CONNECTING THE COOPERATIVE AND COMPETITIVE STRUCTURES OF THE MULTIPLE-PARTNERS ASSIGNMENT GAME

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TWO-SIDED MATCHING MARKETS.

Two-sided matching models were introduced in the literature by Gale and Shapley in 1962, as a rare instance of an exercise in “pure” mathematics (combinatorial theory of ordered sets).

Along the last two decades, the two-sided matching models have moved from being an interesting set of pure mathematical models to being an important part of the emerging field of market design. Through these models a variety of markets has reached a better understanding, what has considerably contributed to the organization of such markets.

Firm- worker

labor markets

buyer-seller markets

cooperative game

structure

competitive market

game structure

How does the competitive equilibrium concept, closely related to the traditional concept of equilibrium from standard microeconomic theory, compare with the cooperative equilibrium concept?

MANY-TO-MANY MATCHING GAME WITH ADDITIVELY SEPARABLE UTILITIES

B, Q – sets of players

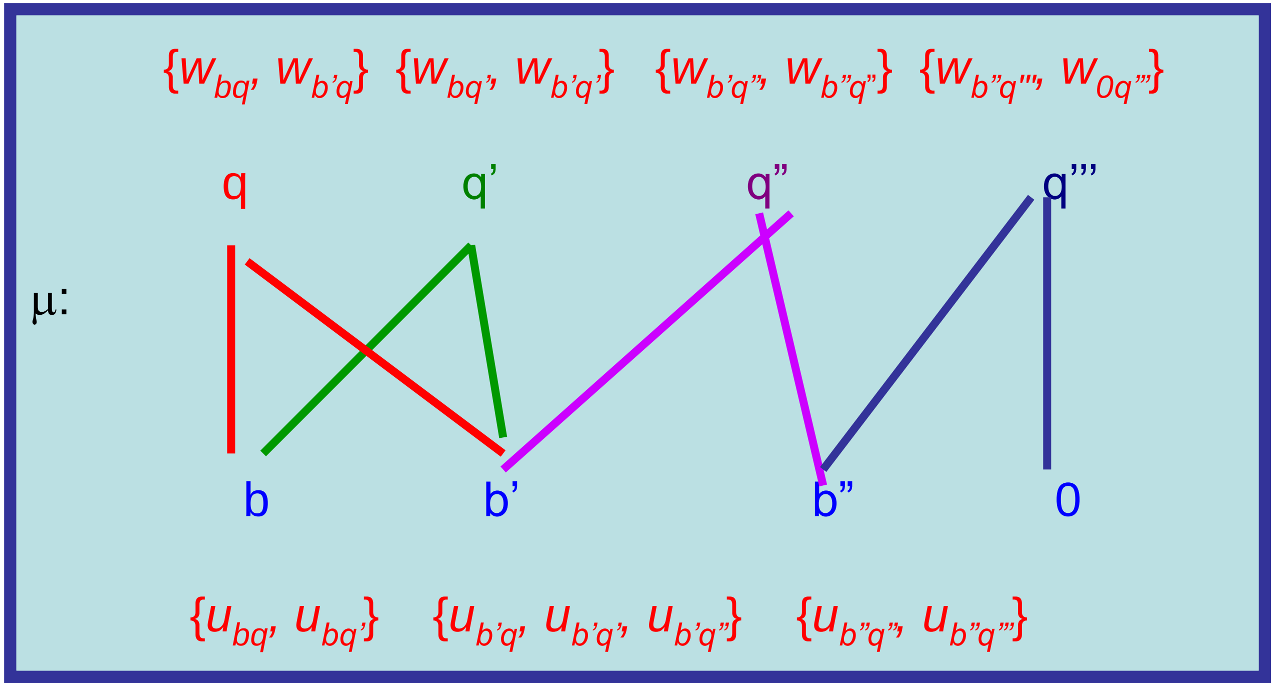
A player of a set may form more than one partnership with different players of the other set.

