STABILITY CONCEPT: A CONTRIBUTION OF THE TWO-SIDED MATCHING MARKETS TO GAME THEORY.

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It is well-known that the theory that has been originated from the study of the two-sided matching markets has had many applications in real markets, helping to better understand these markets and influencing the design of allocation mechanisms.

In the present work, the idea of stability of matchings led us to the identification of a solution concept that captures the intuitive idea of cooperative equilibrium.

Indeed, this concept, that will also be called *stability*, applies to more general games than matching market games and is different and stronger than the core concept.
The notion of stability was introduced in Gale and Shapley (1962) for the marriage model and for the college admission model.

What is the property that characterizes the stable allocations in any matching market game? In other words, what is the solution concept that captures the intuitive idea of cooperative equilibrium in any matching market game?
When the matching market game can be represented under the characteristic function form, the stable allocations are identified with the core allocations.

However, there are matching market games which cannot be completely represented under the characteristic function form and for which the core allocations need not be stable allocations. This means that there are games in which some core allocations are not cooperative equilibrium allocations.

\[ x: \begin{array}{c|c}
q_1 & q_2 \\
p_1 & 1, 2 \\
p_2 & 1, 0 \end{array} \]

\[ y: \begin{array}{c|c}
q_1 & q_2 \\
p_1 & 1, 2 \\
p_2 & 1, 0.5 \end{array} \]

\[ a = \begin{pmatrix}
3 & 2 \\
3 & 3 \end{pmatrix} \]

\[ p_1 \]

\[ p_2 \]

\[ 2.5 \]

\[ 0, 1 \]

\[ 1, 1 \]

\[ 0.5 \]

\[ x \] is in the core but we cannot expect to see it as a prediction of the theory, so it is not a cooperative equilibrium allocation.
Basically we do the following:

- We define a new game form, which will be called *deviation function form*, that is more general than the characteristic function form and can be used to represent the matching models;
- for this game form we define the *quasi-dominance relation* on the set of feasible allocations.
- The set of feasible allocations that are not quasi-dominated by any other feasible allocation via some coalition includes the core and coincides with the set of stable allocations for the matching markets.
- These allocations, which will be called *stable allocations*, capture the intuitive idea of cooperative equilibrium allocation.
Consider a situation in which agents form coalitions and interact among them, by acting according to some established rules, aiming to reach an agreement on the terms that will regulate their participation in these coalitions.

Such a situation will be called game.
**RULES OF THE GAME**

A = Set of actions allowed to S

S + Agreement = Active coalition via a set A of actions

**Definition:** An active coalition via a set A of actions is *minimal* if no proper sub-coalition is active via a subset of A.

That is, S is a *minimal active coalition* if it is active via a set of actions A, and the restriction to any proper subset T of S, of the agreement reached in S via A, cannot be reached through actions in A that only involve players in T.

The agreements reached in each minimal active coalition are independent.
(a) Consider the College Admission model of Gale and Shapley (1962). If some coalition is a minimal active coalition, then it is formed with one student and one college, or only a single student, or only a single college.

(b) When agents make agreements in block, then every agent enters only one minimal active coalition and all members of the coalition interact only among themselves.

(c) In two-sided matching models in which players form multiple partnerships and make individual and independent agreements, a minimal active coalition is formed by a pair of players from opposite sides or by single players.
We will be assuming that if some minimal active coalition has only one player then this player is making an agreement with himself. In this case we say that the player is *single* in this coalition. A *single player* is a player that enters only one minimal active coalition and is single in that coalition.

The outcome that results from the agreements reached in all minimal active coalitions is called *feasible allocation*.

Agents have **preferences** over feasible allocations.

