EXCHANGE ECONOMY WITH MULTIPLE TYPES OF HETEROGENEOUS AND INDIVISIBLE GOODS: OPTIMAL COOPERATIVE AND OPTIMAL COMPETITIVE BEHAVIORS

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1. PRELIMINARIES

1.1. EXCHANGE ECONOMY WITH INDIVISIBLE GOODS

1. A finite set \( N = \{1, 2, \ldots, n\} \) (set of agents);
2. A positive integer \( m \) (number of goods);
3. For each \( j \in N \), a vector \( w_j \in \mathbb{Z}^m_+ \) (\( \mathbb{Z} \) = set of integer numbers) such that every component of \( \sum_{j \in N} w_j \) is positive;
4. For each agent \( j \in N \) a preference relation \( \succeq_j \) on the set of bundles of goods.

 Allocation: \( x = (x_1, \ldots, x_n), \ x_j \in \mathbb{Z}^m_+ \ \forall j \in N. \)

An allocation \( x \) determines a bundle of goods for each agent. It is feasible if the total amount allocated of each good is exactly that existent in the economy: \( \sum_{j \in N} x_j = \sum_{j \in N} w_j. \)
A vector of $\mathbb{R}^m_+$ is called feasible price.

**Definition 1.** The demand set of agent $j$ at prices $p$ is defined by

$$D_j(p) = \{ x \in \mathbb{Z}^m_+; \ p.x \leq p.w_j \text{ and } x \geq y, \ \forall y \in \mathbb{Z}^m_+ \text{ such that } p.y \leq p.w_j \}.$$

**Definition 2.** Let $p$ be a feasible price. The pair $(p,x)$ is a competitive equilibrium if $x$ is a feasible allocation of goods and $x_j \in D_j(p), \ \forall j \in N$.

If $(p,x)$ is a competitive equilibrium then the feasibility of $x$ implies that $\Sigma_{j \in N} x_j = \Sigma_{j \in N} w_j$ (total demand=total surplus). The vector of prices is called competitive equilibrium price and the allocation is called competitive equilibrium allocation.
We can think of an exchange economy as a market operating in an environment in which all agents can freely communicate and their main activity is to form groups (coalitions). If a coalition is formed, its members will want to make an agreement on the exchange of their initial endowments among them. The rules which govern such coalitional interactions require that no agent is compelled to accept a transaction. The allocations which could occur at some step of this negotiation process are the feasible allocations of the economy.
If a feasible allocation is proposed, every rational agent will have a cooperative optimal behavior, in the sense that he will only accept the bundle that the allocation offers to him if it is not possible to negotiate some preferred bundle with some group of other agents. This cooperative decision situation that takes place in such an environment constitutes a cooperative game.

**Definition 3 (Sotomayor 2015):** A feasible allocation will be a cooperative equilibrium if no coalition of players can profitably deviate from the given allocation, by acting according to the rules of the game.

**Definition 4:** A feasible allocation will be a core allocation if no coalition of players can profitably deviate from the given allocation, by interacting among them.
**PREDICTION**: The players reach the cooperative equilibrium when they negotiate at fixed prices.

This prediction is founded on some well known result of the Economic Theory which asserts that

**the competitive allocations are in the core,**

and by the belief that **the core characterizes the cooperative equilibria in an exchange economy.**

This result has, along the years, established **the connection between the competitive and the cooperative structures of an exchange economy.**

The problem that I intend to investigate along this work is if this relationship, between the competitive equilibrium allocations of an exchange economy and the cooperative equilibria of the game associated to it, is in fact so fundamental as it appears.
The reason why one has believed, along the years, that the core characterizes the cooperative equilibria in an exchange economy is that, traditionally, the cooperative game associated to such economy has been represented in the characteristic function form.
CHARACTERISTIC (OR COALITIONAL) FUNCTION FORM, WITHOUT TRANSFERABLE UTILITY: \((N, X, V, (\succeq_i)_{i \in N})\)

\(N\) = set of players;
\(X\) = set of feasible allocations of the economy;
\(V(S) = \{x \in X; \sum_{i \in S} x_i = \sum_{i \in S} w_i\}\)

\((V(S)\) is the set of feasible allocations that can be enforced by \(S\)).

\((\succeq_i)_{i \in N}\) = profile of preference relations of the players.

Therefore, this model serves as vehicle for an analysis of a cooperative decision situation, based on the domination relation.
Definition 5. The feasible allocation $x$ dominates the feasible allocation $y$ via coalition $S$ if

i) $x_i >_i y_i \ \forall i \in S$;

ii) $x \in V(S)$.

Definition 6. The feasible allocation $x$ weakly dominates the feasible allocation $y$ via coalition $S$ if

i) $x_i \geq_i y_i \ \forall i \in S$ and $x_i >_i y_i$ por algum $i \in S$;

ii) $x \in V(S)$.

The core of a game in the coalitional function form is then characterized as the set of feasible allocations which are not dominated by any feasible allocation via some coalition. The set of the feasible allocations which are not weakly dominated by any feasible allocation via some coalition is called strong core.
It is worth to remember that the purpose of a game form is to offer a mathematical model, which serves as vehicle for identifying the cooperative equilibria of the game associated to a given cooperative decision situation. For this purpose, this model must capture, for each feasible allocation $x$, the actions that the coalitions can take against $x$.

