MODELING COOPERATIVE DECISION SITUATIONS: THE DEVIATION FUNCTION FORM AND THE EQUILIBRIUM CONCEPT

by

MARILDA SOTOMAYOR

Universidade de São Paulo

e-mail: marildas@usp.br

This paper was partially supported by CNPq-Brazil and by FIPE. A draft of it was presented in the conference "Roth and Sotomayor: 20 years after", May, 2010, Duke University.

We idealize an environment in which a cooperative decision situation takes place, involving a *finite set* N *of agents* who are able to freely communicate and want to form *coalitions*. These agents will be called players. The players involved in a coalition interact among themselves, by acting according to established *rules*, aiming to reach an agreement (or to sign a contract) on the terms that will regulate their participation in the given coalition.

An *outcome* is a set of coalitions, whose union is the whole set of players (*coalition structure*), together with the set of agreements reached by the coalitions in the negotiation process. An outcome is *feasible* if it does not violate the established rules.

One of the features of this cooperative decision situation is that a player might want to enter in more than one coalition, so **a coalition structure is not necessarily a partition of *N*.** Also, the agreements reached in a coalition are independent of the agreements reached in any other coalition. Of course the players derive a utility level in each coalition they enter and have preferences over possible outcomes.

What outcomes can one predict that will occur?

The answer to this question involves the assumption that the players should take their decisions based on some **criterion of rationality**, taking into account the consequences of the possible agreements they could make in each coalition they could form. More specifically, we idealize the cooperative decision situation by assuming that all agents are rational and we postulate that the cooperative behavior of the players should be governed by the following line of reasoning:

“Facing a feasible outcome x, a coalition of agents ***will take*** *a joint action against x (this joint action may involve current partners out of the coalition), whenever* ***such action is allowed by the established rules*** *and all the outcomes that might arise from this particular joint action are preferred to x by all players in the coalition”.*

The consequences of this line of reasoning for the players lead to some kind of equilibrium, which we will call *cooperative equilibrium*. The intuitive idea is that a *feasible outcome x is a* ***cooperative equilibrium*** *if there is no coalition whose members can profitably deviate from x,* ***by taking actions that are allowed by the established rules.***

An outcome x is in the ***core*** *if there is no coalition whose members can profitably deviate from x by* ***interacting only among themselves****.*

Therefore, **every cooperative equilibrium is a core outcome.**

Since the players are free **to interact coalitionally** and they **take rational decisions** we can expect that the outcomes that will occur should be stable against any coalitional deviation. Thus the prediction will be that **only cooperative equilibria will occur.**

Well known special cases of such a cooperative decision situation are the **matching markets.** The coalition structure is given by a matching and the individual payoffs of the players only depend on their agreements with their partners. In these markets the intuitive idea of cooperative equilibrium is captured by the concept of ***stability,*** which has been defined locally, for every matching model that has been studied, since Gale and Shapley (1962).

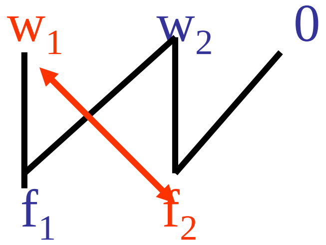
EXAMPLE 1: $F=\{f_1, f_2\}$, $W=\{w_1, w_2\}$

$P(f_1)=P(f_2)= \{w_1, w_2\}, w_2, w_1$

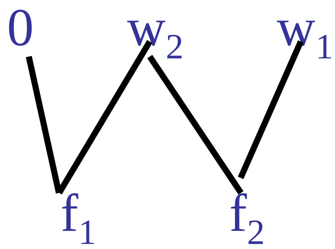
$P(w_1) = f_2,\ f_1$

$P(w_2) = \{f_1, f_2\},\ f_2,\ f_1$

x:



y:



x is not stable.

x is in the core.

