwedge a = a$. ■

Theorem 63 *For each $a \in B$, $a \bigvee u = u$ and $a \bigwedge z = z$.*

Theorem 64 *ABSORPTION LAW: If $a, b \in B$, then $a \bigwedge (a \bigvee b) = a$ and $a \bigvee (a \bigwedge b) = a$.*

Theorem 65 *CANCELLATION LAW: If $a, b, c \in B$ and $a \bigwedge c = b \bigwedge c$, and $a \bigvee c = b \bigvee c$, then $a = b$.*

Theorem 66 *If $a \in B$ then a has a unique complement; i.e. if $a \bigvee x = u$ and $a \bigwedge x = z$ then $x = a'$.*

Theorem 67 *If $a, b, c \in B$, $a \bigvee c = b \bigvee c$ and $a \bigvee c' = b \bigvee c'$ then $a = b$. If $a \bigwedge c = b \bigwedge c$ and $a \bigwedge c' = b \bigwedge c'$ then $a = b$.*

Definition 68 *Suppose $a, b \in B$. Then $a \leq b$ if and only if $a \bigvee b = b$. "$a \leq b$" is read as "a precedes b."*

PROPERTIES OF $\leq$:

1. REFLEXIVE PROPERTY: If $a \in B$, then $a \leq a$.
2. ANTISYMMETRIC PROPERTY: If $a, b \in B, a \leq b$ and $b \leq a$, then $a = b$.
3. TRANSITIVE PROPERTY: If $a, b, c \in B$, $a \leq b$ and $b \leq c$, then $a \leq c$.

Theorem 69 *If $a \in B$ then $z \leq a$ and $a \leq u$.*

Theorem 70 *If $a, b, c \in B$ and $a \leq b$, then $a \bigvee c \leq b \bigvee c$ and $a \bigwedge c \leq b \bigwedge c$.*

Theorem 71 *If $a, b \in B$, then $a \bigwedge b \leq a \leq a \bigvee b$.*

Theorem 72 *$z' = u$ and $u' = z$.*

Theorem 73 *If $a \in B$ then $(a')' = a$*

Theorem 74 *If $a, b \in B$, then $(a \bigvee b)' = a' \bigwedge b'$ and $(a \bigwedge b)' = a' \bigvee b'$.*

Theorem 75 *If B is a Boolean Algebra with a finite number of elements and $\mid B \mid \geq 2$, then $\mid B \mid$ is even.*

Theorem 76 *There exists no Boolean Algebra with* 6 *elements.*

Theorem 77 *Let* B *be a Boolean Algebra. Define a binary operation* $+$ *on* B *so that for every* $a, b \in B$, $a + b = (a \bigvee b) \bigwedge (a' \bigvee b')$. *Then:*

1. $(B, +)$ *is commutative.*

2. $(B, +)$ *is associative.*

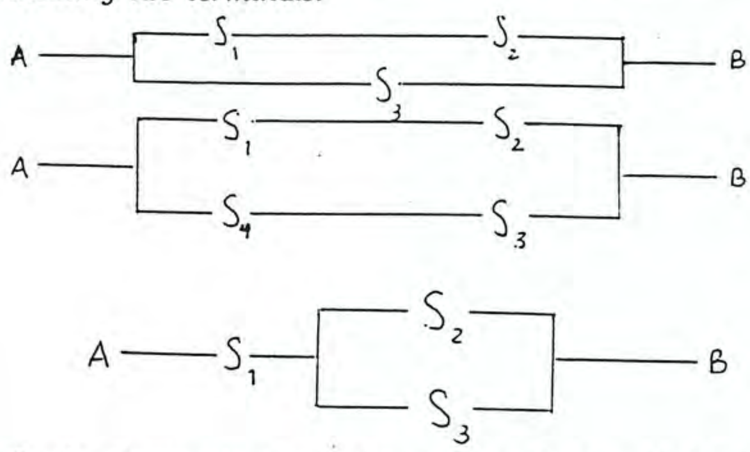
3. $(B, +)$ *has an operational identity.*

4. *If* $a \in B$, *then* a *has an operational inverse in* $(B, +)$ *for the identity in* (3).

5. *If* $a, b \in B$, *then* $a + b = (a \bigwedge b') \bigvee (a' \bigwedge b)$.