Roughly speaking, the equilibrium occurs in some feasible allocation when, *ex post*, no coalition of players “regrets” the agreements reached by its members in the minimal active coalitions in which they are participating. That is, after the feasible allocation has been obtained, for every set $S$ of players, there is no set of feasible actions that players in $S$ are allowed to take to profitably deviate from the given allocation.
The rules of the game must incorporate all details that are relevant to the phenomenon which we want to observe. In order to observe the equilibrium allocations these rules must specify, for example:

- the maximum number of minimal active coalitions a player can enter;
- the maximum number of players that is allowed in a minimal active coalition;
- which coalitions a player is allowed to form, etc.
The possibility of *reformulation* of an agreement inside a minimal active coalition occurs when this agreement consists of multiple and independent sub-agreements. In this case,

**Definition:** An agreement $A$ is a *reformulation* of an agreement $B$ if $A$ is obtained from $B$ by (i) nullifying some, but not all, sub-agreements of $B$; (ii) by keeping the sub-agreements that were not nullified and (iii) by replacing the nullified sub-agreements by new sub-agreements.

Therefore, the rules of the game also must indicate if a kind of reformulation of an agreement is or is not allowed.
In a market game where players may be thought of as being firms and workers operating in the entrance level, the rules must specify, for example:

- whether a firm negotiates in block or individually with the workers,
- whether a contract already done between a firm and a worker can or cannot be reformulated, by reducing the time of work without affecting the sub-agreement already reached by both agents on the monetary gains per unit of labor time.

Analogously, if players may be thought of as being buyers and sellers of a finite number of indivisible goods, the rules must specify, for example:

- if a contract already done between a buyer and a seller can or cannot be reformulated, by decreasing the number of items to be negotiated without affecting the sub-agreement already reached on the price for one unit of the good, etc.
In sum, the rules of the game must specify:

(a) the set of feasible actions, i.e., the set of actions that the players are allowed to take in the process of reaching the feasible allocations and,
(b) for each feasible allocation \( x \) and each coalition \( S \), the set of feasible actions for \( S \) given \( x \), i.e., the set of feasible actions that players in \( S \) are allowed to take *ex post* (given their agreements under \( x \)) in order to deviate from \( x \).
• It is worthwhile to point out that when, at a feasible allocation, a player enters more than one minimal active coalition, “to nullify all agreements” and “to make new agreements” are always feasible actions for him given the feasible allocation, as well as “to nullify the agreements in some, but not all, minimal active coalitions”. This is because the agreements reached in each minimal active coalition are independent.

• When the agreements (independent or not) are not formed with independent sub-agreements, the set of feasible actions for a coalition given some allocation coincides with the set of feasible actions for that coalition. Thus, if some sub-agreement is broken then the whole agreement is nullified.

An agreement at the feasible allocation $y$ is a **new agreement with respect to** $x$ if it is reached in some minimal active coalition at $y$, which is not minimal active at $x$, or which is also minimal active at $x$, but the players in this active coalition have nullified the whole agreement at $x$ and made a different agreement at $y$. Then, the agreement at $y$ is distinct from the agreement at $x$ but it is not a reformulation of the agreement at $x$. 
Given a coalition $S$ and a feasible allocation $x$, a feasible allocation $y$ is a **feasible deviation from $x$ via $S$** if

(a) when some minimal active coalition at $y$ involves a **non-empty set** $T \subseteq S$ and also players out of $S$, then this coalition is also minimal active at $x$. Furthermore, if the agreements reached by this coalition in the two allocations are distinct, then the agreement at $y$ is a reformulation of the agreement at $x$ via feasible actions for $T$ given $x$;

(b) every player in $S$ is involved in some new agreement at $y$ and

(c) if some **new agreement** at $y$ involves elements of $S$ then all players involved in such agreement belong to $S$. 
Thus, for example, if \( y \) is a feasible deviation from \( x \) via \( S \) and if the rules of the game do not allow that a player enters more than one minimal active coalition, then there is no minimal active coalition at \( y \) that contains elements of \( S \) and elements out of \( S \).