In general terms, the way game theorists use to approach a cooperative decision situation is by constructing a mathematical model. They do that by **abstracting from the negotiation process and focusing on what each coalition can obtain, without specifying how to obtain**. The model is called a **cooperative game**. How much of the details of the rules of the game should be retained is the central issue in the modeling of a cooperative game situation. Certainly, this depends on the purpose of the analysis.

Basically, the actions players can take to play the game are modeled by the set of **feasible outcomes**. However, the feasible outcomes are too general to capture all the relevant details of the rules of the game for the purpose of observing **cooperative equilibria.** It turns out that no cooperative equilibrium analysis can ignore the set of feasible actions that the members of a coalition are allowed to take in order to deviate from a given feasible outcome.

Taking this into account, the game theorists proposed *forms*, which represent special classes of cooperative games, to serve as vehicle for the equilibrium analysis of these games.

1) COOPERATIVE NORMAL FORM

Each player participates in only one coalition and the payoff of a player is conditioned to the actions taken in all coalitions formed.

An **outcome** is specified by

(i) a partition of N *(each player participates in one coalition)* and

(ii) a joint strategy for each partition set.

2) CHARACTERISTIC FUNCTION FORM

The rules of the game are specified by a function ***V,*** which associates each coalition *S* to a set of **|*S*|-dimensional payoff-vectors,** each of which coalition *S* can "assure" itself in some sense, through ***interactions only among its members***.

- **An outcome is represented by the payoffs of the players,** so the information with respect to the **actions** the players take to reach these payoffs is lost.
- The **conditionality** that characterizes the payoffs of the players in the cooperative games in the normal form is also lost.

Consequently, as observed by Rosenthal (1972), we may have an outcome which **is in the core** of the game in the **characteristic function form** but it **is not in the core** of the game in the **normal form**.

3) EFFECTIVENESS FORM

In an attempt to correct the imperfections of the characteristic function representation, Rosenthal (1972) proposed the **effectiveness form,** which is enough general to model cooperative games in normal form, so an outcome might consider the actions which support the payoffs and the payoffs of a coalition might depend on the actions taken by the players out of the coalition.

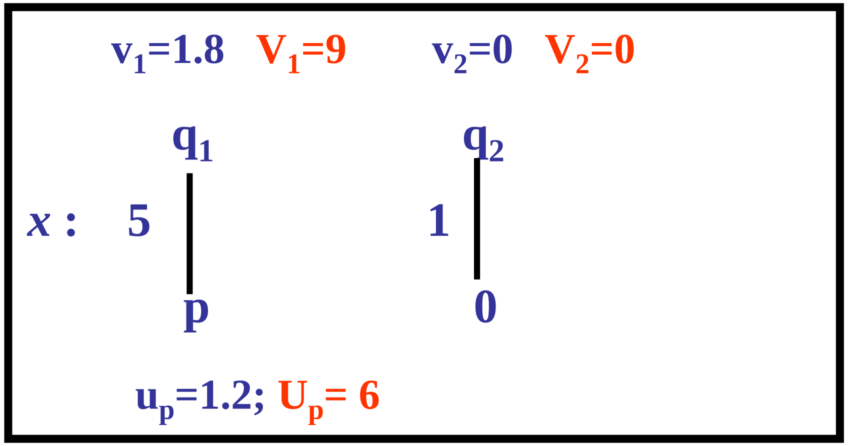
For a given coalition ***T*** and a given outcome ***x*** there is a set of alternative outcome subsets, ***E (x,T),*** which the memers of ***T*** **can enforce against *x*, by interacting only among themselves**. .

In the characteristic function form and in the effectiveness form, the joint actions that the members of a coalition can take against a proposed outcome, and that can be captured by these models, are restricted to **"interactions among themselves",** so the cooperative analysis is based on the core.