Quota –r(b), s(q) - maximum number of partners.

If *b* and *q* become partners, they undertake an activity together that produces a gain v_{bq}, which is split between them the way both agree:

$$u_{bq} \geq 0 \text{ for } b \text{ and } w_{bq} = v_{bq} - u_{bq} \geq 0 \text{ for } q.$$

The game is described by **(B,Q,v,r,s).**



$\mathbf{u_{bq} + w_{bq} = v_{bq}}$ $\mathbf{u_{bq} \geq 0,\ w_{bq} \geq 0}$

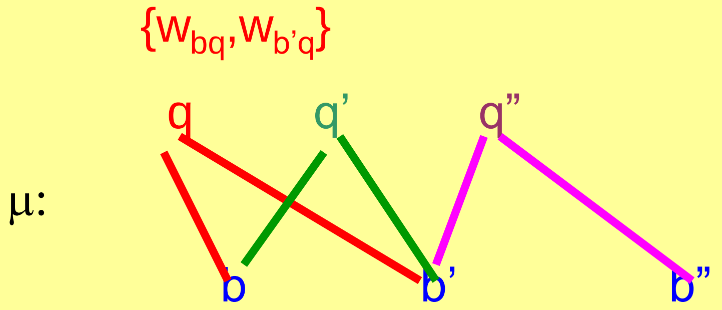
Outcome =matching **+** payoffs u_{bq}'s, w_{bq}'s.

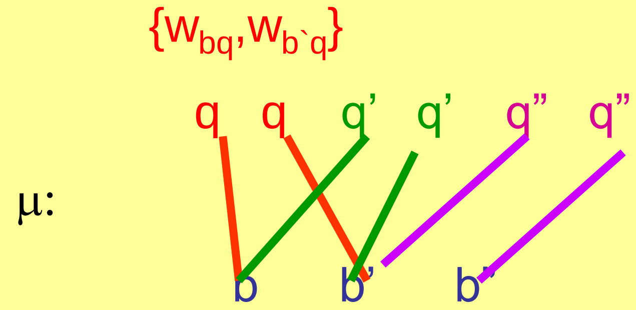
Assignment Game of Shapley and Shubik (1972) - a special case

INTERPRETATION 1: *B= set of firms; Q= set of workers.*

quota of a firm: the maximum number of workers it can hire;