This is the case of the games which can be represented in the characteristic function form \((N,V)\). For these games, we can redefine \( V(S) \) as the set of feasible allocations that can be forced by \( S \), so \( V(N) \) is the set of feasible allocations. Clearly, an element of \( V(S) \) is a feasible deviation from \( x \) via \( S \) for some feasible allocation \( x \) (take \( x \), for example, where all members of \( S \) are single players and let \( x \) agree with the given element of \( V(S) \) for the players who are not in \( S \)). Conversely, for this kind of game, if \( y \) is a feasible deviation from \( x \) via \( S \) for some feasible allocation \( x \) then \( y \) is in \( V(S) \).
Example: (Time-sharing Assignment game, Sotomayor, 2010) Agent $p$ is a buyer; agents $q_1$ and $q_2$ are sellers. Seller $q_1$ has 5 units of a good to sell and seller $q_2$ has only 1 unit of the same good. Buyer $p$ values one unit of the good in $3$ and is not allowed to acquire more than 5 units.

The agreements are negotiated independently, by each seller and the buyer.

\[
\text{allocation: }\begin{array}{c}
\text{price of one unit} \\
\text{number of units}
\end{array}
\]

\[
v_1 \quad v_2
\]

\[
q_1 \quad q_2
\]

\[
k_1 \quad k_2
\]

\[
p
\]

\[
u_1 = (3-v_1), \quad u_2 = (3-v_2) \quad U=(3-v_1)k_1+(3-v_2)k_2
\]
\( v_1=2 \quad V_1=10 \quad v_2=0 \quad V_2=0 \quad v'_1=2 \quad V'_1=8 \quad v'_2=1 \quad V'_2=1 \)

\[
x: \begin{array}{c|c}
q_1 & q_2 \\
5 & 1 \\
p & (q_2)
\end{array}
\]

\[
u_1=(3-2)=1 \quad u_2=0
\]

\( U=(1)5+0=5 \)

\[
y: \begin{array}{c|c}
q_1 & q_2 \\
4 & 1 \\
p & (q_2)
\end{array}
\]

\[
u'_1=(3-2)=1, \quad u'_2=(3-1)=2
\]

\( U'=(1)4+(2)=4+2=6 \)

1) The sub-agreements are independent (the price of one item does not depend on the number of items acquired by the buyer).

The rules specify that given allocation \( x \), it is allowed to \( p \) to reformulate his current agreement with \( q_1 \), by reducing the number of units that he had agreed to negotiate, but by keeping the same price per unit.

\( y \) is a feasible deviation from \( x \) via \( S=\{p,q_2\} \). This deviation is profitable for both agents. Then, \( x \) is quasi-dominated by \( y \) via \( \{p,q_2\} \).

This means that coalition \( \{p,q_2\} \) is able to upset allocation \( x \), so \( x \) cannot be a cooperative equilibrium.
2) The sub-agreements are not independent. Given that the buyer and the seller agree on the number of units to be sold, they must agree about the price of each unit. Under this assumption, the action "to reformulate current agreements" is not allowed.

\( y \) is not a feasible deviation from \( x \) via \( \{p,q_2\} \). Observe, however, that \( y \) is a feasible deviation from \( x \) via \( \{p,q_1, q_2\} \), which is non-profitable to seller \( q_1 \).

There is no coalition of players that regrets the current agreements at \( x \).

Allocation \( x \) is a cooperative equilibrium.

It is a matter of verification that under both rules \( x \) is a core allocation. ■
We will abstract from the actions and will focus on the allocations. Thus, we reach the feasible allocations and, for each feasible allocation \( x \) and for each coalition \( S \), we get the set of \textit{feasible deviations} from \( x \) via \( S \), leading to the \textit{deviation function form} of representing a game with \( n \) players.

\( N = \{1, \ldots , n\} \) - set of players.

Each non-empty subset of \( N \) is called a \textit{coalition}. The set \( N \) is referred to as the \textit{grand coalition}.

\( X \) - set of \textit{feasible allocations}.

For each player \( j \),

\( R_j \) - \( j \)'s preference relation on set \( X \).