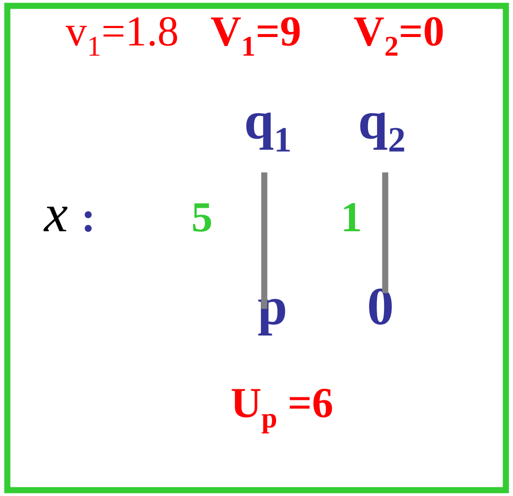
However, when the rules of the game allow the coalitions to do more than to **merely interact among themselves**, we may expect that some core outcomes will not occur. (Sotomayor, 1992, 1999, 2010). In these cases the cooperative analysis only based on the core is not the most appropriate approach.

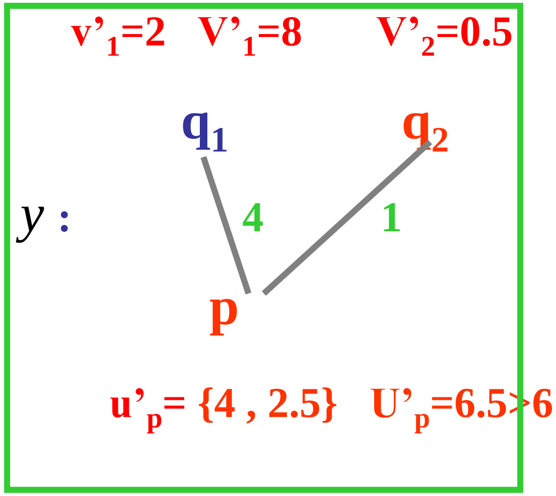
The following example illustrates that a cooperative equilibrium analysis may be deficient if it uses as vehicle the characteristic function form or the effectiveness form.

EXAMPLE 2. N={p, q_1, q_2}; p is a buyer and the other agents are sellers. Seller q_1 has **5** units of a good to sell and seller q_2 has **1** unit of the same good. The maximum amount of money buyer p considers to pay for one unit of the good is **$3**. This agent has no utility for more than **5** units of the good. The negotiations between the buyer and each seller are made independently. Furthermore, the market allows some kind of flexibility on the number of items negotiated between the buyer and seller q_1: Once the price of one item is negotiated, **the buyer gets a discount of k% over that price if he acquires 5 units of the good.**

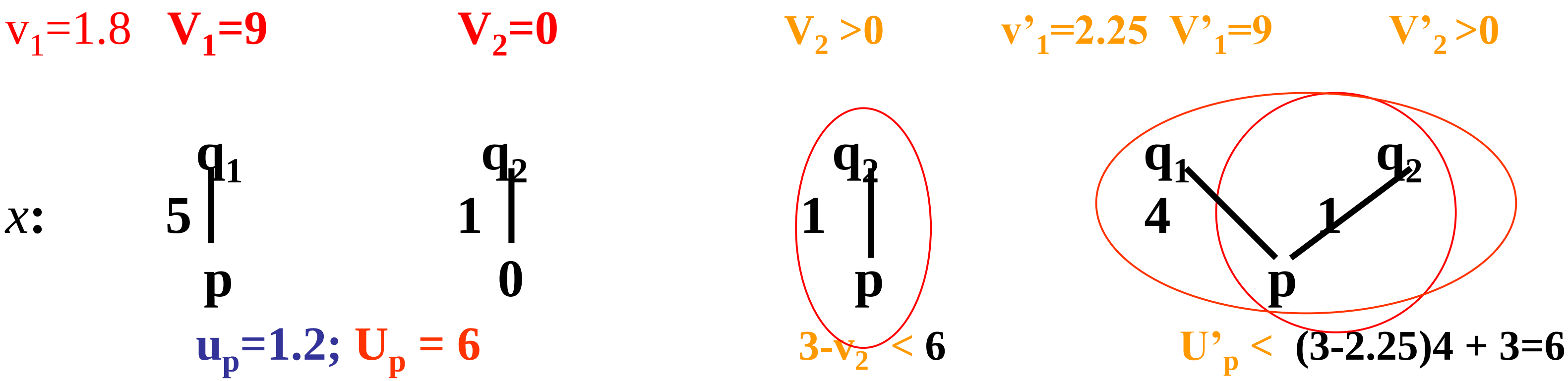


x is a cooperative equilibrium when k%=20% and it is not a cooperative equilibrium when k%=10%. Furthermore x is in the core for any k.