INTERPRETATION 2: *B= set of buyers; Q= set of sellers.*





INTERPRETATION 1

feasible matching

INTERPRETATION 2

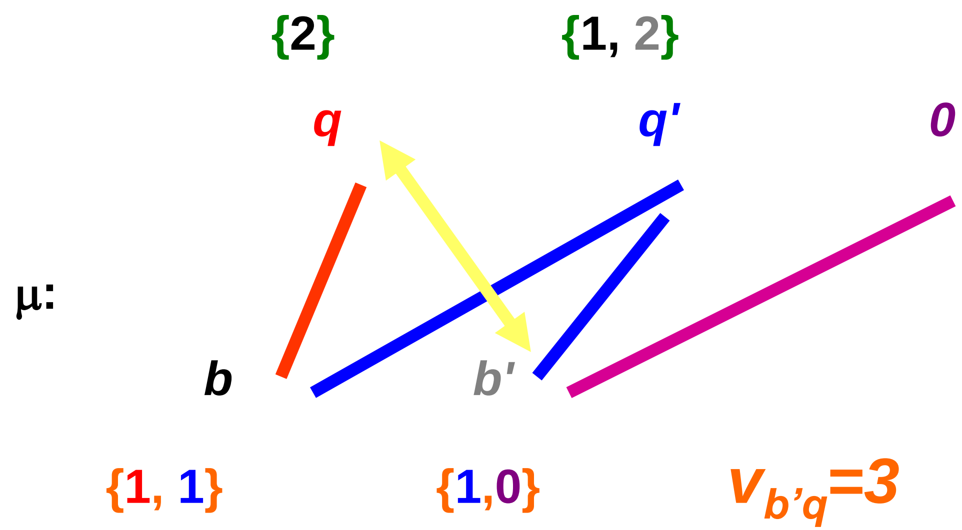
feasible allocation

EQUILIBRIUM CONCEPTS

The outcome $x=(u,w;\mu)$ will be called ***unstable*** if there are agents ***b*** and ***q*** that do not form a partnership at ***x***, but that can increase their total payoff, becoming partners and at the same time keeping and/or leaving some of their old partners, if necessary, in order to remain within their quotas. The outcome $(u,w;\mu)$ is ***stable*** if it is not unstable.

1. Labor market

Setwise-stability concept ≅ pairwise-stability concept (Sotomayor, 1999)



$(u,w;\mu)$ is unstable: $2= u_{b'0} + w_{bq} < v_{b'q}=3$

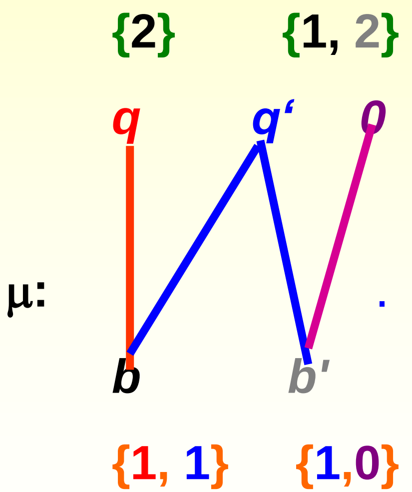
For some markets, instabilities can be restricted to pairs of agents of opposite sides and then setwise-stability is equivalent to pairwise-stability. For some other markets the setwise-stability concept is given by corewise-stability. Sotomayor (1999) proved that setwise-stability is a new cooperative equilibrim concept, stronger than pairwise-stability plus corewise-stability. The main point however is that any stable outcome must be in the core.

Shapley and Shubik proved that the core of the one-to-one buyer-seller market coincides with the set of stable payoffs.

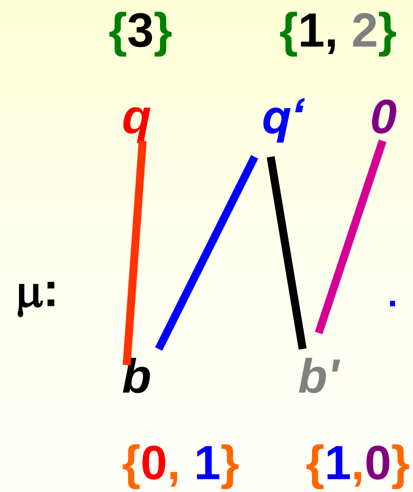
However **both outcomes are** in the core*!*