For each feasible allocation \( x \) and coalition \( S \),

\( V_x(S) \) - set of \textit{feasible deviations from} \( x \) \textit{by} \( S \). \( V_x \) is called \textit{deviation function from} \( x \).
The intuitive idea that underlies the concept of cooperative equilibrium allocation is that, in a context in which agents freely interact and form coalitions, *ex-post*, i.e., after an allocation $x$ is reached, there is no group of players who regret the coalitional interactions they performed at $x$. Of course, such regret only exists for a given set of players $S$, if the players in $S$ identify new coalitional interactions, which are **feasible for $S$ given $x$**, such that, if these coalitional interactions are performed, the resulting outcome will be a new feasible allocation that is more profitable to all players in $S$ than it is the current allocation.

Generally, for every coalition $S$, $V(S) \subseteq \bigcup V_x(S)$ over all feasible allocations $x$. Every game that can be represented in the form $(N,V)$ has $V(S) = \bigcup V_x(S)$, so it can be represented in the deviation function form. The converse is not true. The multiple-partners assignment game can be represented in the deviation function form, but it does not have a representation in the form $(N,V)$. 

The intuitive idea that underlies the concept of cooperative equilibrium allocation is that, in a context in which agents freely interact and form coalitions, *ex-post*, i.e., after an allocation $x$ is reached, there is no group of players who regret the coalitional interactions they performed at $x$. Of course, such regret only exists for a given set of players $S$, if the players in $S$ identify new coalitional interactions, which are **feasible for $S$ given $x$**, such that, if these coalitional interactions are performed, the resulting outcome will be a new feasible allocation that is more profitable to all players in $S$ than it is the current allocation.
The solution concept that captures this idea of equilibrium will be called *stability*. Roughly speaking, a feasible allocation \( x \) is *stable* if there is no coalition \( S \) of players and a feasible deviation from \( x \) via \( S \), which is more profitable to all players in \( S \) than it is the current allocation \( x \). This concept can be formally defined by using the *quasi-domination relation* on the feasible allocations defined below.

**Definition 1.** The feasible allocation \( y \) *quasi-dominates* the feasible allocation \( x \) via coalition \( S \) if:

(a) all players in \( S \) prefer \( y \) to \( x \) and

(b) \( y \in V_x(S) \).

**Definition 2.** The allocation \( x \) is *stable* if it is not quasi-dominated by any feasible allocation via some coalition.
When agents negotiate in block, each one obtains a one-dimensional payoff, given by his total gain in the trades. The allowable actions to a coalition to deviate from a feasible allocation are either to nullify all agreements and to perform new agreements or to keep unchanged all agreements. Thus, the core concept is the concept of cooperative equilibrium and a deviating coalition is a blocking coalition. Thus, if $y$ is a feasible deviation from $x$ via $S$, all agreements at $y$, which involve players in $S$, are new agreements.

This is the traditional approach. The characteristic function captures all the relevant details of the market game.
The stability concept has been established for two-sided matching models as the concept that captures some intuitive idea of equilibrium. It was introduced in Gale and Shapley (1962) for the marriage model and for the college admission model. Along the years it has been defined locally for the matching model that is being studied and has evolved according to the complexity of the model.

The main feature of the concept of stability is that when a two-sided matching market is treated as a cooperative game (matching market game), that idea of equilibrium for the matching market is identified with the idea of cooperative equilibrium. Thus, the stable allocations of a matching market are the cooperative equilibrium allocations for the corresponding matching market game.

What is the property that characterizes the stable allocations in any matching market game? In other words, what is the cooperative solution concept that captures the intuitive idea of cooperative equilibrium in the matching market games?
For each coalition \( S \) there is a set of actions allowed to \( S \) by the specified rules.

If an agreement among the players of \( S \) was reached via a set of actions that only involve players in \( S \), then \( S \) is called \textit{active coalition}.