If k%=10% and x is proposed, then buyer p and seller q_2 can counter-propose an alternative outcome that both prefer. At this outcome buyer p reduces, from 5 to 4, the number of units to be acquired from q_1, in order to trade with q_2. Then he pays $2 for each unit of the good of q_1. These actions are allowed by the rules of the market. The outcome y might be the resulting outcome if q_2 sells his item to p for $0.50. The power of p of increasing his payoff is due to the concurs of q_1, which is assured by the flexible nature of the agreement with respect to the number of units negotiated. Therefore, x **cannot be considered a cooperative equilibrium when k=10.**



If k%=20%, it is easy to verify that there is no way for *p* to increase his total payoff by only trading with q_2. If *p* reduces from 5 to 4 the number of units negotiated with q_1, he will have to pay \$2.25 for each unit of the good of q_1. In this case there is no price that can increase the current total payoffs of *p* and q_2. Also there are no prices that can increase the current total payoffs of the three agents. **Therefore, the outcome *x* is a cooperative equilibrium, so it is in the core, when** k=20.

We can also observe that, for any *k*, there are no prices that can increase the current total payoffs of the three agents, so *x* **is in the core for any *k*. Then, the information about the discount is not relevant when the analysis is based on the core.**

The point is that a cooperative equilibrium analysis for the market of this example cannot ignore the type of flexibility of the agreements that can be reached. On the other hand, the type of flexibility of the agreements cannot be modeled, either by the effectiveness form or by the characteristic function form. Therefore, **there is no way to conclude from these representations if x is or is not a cooperative equilibrium.**

It is for solving problems as the one presented in this example that we propose the **deviation function form**. This is a mathematical model to serve as vehicle for cooperative equilibrium analysis of cooperative decision situations. Our framework is more general than the effectiveness form and it complements that form by also capturing the kinds of coalitional interactions that support the agreements proposed by **deviating coalitions**.

THE DEVIATION FUNCTION FORM: *(N, C, X, U, u, ϕ)*.

*a) a **set $N = \{1, . . . ,n\}$ of players**;*

*b) a set C of **feasible coalition structures;***

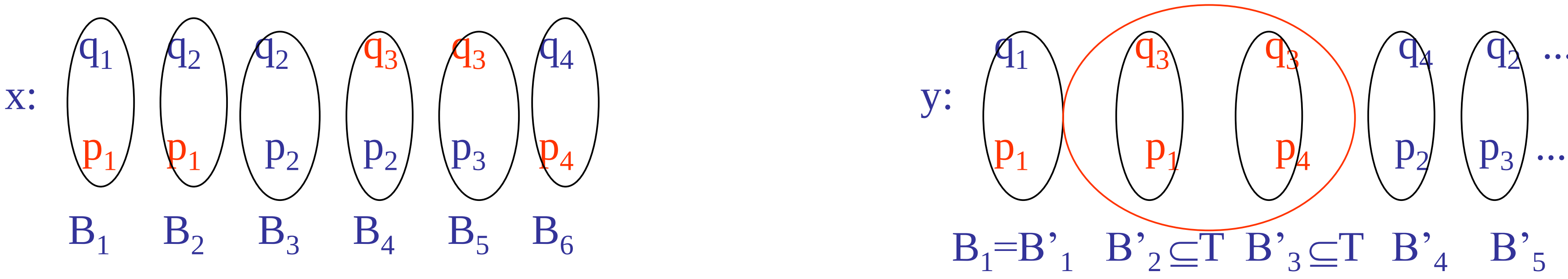
*c) For each coalition structure β in C, a set X_β of **feasible outcomes compatible with β.***