Stability concept ≠ Core concept

		q	q'	0
v:	b	3	2	0
	b'	3	3	0



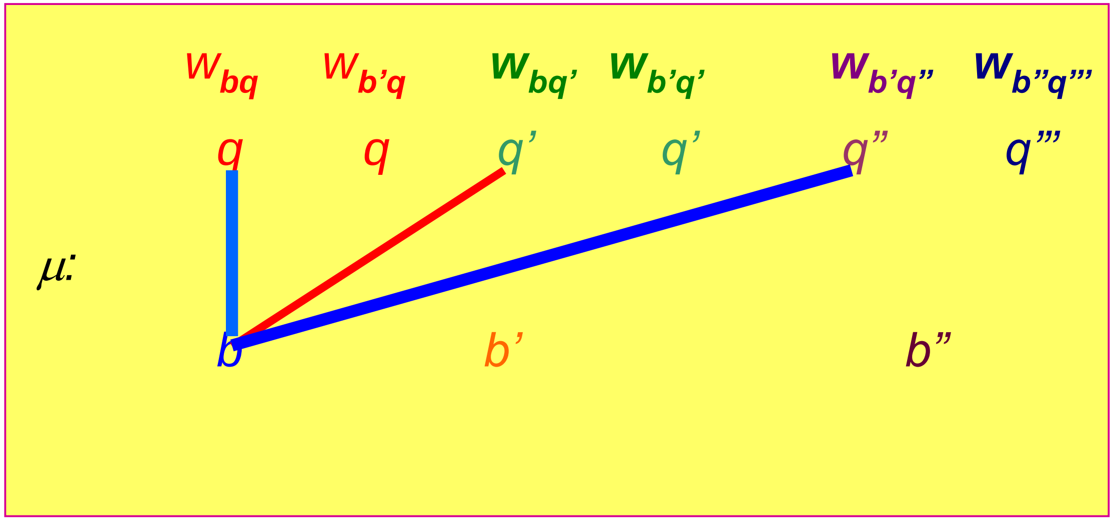
It is unstable



It is stable

EQUILIBRIUM CONCEPTS

2. Buyer-seller market: competitive equilibrium



$$v_{bq} - w_{bq} > v_{bq'} - w_{bq'} = v_{bq''} - w_{b'q''} > v_{bq'''} - w_{q'''}$$

Buyer *b, with quota 2,* wants to buy any of the sets

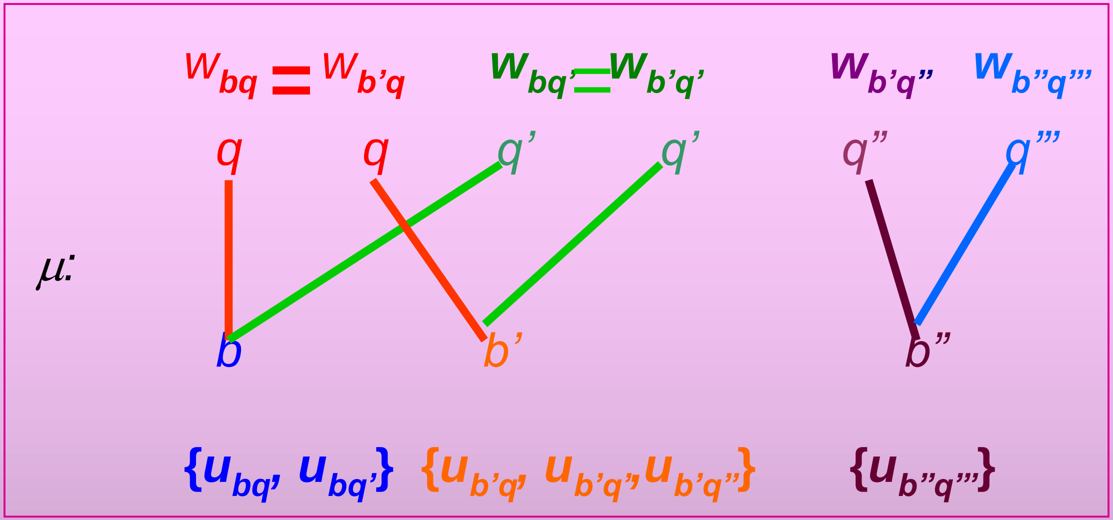
{*q,q'*} and {*q,q''*}.

EQUILIBRIUM CONCEPTS

2. Buyer-seller market: competitive equilibrium

The outcome **($u,w;\mu$)** is a *competitive equilibrium outcome* if

- μ is a feasible matching;
- each active buyer receives one of her demanded sets of items at prices ***w*** (i.e. a set of items that, given prices, maximizes her additive utility payoff);
- every inactive buyer has zero payoff; every unsold object has zero price.



Unlike the cooperative model, **every seller sells all of his items at the same price.**

In fact, if a seller has two identical objects, q and q', and $p_q > p_{q'}$ for some price vector p, then no buyer b will demand, at prices p, a set S of objects that contains object q. This is because, by replacing q with q' in S, b gets a more preferable set of objects. But then, q will remain unsold with a positive price.