An active coalition that reached an agreement via a set \( A \) of actions is \textit{minimal} if no proper sub-coalition is active via a subset of \( A \). That is, \( S \) is a \textit{minimal active coalition} if it is active, and the restriction, to any \( T \subseteq S, T \neq S \), of the agreement reached in \( S \) via a set of actions \( A \) cannot be reached through actions in \( A \) that only involve players in \( T \).

The agreements reached in each minimal active coalition are independent.

We will be assuming that if some minimal active coalition has only one player then this player is making an agreement with himself. In this case we say that the player is \textit{single} in this coalition. A \textit{single player} is a player that enters only one minimal active coalition and is single in that coalition.
**Example.** Consider a labor market with two sellers, $q_1$ and $q_2$, and one buyer $p$. Seller $q_1$ has 5 units of a good to sell and seller $q_2$ has only 1 unit of the same good. Buyer $p$ values one unit of the good in $3 and is not allowed to acquire more than 5 units. The agreements are negotiated independently, by each seller and the buyer. An agreement between a seller and the buyer includes two sub-agreements: one sub-agreement on the price of one unit of the good and another one on the number of units to be acquired by the buyer. A transaction between some seller and buyer $p$ only occurs if the price of one unit of the good is any non-negative number less than or equal to $3 and the number of units sold by the seller does not exceed the minimum between the number of units he owns and 5 minus the number of units sold by the other seller. In this case each seller receives the product of his negotiation with the buyer. If buyer $p$ acquires $k$ items from a seller, for $t$ each, then he receives an individual payoff given by $(3-t)k$ and the seller receives the payoff of $tk$. An allocation specifies one payoff for each seller, a pair of individual payoffs for the buyer, corresponding to the transactions done with each seller, and the number of units sold by each seller. It is feasible if it results from transactions between the buyer and each seller. This is an instance of the matching model introduced in Sotomayor (2010) and called Time-sharing Assignment game.
Now consider the feasible allocation \( x \) where buyer \( p \) acquires 5 units of the good of seller \( q_1 \) and pays $2 for each unit. At the feasible allocation \( y \), agent \( p \) acquires 4 units of the good of seller \( q_1 \), for the same $2 for each unit, and \( q_2 \) sells his item to \( p \) for $1. Suppose that the sub-agreements are independent. This means that the price of one item does not depend of the number of items acquired by the buyer. Thus it is reasonable that the rules of the game specify that a feasible action for buyer \( p \) given allocation \( x \) is allowed to reformulate his current agreement with \( q_1 \), by reducing the number of units that he had agreed to negotiate, but by keeping the same price per unit. Thus, if \( x \) was reached, it would be allowed to \( p \) to reduce the number of units acquired from \( q_1 \) from 5 to 4, without changing the sub-agreement with \( q_1 \) with respect to the price of each unit (observe that it is not required that this reformulation be of \( q_1 \)’s interest). Then, if \( q_2 \) nullified the agreement with himself (this action is a feasible action given \( x \)) and sold his item to the buyer for $1, \( y \) would be a feasible deviation from \( x \) via \( \{p,q_2\} \). In fact, observe that coalition \( \{p,q_1\} \) is minimal active at \( y \) and \( q_1 \notin S \). As required in (a), \( \{p,q_1\} \) is minimal active at \( x \) and the whole agreement has been reformulated by \( p \), according to the rules of the market. Also, coalition \( \{p,q_2\} \) is active at \( y \) and is not active at \( x \), so the whole agreement between the two agents is new. As it is required in (b), every element of \( S \) is part of an active coalition at \( y \), and as it is required in (c), both players are in \( S \).
This deviation is profitable for both agents. We will say that \( x \) is quasi-dominated by \( y \) via \( \{p,q2\} \). This means that coalition \( \{p,q2\} \) (regrets the current agreements at \( x \)) is able to upset allocation \( x \), so \( x \) cannot be a cooperative equilibrium.