*d) for each $p \in N$, for each coalition structure β and $B \in \beta$, with $p \in B$, a **utility function** $U_{pB}: X_\beta \to R$;*

*e) **an ordinal, vector-valued utility function** $u: X \to R^n$, where X is the union of the sets X_β's for all coalition structures β;*

*f) for each $x \in X$, a **deviation function** ϕ_x from x, which maps every coalition $T \subseteq N$ into a set of feasible outcomes, called **feasible deviations from x via T**.*

The outcomes in $\phi_x(T)$ intend to reflect, in some sense, **which feasible actions the members of T can take against x.**



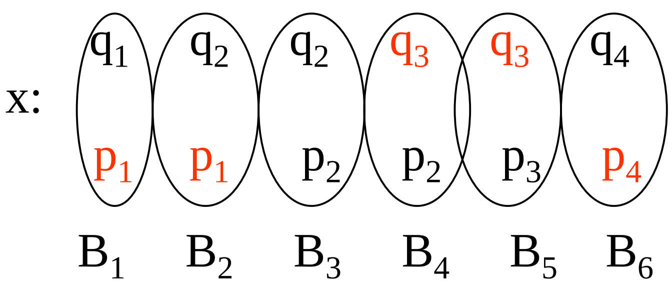
$T=\{p_1,q_3,p_4\}$

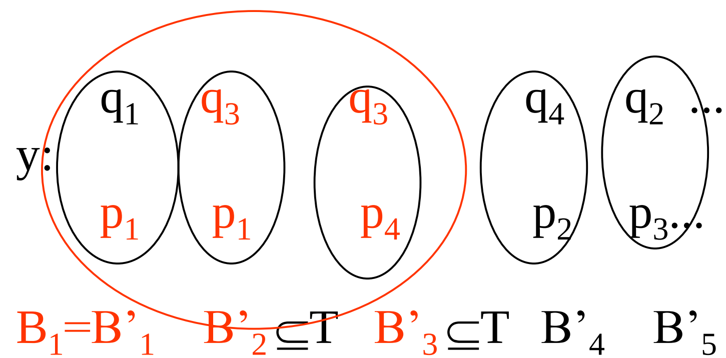
We identified the structure that a feasible deviation from an outcome x via some coalition T should have for capturing the relevant details of the rules of the game for the cooperative equilibrium analysis purpose.

Roughly speaking, if **y** is a feasible deviation from **x** via **T**, (i) **the members of T make new agreements and only among them**; (ii) **any coalition formed with players in T and players out of T must be some current coalition of x;** (iii) **the interaction inside such coalition keeps the current agreements or reformulates some of the terms of them**.

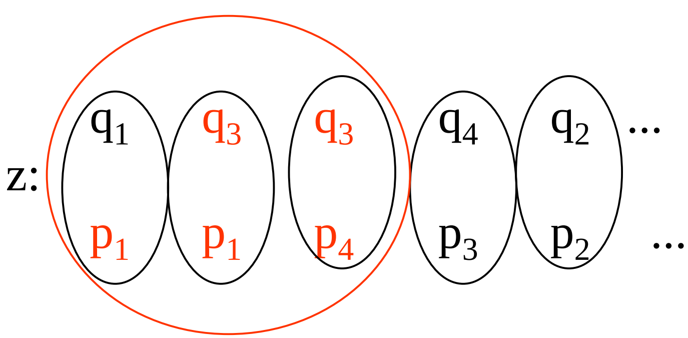
Then, $\phi_x(T)$ **contains** $E(x,T)$ **and may be bigger than** $E(x,T)$**.**