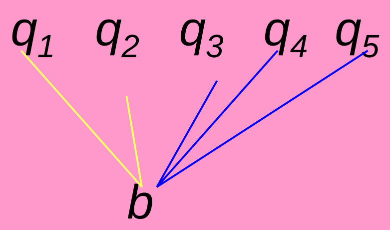
PARTIAL ORDER RELATION

This representation is

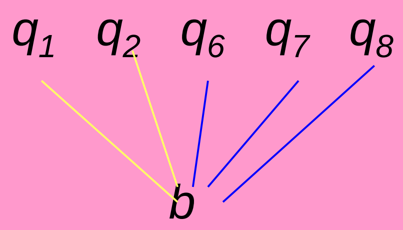
It

Now, suppose μ' is another matching that is compatible with some other stable payoff:

μ:



μ':



$\{u_{b1}, u_{b2}, u_{b3}, u_{b4}, u_{b5}\}$ $\{u'_{b1}, u'_{b2}, u'_{b6}, u'_{b7}, u'_{b8}\}$

$u_{b3} = u_{b4} = u_{b5} = u_b(min) \equiv \lambda.$ $u'_{b6} = u'_{b7} = u'_{b8} = u'_b(min) \equiv \lambda'.$

$u_b = (u_{b1}, u_{b2}, \lambda, \lambda, \lambda)$ $u'_b = (u'_{b1}, u'_{b2}, \lambda', \lambda', \lambda').$

Therefore, by ordering the players in *B* (re...

immerse the stable payoffs of these ...

whose dimension is the sum of th...

(respectively, *Q*). Then, the natu...

Euclidean space induces the parti...

$\geq_B$ (respectivel...

in the set of stable payoffs

Observe that $\geq_B$ (respectively $\geq_Q$) does not correspond to the preferences of the buyers (respectively, sellers).

$$(u,w) \geq_B (u',w') \text{ if and only if } u_b \geq u'_b, \text{ for all } b \in B.$$

$$(u,w) \geq_Q (u',w') \text{ if and only if } w_q \geq w'_q, \text{ for all } q \in Q.$$

Although the preferences of the players do not define the partial orders $\geq_B$ and $\geq_Q$, the maximal element under $\geq_B$ (respectively, $\geq_Q$) is the *B*-optimal (respectively *Q*-optimal) stable payoff. The best for one side is the worst for the other side.

payoffs is a non-empty convex and complete lattice under ***$\geq_B$ and $\geq_Q$.***

xist one and only one *B*-optimal stable payoff and one and only one *Q*-optimal stable payoff.

ALGEBRAIC STRUCTURE OF THE SET OF COMPETITIVE EQUILIBRIUM PAYOFFS

Unlike the one-to-one case, the set of stable payoffs and the set of competitive equilibrium payoffs are distinct:

The set of competitive equilibrium payoffs can be obtained by "shrinking" the set of stable payoffs through an isotone function *f.*

For each stable outcome, *f* reduces the total payoff to every seller by reducing the price of each of his items to his minimal individual payoff, to create a competitive equilibrium.

The competitive equilibrium payoffs are exactly the fixed points of *f*. However, the function *f* **is not the identical map**, so there are stable payoffs that **are not fixed points**, and so there are stable payoffs that **are not competitive**. Hence the set of competitive equilibrium payoffs is a **proper subset** of the set of stable payoffs.

Teorema (Sotomayor, 2006). ***The set of competitive equilibrium payoffs is a non-empty complete sub-lattice of the set of stable payoffs. In addition, it reflects the same kind of polarization of interests that characterizes the stable payoffs.***

Then, the B-optimal and the Q-optimal competitive equilibrium payoffs exist.