Now, suppose that the sub-agreements are not independent: given that the buyers and seller agree on the number of units to be sold, they must agree about the price of each unit. Under this assumption, the action “to reformulate current agreements” is not allowed. In this case, \( y \) is not a feasible deviation from \( x \) via \( \{p,q2\} \). Observe, however, that \( y \) is a feasible deviation from \( x \) via \( \{p,q1,q2\} \), which is non-profitable to seller \( q1 \). In this case the agreement between \( p \) and \( q1 \) at \( y \) is not a reformulation of the agreement at \( x \), but a new agreement between these two agents, which has been obtained after the current agreement has been nullified. It is easy to verify that there is no way for \( p \) to increase his total payoff by trading only with \( q2 \). In order to increase his total payoff \( p \) must trade with both agents, but there are no prices that can increase the current total payoffs of the three agents. There is no coalition of players that regrets the current agreements at \( x \). Allocation \( x \) is a cooperative equilibrium. It is a matter of verification that under both rules \( x \) is a core allocation. ■
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When the matching market game can be represented under the characteristic function form, the stable allocations are identified with the core allocations.

However, there are matching market games which cannot be completely represented under the characteristic function form and for which the core allocations need not be stable allocations. This means that there are games in which some core allocations are not cooperative equilibrium allocations.

\[
\begin{align*}
\text{x:} & \quad \begin{array}{c}
q_1 \quad 2 \quad 1, 2 \\
p_1 \quad 1, 1
\end{array} & \quad \begin{array}{c}
q_2 \quad 1, 0 \\
p_2 \quad 0, 1
\end{array} \\
\text{a=} & \quad \begin{array}{c}
p_1 \quad 3 \quad 2 \\
p_2 \quad 3 \quad 3
\end{array} \\
\text{y:} & \quad \begin{array}{c}
q_1 \quad 1, 2 \\
p_1 \quad 0, 1
\end{array} & \quad \begin{array}{c}
q_2 \quad 1, 0.5 \\
p_2 \quad 1, 0.5
\end{array}
\end{align*}
\]

\(x\) is in the core but we cannot expect to see it as a prediction of the theory, so it is not a cooperative equilibrium allocation.
Basically we do the following:

- We define a new game form, which will be called *deviation function form*, that is more general than the characteristic function form and can be used to represent the matching models;
- for this game form we define the *quasi-dominance relation* on the set of feasible allocations.
- The set of feasible allocations that are not quasi-dominated by any other feasible allocation via some coalition includes the core and coincides with the set of stable allocations for the matching markets.
- These allocations, which will be called *stable allocations*, capture the intuitive idea of cooperative equilibrium allocation.
Consider a situation in which agents form coalitions and freely interact inside each coalition that is formed, by acting according to some established rules, aiming to reach an agreement on the terms that will regulate their participation in these coalitions.

Such a situation will be called *game*. 
In this context, the intuitive idea that underlies the concept of cooperative equilibrium allocation is that, *ex-post*, i.e., after an allocation $x$ is reached, there is no group of players who *regret* the coalitional interactions they performed at $x$. Of course, such regret only exists for a given set of players $S$, if the players in $S$ identify new coalitional interactions, which are *feasible for $S$ given $x*", such that, if these coalitional interactions are performed, the resulting outcome will be a new feasible allocation that is more profitable to all players in $S$ than it is the current allocation.

**Feasible actions for $S=\{p_1, q_3\}$ given $x$:**

- $p_1$ keeps its partnership with $q_1$, $q_3$ breaks his partnership with $p_2$ and $p_1$ and $q_3$ enter a new partnership.

  $y$ is a feasible deviation from $x$ via $S$.  

Example: *(Time-sharing Assignment game, Sotomayor, 2010)* Agent \( p \) is a buyer; agents \( q_1 \) and \( q_2 \) are sellers. Seller \( q_1 \) has 5 units of a good to sell and seller \( q_2 \) has only 1 unit of the same good. Buyer \( p \) values one unit of the good in $3 and is not allowed to acquire more than 5 units.