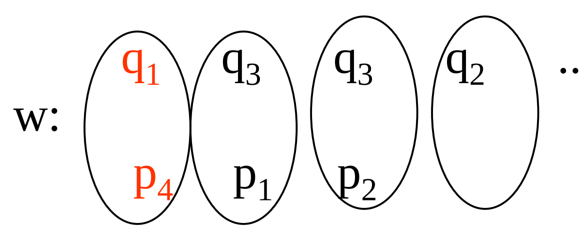
Some internal consistency is required for the set $\phi_x(T)$. If $y \in \phi_x(T)$, then any feasible outcome at which the players in T take the same actions as at y is also a feasible deviation from x via T. We say that this outcome is generated by y. Also, if the members of a coalition S only interact among themselves, and their payoffs only depend on the coalitions they form, then the outcome is a feasible deviation from any feasible outcome via S.





$S=\{p_4, q_1\}$





$w \in \phi_x(S), \ \forall x$

$y \in \phi_x(T) \leftrightarrow z \in \phi_x(T)$ **$z \in \phi^*_x(T,y)$= subset of outcomes in $\phi_x(T)$ generated by *y***

COOPERATIVE EQUILIBRIUM

Definition 1: *Let (N, C, X, U, u, ϕ) be a game in the deviation function form. Let x and y be in X. Outcome y* ***ϕ-dominates*** *outcome x via coalition T if:*

(a) $u_p(y) > u_p(x)$ for all players $p \in T$ and

(b) $y \in \phi_x(T)$

Definition 2: *Let (N, C, X, U, u, ϕ) be a game in the deviation function form. The feasible outcome x is* ***destabilized*** *by coalition T if there is some $y \in \phi_x(T)$ such that* ***x is ϕ-dominated by every outcome in $\phi^*_x(T,y)$, via coalition T****. An outcome $x \in X$ is* ***stable*** *for (N, C, X, U, u, ϕ) if it is not destabilized by any coalition.*

Thus, the **cooperative equilibria** for the cooperative games which can be fully represented in the *df*- form are the **stable outcomes.**

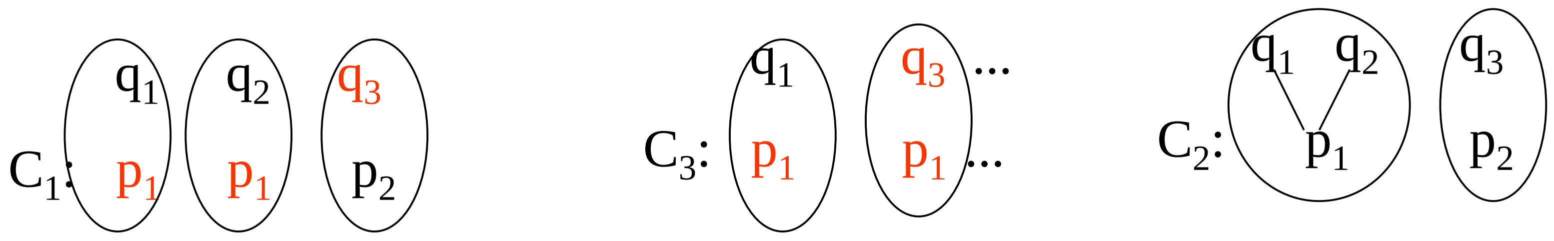
Another approach distinguishes two types of instabilities:

1. *The feasible outcome* x *is* ***destabilized in the strong sense*** *by coalition* T *if there is some* $y \in \phi_x(T)$ *such that* ***x is ϕ-dominated by every outcome in $\phi^*_x(T,y)$, via coalition T.***
2. *The feasible outcome* x *is* ***destabilized in the weak sense*** *by coalition* T *if there is some* $y \in \phi_x(T)$ *which* ***ϕ-dominates x via T.***

The outcome x is **stable in the strong sense** (resp. weak sense) if **it is not destabilized in the strong sense** (resp. weak sense) by any coalition.

When the payoffs of the members of the coalitions only depend on the agreements taken inside the coalitions, the two concepts are equivalent.