SWITCHING NETWORKS

Definition 78 *A* ***switching network*** *is an arrangement of wires and switches connecting two terminals.*



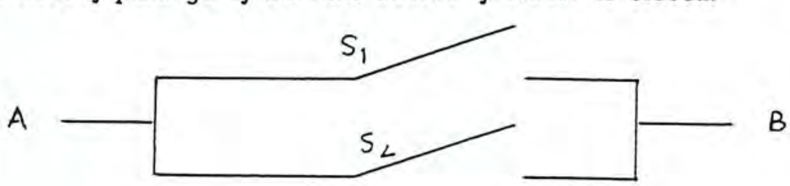
In the above examples, A and B are the terminals and S_1, S_2, S_3 and S_4 are the switches. The connecting line segments represent the wires.

Definition 79 *A switch is* ***closed*** *if it is in a position to permit passage of current; otherwise it is* ***open.***

OPEN SWITCH

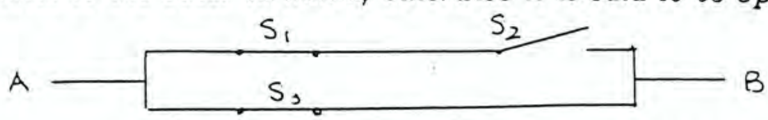
CLOSED SWITCH

Definition 80 *Two switches S_1 and S_2 are said to form a* ***parallel connection*** *if passage of current occurs if either is closed.*

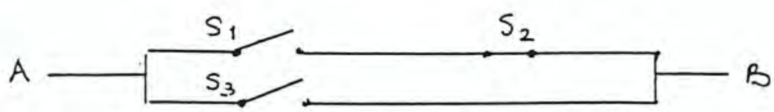


Definition 81 *Two switches, S_1 and S_2, are said to form a* **series connection** *if passage of current occurs only when both switches are closed.*

Definition 82 *A network is said to be* **closed** *if current is passing from one terminal to the other terminal; otherwise it is said to be* **open.**



This is an example of a closed network



This is an example of an open network

State Tables for Switching Networks

Just like truth tables for statements, with switching networks we have **state tables:**

If two switches, S_1 and S_2, are connected in parallel, this situation will be denoted by $S_1 \oplus S_2$.

If two switches, S_1 and S_2, are connected in series, this situation will be denoted by $S_1 \odot S_2$.

The opposite of closed is open, and the opposite of open is closed.

C and O will represent closed and open states respectively.

S_1	S_2	$S_1 \oplus S_2$	$S_1 \odot S_2$	$\bar{S}_1$
C	C	C	C	O
C	O	C	O	O
O	C	C	O	C
O	O	O	O	C

Examining the state tables, our analogy between switches and statements is made very clear:

SWITCHES	STATEMENTS
$\oplus$	$\bigvee$
$\odot$	$\bigwedge$
$-$	$\sim$
C	T
O	F

If S represents a complete system of switches and networks with the operators $\oplus, \odot$, and $-$, then S is a Boolean Algebra.

Sometimes, the symbol "1" is used for "C" and "0" is used for "O." In this case, our state table would look like:

S_1	S_2	$S_1 \oplus S_2$	$S_1 \odot S_2$	$\bar{S}_1$
1	1	1	1	0
1	0	1	0	0
0	1	1	0	1
0	0	0	0	1

In general, given a network of switches:

If current runs through a network, it must have a state value of 1; otherwise it has a state value of 0.

A switch which is always on has a state value of 1.

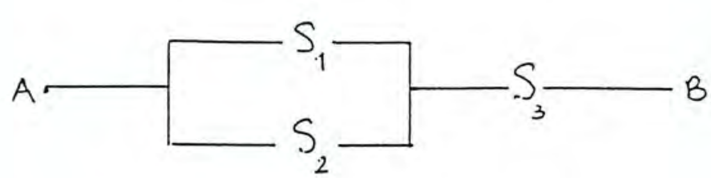
A switch which is always off has a state value of 0.

Switching Statements

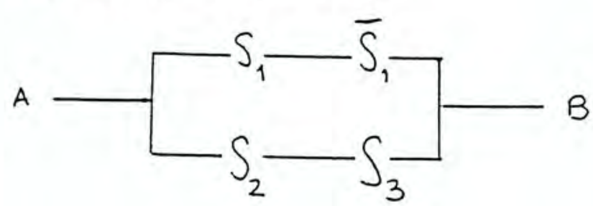
Switching statements are analogous to Logic statements.

EXAMPLES:

1. $(S_1 \oplus S_2) \odot S_3$



2. $(S_1 \odot \bar{S}_1) \oplus (S_2 \odot S_3)$



3. $[(S_1 \odot \bar{S}_2) \oplus (\bar{S}_1 \odot S_2)] \odot \bar{S}_2$