The agreements are negotiated independently, by each seller and the buyer.

**Allocation:**

\[
\begin{align*}
\text{price of one unit} & \quad \text{number of units} \\
\downarrow & \quad \downarrow \\
v_1 & \quad v_2 \\
k_1 & \quad k_2 \\
k_1 \leq 5, \ k_2 \leq 1 \\
k_1 + k_2 \leq 5 \\
p
\end{align*}
\]

\[
\begin{align*}
u_1 &= (3-v_1), \quad u_2 = (3-v_2) \quad U = (3-v_1)k_1 + (3-v_2)k_2
\end{align*}
\]
The sub-agreements are independent (the price of one item is independent of the number of items acquired by the buyer).

The rules specify that given allocation $x$, it is allowed to $p$ to reformulate his current agreement with $q_1$, by reducing the number of units that he had agreed to negotiate, but by keeping the same price per unit. (feasible actions for $p$ given $x$).

$y$ is a feasible deviation from $x$ via $S=\{p, q_2\}$. This deviation is profitable for both agents. Then, $x$ is not a cooperative equilibrium.
2) The sub-agreements are not independent. Given that the buyer and the seller agree on the number of units to be sold, they must agree about the price of each unit. Under this assumption, the action “to reformulate current agreements” is not allowed. If some sub-agreement is broken then the whole agreement is nullified.

\[ y \text{ is not a feasible deviation from } x \text{ via } \{p, q_2\}. \]

Observe, however, that \( y \) is a feasible deviation from \( x \) via \( \{p, q_1, q_2\} \), which is non-profitable to seller \( q_1 \).

There is no coalition of players that regrets the current agreements at \( x \).

Allocation \( x \) is a cooperative equilibrium.

It is a matter of verification that under both rules \( x \) is a core allocation. ■
The rules of the game must incorporate all details that are relevant to the phenomenon which we want to observe. In order to observe the equilibrium allocations these rules must specify:

(a) the set of feasible actions, i.e., the set of actions that the players are allowed to take in the process of reaching the feasible allocations and,
(b) for each feasible allocation $x$ and each coalition $S$, the set of feasible actions for $S$ given $x$, i.e., the set of feasible actions that players in $S$ are allowed to take ex post (given their agreements under $x$) in order to deviate from $x$. 

Rules of the game
The rules of the game specify:

a) The set of **feasible allocations**.

b) For each feasible allocation $x$ and coalition $S$, the set of **feasible deviations from $x$ given $S$**.
**Definition:** An active coalition via a set $A$ of actions is *minimal* if no proper sub-coalition is active via a subset of $A$.

That is, $S$ is a *minimal active coalition* if it is active via a set of actions $A$, and the restriction to any proper subset $T$ of $S$, of the agreement reached in $S$ via $A$, cannot be reached through actions in $A$ that only involve players in $T$.

The agreements reached in each minimal active coalition are independent.
Consider the College Admission model of Gale and Shapley (1962). If some coalition is a minimal active coalition, then it is formed with one student and one college, or only a single student, or only a single college.

When agents make agreements in block, then every agent enters only one minimal active coalition and all members of the coalition interact only among themselves.

In two-sided matching models in which players form multiple partnerships and make individual and independent agreements, a minimal active coalition is formed by a pair of players from opposite sides or by single players.
The possibility of reformulation of an agreement inside a minimal active coalition occurs when this agreement consists of multiple and independent sub-agreements. In this case,

**Definition:** An agreement $A$ is a reformulation of an agreement $B$ if $A$ is obtained from $B$ by (i) nullifying some, but not all, sub-agreements of $B$; (ii) by keeping the sub-agreements that were not nullified and (iii) by replacing the nullified sub-agreements by new sub-agreements.