We can use this model to represent a cooperative decision situation. The key observation is that there might exist more than one way to represent an outcome, and some of these representations might lead to incorrect conclusions. For example, consider the outcome at which each of the following pairs of agents, $\{p_1, q_1\}$, $\{p_1,q_2\}$ and $\{p_2,q_3\}$, agrees to work together. Suppose these agreements are independent.



Clearly, C1 and C2 can be used to represent the given outcome. Now observe that the alternative outcome C_3, at which p_1 keeps its partnership with q_1, and p_1 and q_3 form a new coalition, is a feasible deviation from the given outcome via $T=\{p_1, q_3\}$. However, this outcome cannot be identified with a feasible deviation from C2 via T, since **$\{p_1,q_1\}$ is not one of the current coalitions of C_2.**

The way we found to solve this problem was to require that in the modeling of a feasible outcome, the coalitions be *minimal for the respective agreements*. Roughly speaking, a coalition is minimal if its members *cannot reach the part of the agreement due to them by rearranging themselves in proper sub-coalitions.*

The given outcome cannot be represented by C2 because $S_1 \cup S_2$ is not minimal. The players in S_1 and S_2 get the same agreements they get in $S_1 \cup S_2$, but in two proper sub-coalitions.

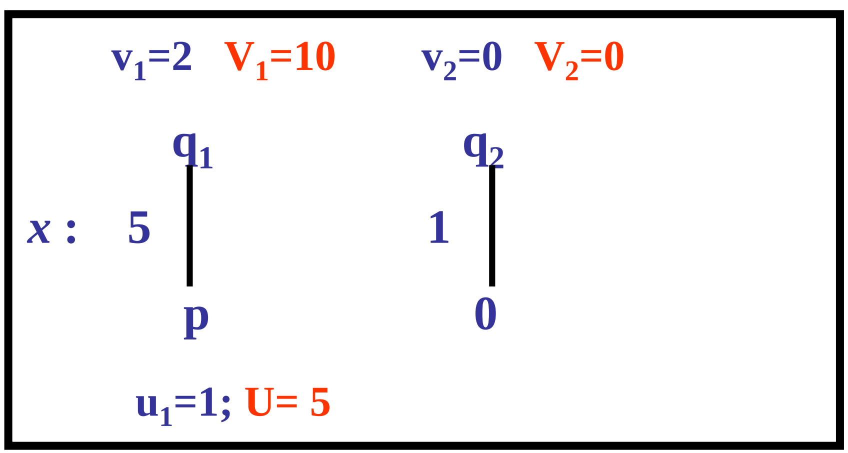
EXAMPLE 2. N={p, q_1, q_2}; p is a buyer and the other agents are sellers.

Seller q_1 has **5** units of a good to sell and seller q_2 has **1** unit of the same good. The maximum amount of money buyer p considers to pay for one unit of the good is **$3**. This agent has no utility for more than **5** units of the good.

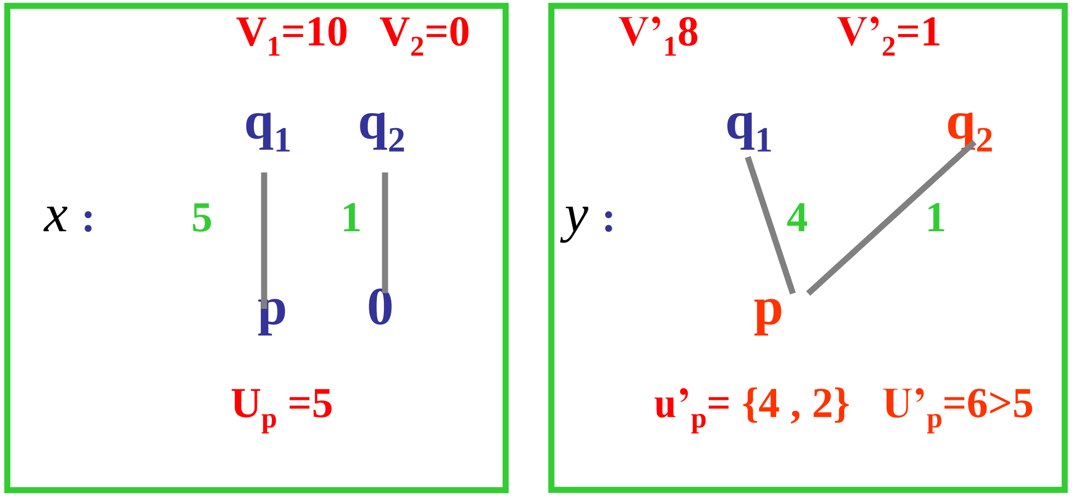
The negotiations are made between the buyer and each seller, independently.