Theorem (Tarski, 1955). *Let E be a complete lattice with respect to some partial order $\geq$, and let f be an isotone function from E to E. Then the set of fixed points of f is non-empty and is itself a complete lattice with respect to the partial order $\geq$.*

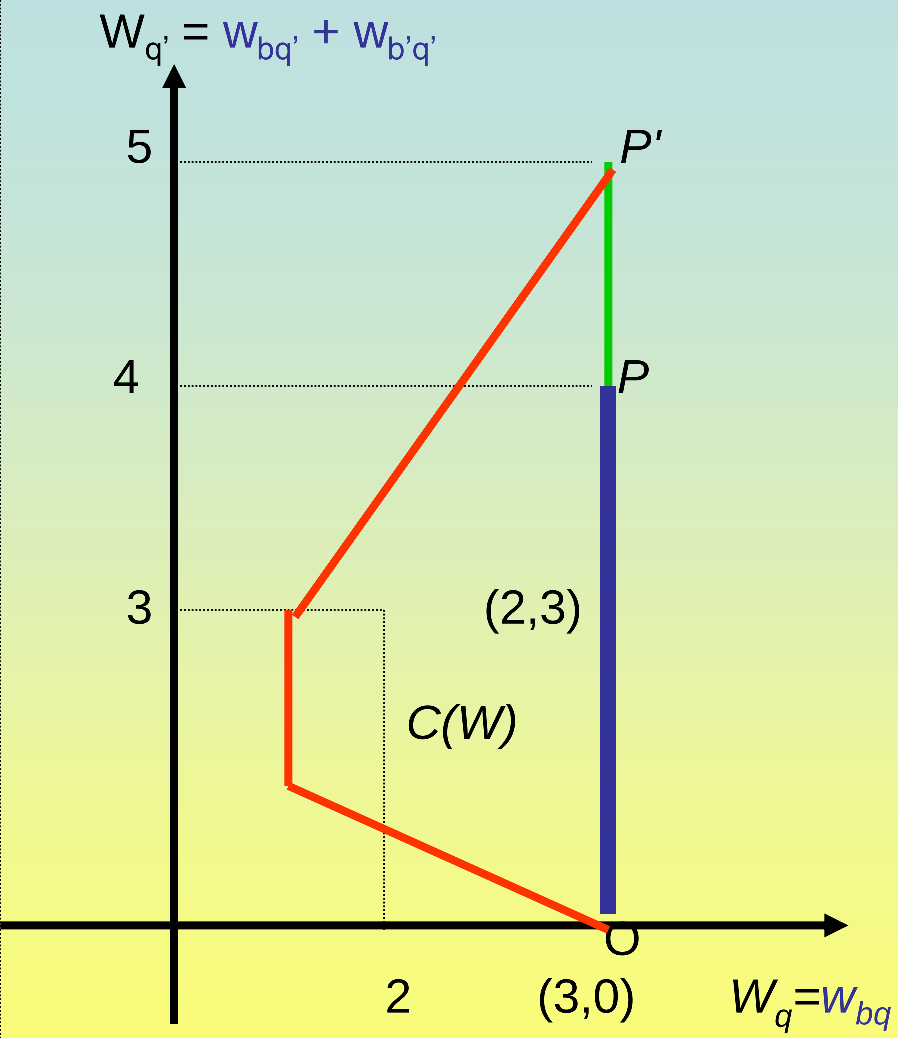
Theorem (Sotomayor, 2006) The function f maps the extreme points of the lattice of stable payoffs to the corresponding extreme points of the lattice of competitive equilibrium payoffs. In addition, the *B*-optimal stable payoff *is equal to the* *B*-optimal competitive equilibrium payoff.

f: **{stable payoffs} ⟶ {stable payoffs}**

B-optimal stable payoff ⟶ *B*-optimal competitive equilibrium payoff

Q-optimal stable payoff ⟶ *Q*-optimal competitive equilibrium payoff.

EXAMPLE.



The best core payoff for players in B is not the worst core payoff for players in Q.

Indeed, there is no minimum core payoff for the sellers.