Therefore, the rules of the game also must indicate if a kind of reformulation of an agreement is or is not allowed.
Definition: An agreement at the feasible allocation \( y \) is a new agreement with respect to \( x \) if it is reached in some minimal active coalition at \( y \), which is not minimal active at \( x \), or which is also minimal active at \( x \), but the players in this active coalition have nullified the whole agreement at \( x \) and made a different agreement at \( y \). Then, the agreement at \( y \) is distinct from the agreement at \( x \) but it is not a reformulation of the agreement at \( x \).
Feasible deviation from \( x \) via \( S \)

Given a coalition \( S \) and a feasible allocation \( x \), a feasible allocation \( y \) is a feasible deviation from \( x \) via \( S \) if

(a) when some minimal active coalition at \( y \) involves a non-empty set \( T \subseteq S \) and also players out of \( S \), then this coalition is also minimal active at \( x \).

Furthermore, if the agreements reached by this coalition in the two allocations are distinct, then the agreement at \( y \) is a reformulation of the agreement at \( x \) via feasible actions for \( T \) given \( x \);

(b) every player in \( S \) is involved in some new agreement at \( y \) and

(c) if some new agreement at \( y \) involves elements of \( S \) then all players involved in such agreement belong to \( S \).
We will abstract from the actions and will focus on the allocations. Thus, we reach
the feasible allocations and, for each feasible allocation $x$ and for each coalition
$S$, we get the set of feasible deviations from $x$ via $S$, leading to the deviation
function form of representing a game with $n$ players.

$N = \{1, \ldots, n\}$ - set of players.

Each non-empty subset of $N$ is called a coalition. The set $N$ is referred to as the
grand coalition.

$X$ - set of feasible allocations.

For each player $j$,

$R_j$ - $j$'s preference relation on set $X$.

For each feasible allocation $x$ and coalition $S$,

$V_x(S)$ - set of feasible deviations from $x$ by $S$. $V_x$ is called deviation function
from $x$. 

For the games which can be represented in the characteristic function form \((N,V)\), we can redefine \(V(S)\) as the set of feasible allocations that can be forced by \(S\), so \(V(N)\) is the set of feasible allocations. Clearly, an element of \(V(S)\) is a feasible deviation from \(x\) via \(S\) for some feasible allocation \(x\) (take \(x\), for example, where all members of \(S\) are single players and let \(x\) agree with the given element of \(V(S)\) for the players who are not in \(S\)). Conversely, for this kind of game, if \(y\) is a feasible deviation from \(x\) via \(S\) for some feasible allocation \(x\) then \(y\) is in \(V(S)\). Thus, \(V(S) = \bigcup V_x(S)\), over all feasible allocations \(x\), so every game that can be represented in the form \((N,V)\) can be represented in the deviation function form.

The converse is not true. The multiple-partners assignment game can be represented in the deviation function form, but it does not have a representation in the form \((N,V)\).
The solution concept that captures this idea of equilibrium will be called *stability*. Roughly speaking, a feasible allocation $x$ is *stable* if there is no coalition $S$ of players and a feasible deviation from $x$ via $S$, which is more profitable to all players in $S$ than it is the current allocation $x$. This concept can be formally defined by using the *quasi-domination relation* on the feasible allocations defined below.

**Definition:** The feasible allocation $y$ quasi-dominates the feasible allocation $x$ via coalition $S$ if:

(a) all players in $S$ prefer $y$ to $x$ and

(b) $y \in V_x(S)$.

**Definition:** The allocation $x$ is *stable* if it is not quasi-dominated by any feasible allocation via some coalition.
When agents negotiate in block, each one obtains a one-dimensional payoff, given by his total gain in the trades. The allowable actions to a coalition to deviate from a feasible allocation are either to nullify all agreements and to perform new agreements or to keep unchanged all agreements. Thus, the core concept is the concept of cooperative equilibrium and a deviating coalition is a blocking coalition. Thus, if $y$ is a feasible deviation from $x$ via $S$, all agreements at $y$, which involve players in $S$, are new agreements.

This is the traditional approach. The characteristic function captures all the relevant details of the game.