Rigid agreements: if the term with respect to the number of items is broken then the whole agreement is nullified.

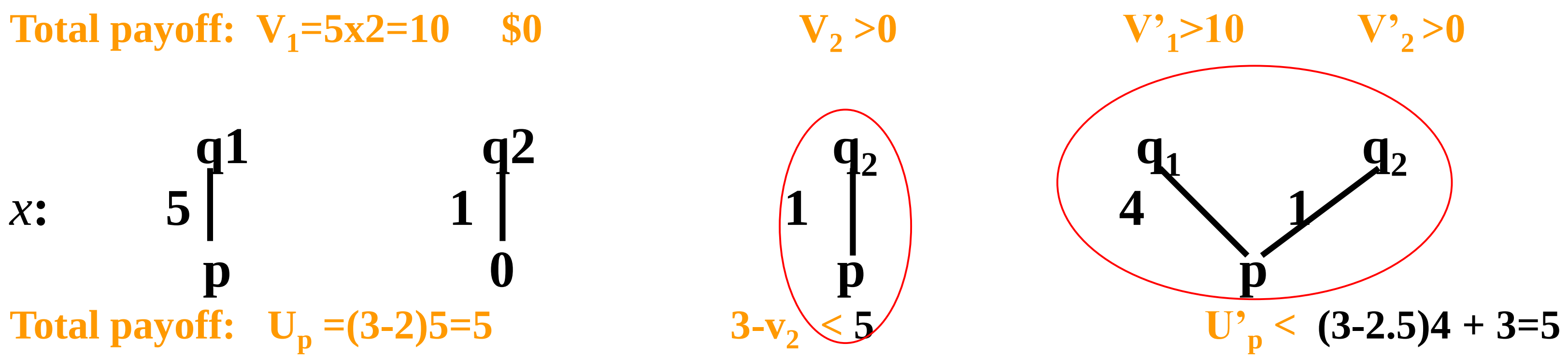
A **flexible agreement** allows the buyer to decrease the number of units without breaking the agreement corresponding to the price.



x **is a cooperative equilibrium under rigid agreements and it is not under flexible agreements. Furthermore** ***x*** **is in the core.**



If the agreements are flexible and x is proposed, then buyer p and seller q_2 can counter-propose an alternative outcome that both prefer. At this outcome buyer p reduces, from 5 to 4, the number of units to be acquired from q_1, in order to trade with q_2. These actions are allowed by the rules of the market. The outcome y might be the resulting outcome if q_2 sells his item to p for $1. The power of p of increasing his payoff is due to the concurs of q_1, which is assured by the flexible nature of the agreement with respect to the number of units negotiated. Therefore, ***x*** **cannot be considered a cooperative equilibrium** when the agreements are flexible. (Baiou and Balinski (2002), Alkan and Gale (2003))



If the agreements are **rigid**, it is easy to verify that there is no way for *p* to increase his total payoff by only trading with q_2.

In order to increase his total payoff, *p* must trade with both sellers, but there are no prices that can increase the current total payoffs of the three agents.

Therefore, the outcome *x* is a cooperative equilibrium when the agreements are rigid. The nature of the agreements is not relevant if we want to observe core allocations. **The outcome *x* is in the core of both markets.**