The Q-optimal stable payoff is not a competitive equilibrium payoff: *P'=(3,5) corresponds to* $(w_q = 3; w_{bq'} = 2, w_{b'q'} = 3)$.

The set of sellers core payoffs is bigger than the set of sellers stable payoffs and is not a lattice.

Although the preferences of the players do not define the partial orders $\geq_B$ and $\geq_Q$, the maximal element under $\geq_B$ (respectively , $\geq_Q$) is the *B*-optimal (respectively *Q*-optimal) stable payoff.

P1. *A matching is compatible with a stable payoff if and only if it is optimal.*

P2. *… and (u',w') are stable payoffs then* ***(u,w)*** $\geq_B$ ***(u',w')*** *if and only if* ***(u',w')*** $\geq_Q$ ***(u,w).***

P3. *The set of stable payoffs is a non-empty convex and complete lattice under* $\geq_B$ ***and*** $\geq_Q$.

P4. There exist one and only one *B*-optimal stable payoff and one and only one *Q*-optimal stable payoff.

Let the game in coalitional function form (N,v) such that $N=B\cup Q$ and

(a) $v(\phi)=0$, (b) $v(S)=0$ if $S\subseteq B$ or $S\subseteq Q$, (c) $v(S)\leq v(T)$ if $S\subseteq T$.

Let r(b) be the smallest integer number such that, for all sets $S\subseteq Q$ with $|S|\geq r(b)$,

(d) $\mathbf{v(b\cup S)=\max\{ v(b\cup S');\ S'\subseteq S \text{ and } |S'|=r(b) \}}$.

(c) and (d) imply that for all sets $S\subseteq Q$ with $|S|\geq r(b)$,

$\mathbf{v(b\cup S)=v(b\cup S')}$ **for some** $\mathbf{S'\subseteq S}$ **with** $\mathbf{|S'|=r(b)}$

Analogously define s(q).

For every coalition $S=R\cup T$, $R\subseteq B$ and $T\subseteq Q$, define an **S-feasible assignment x** as a |R|x|T|-matrix of zeros and ones such that

$\sum_{q\in T} x_{bq}\leq r(b)$ and $\sum_{b\in R} x_{bq}\leq s(q)$.

The S-feasible assignment x is optimal for S if

$\sum_{(b,q)\in RxT} x_{bq} \geq \sum_{(b,q)\in RxT} x'_{bq}$ for every S-feasible assignment x.

For $S=R\cup T$,

(e) $\mathbf{v(R\cup T) = \sum_{(b,q)\in RxT} x_{bq}.v(b,q)}$, where x is an S-feasible assignment that is optimal for S.

Consequently, for all $S\subseteq Q$ with $|S|\leq r(b)$ and for all $S\subseteq B$ with $|S|\leq s(q)$,

$\mathbf{v(b\cup S) = \sum_{q\in S} v(b,q)}$ and $\mathbf{v(q\cup S) = \sum_{b\in S} v(b,q)}$.

t

The game (N,v) defines the many-to-many Assignment game $(B,Q,r,s,(v_{bq}))$, and vice-versa, where (v_{bq}) is the mxn-matrix of the numbers v(b,q)'s.

Example.

	q_1	q_2	q_3	$\{q_1,q_2\}$	$\{q_1,q_3\}$	$\{q_2,q_3\}$	$\{q_1,q_2,q_3\}$	
b_1	3	2	1	5	4	3	5	$r(b_1)=2$
b_2	2	1	2	3	4	3	5	$r(b_2)=3$
$\{b_1,b_2\}$	5	2	2	7	7	4	9	

$s(q_1)=2$ $s(q_2)=1$ $s(q_3)=1$

$v(b_1,b_2,q_1,q_2)=\max\{v(b_1,q_1,q_2)+v(b_2,q_1),\ v(b_1,q_1)+v(b_2,q_1,q_2),\ v(b_1,q_2)+v(b_2,q_1)\}=7$