Thus, the characteristic function form will be adequate for representing an exchange economy only if $V(S)$ captures all actions that coalition $S$ can take against a given allocation. This requires that the only actions against a given allocation $x$, allowed to any coalition $S$ by the rules of the game associated to the exchange economy are:

(i) To break all its current agreements and
(ii) to interact only among its members, by forming an autonomous sub-economy.
Hence, in an exchange economy that can be fully represented by a game in the coalitional form, the cooperative equilibria are characterized by the allocations of the core. Thus, for these economies, the competitive allocations are also cooperative equilibrium allocations. Such an equivalence means that, given a competitive equilibrium price vector, the allocation yielded when the agents behave as price takers can also be yielded in a scenery without reference to the prices, in which the agents adopt a cooperative optimal behavior.
The present work shows that the representation of an exchange economy in the characteristic function form not always is the correct approach to be used as vehicle for a cooperative equilibrium analysis. We introduce an economy in which the members of a coalition can do more than to interacting only among themselves. In this economy there may exist allocations of the core that are not cooperative equilibria. This phenomenon was first pointed out in Sotomayor (1992), for the Multiple partners assignment game. In that economy, the competitive equilibrium allocations are always cooperative equilibria although the converse is not always true (Sotomayor 2007). The novelty we bring here is that the price taker behavior may not lead to an allocation produced by the optimal cooperative behavior: the competitive allocations are not, necessarily, cooperative equilibria.
DESCRIPTION OF THE MODEL

In the exchange economy we propose there are a set \( N=\{1,2,...,n\} \) of agents, \( k \) types of indivisible goods and \( n \) goods of each type.

For simplicity we will consider \( k=2 \) and we will think of the goods as being houses and cars. Then, there are a set \( H \) with \( n \) houses and a set \( C \) with \( n \) cars. We will denote the market of houses and cars by \( M(H,C) \).

Each agent \( j \) has one house \( h_j \) and one car \( c_j \) and may exchange the bundle \((h_j,c_j)\) of his initial endowments for another bundle. These markets operate together. An agent can negotiate with the same group of agents in both markets or with a different group in each market. However, the trades of an agent in the two markets are independent: if, for example, an agent negotiates the exchange of his house and his car with a group of agents and further breaks the agreement with respect to the exchange of his house, the agreements with respect to the exchange of his car can be maintained. Then a bundle of goods is a pair formed with one house and one car, independently traded in the markets \( M(H) \) and \( M(C) \).
Every agent \( j \) has strict preferences \( \text{PH}(j) \) e \( \text{PC}(j) \) on the set of houses (including his own house) and on the set of cars (including his own car), respectively, and also has strict preference \( \text{PHC}(j) \) on the set of pairs \((h_i,c_k) \in H\times C\). Our assumption on these preferences is the following:

\[
(h_i,c_k) >_j (h_i,c_m) \iff c_k >_j c_m; \quad (h_k,c_i) >_j (h_m,c_i) \iff h_k >_j h_m. \quad (*)
\]

A feasible allocation \( x = (x_H,x_C) \) is a permutation \( x_H \) of all houses and a permutation \( x_C \) of all cars.

**Feasible price** - \( p = (p_H,p_C) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \)

Demand set of \( j \) at prices \( p \):

\[
D_j(p) = \{(h_i,c_k); \ p_H(h_j) + p_C(c_j) \geq p_H(h_i) + p_C(c_k) \text{ and } (h_i,c_k) >_j (h_l,c_q) \text{ for all } (h_l,c_q) \text{ such that } p_H(h_j) + p_C(c_j) \geq p_H(h_l) + p_C(c_q)\}.
\]

\((p,x)\) is a competitive equilibrium if \( p \) and \( x \) are feasible and \( x(j) \in D_j(p) \) \( \forall j \in \mathbb{N} \).
The rules of $M(H,C)$ imply that, facing a feasible allocation $x$, a coalition $S$ can deviate from $x$ as follows: any player in $S$ may keep his agreements made in one of the markets with players who are not necessarily in $S$, while breaking their agreements made in the other market, and making new agreements in that market with players of $S$.

An allocation $x$ is **stable** if it is feasible and no coalition $S$ can benefit all its members by deviating from $x$, the way described above. Clearly, the **stable allocations** characterize the cooperative equilibria in this economy. A **strongly stable allocation** is defined analogously.
A special case of this market is obtained when $k=1$ and it is the well-known “Housing market” introduced by Shapley and Scarf in 1974. The main results for this model are the following:

1. The core (the set of stable allocations) is always non-empty. (Shapley and Scarf, 1974; Alg. TTC of David Gale, 1974; Sotomayor, 2005)

2. The set of competitive equilibria is always non-empty. (Alg. TTC).

3. When the preferences are strict, the strong core (the set of strongly stable allocations) is always non-empty and has only one allocation. (Roth and Postlewaite, 1977).

4. This point in the strong core is the only competitive allocation in the economy. (Roth and Postlewaite, 1977).
Hence, when preferences are strict in the Housing market, the agents price taker behavior, facing a competitive price, always leads to an allocation in which all agents are taking their optimal cooperative behavior.
TTC ALGORITHM OF D. GALE

R(1): 3,4,2,1,5,6  
R(4): 2,3,5,1,4,6  
R(2): 1,3,2,4,6,5  
R(5): 1,2,6,4,3,5  
R(3): 2,1,3,4,5,6  
R(6): 3,1,5,2,4,6

\[
x^*_H : \begin{array}{ccccccc} 
  h_3 & h_1 & h_2 & h_6 & h_5 & h_4 \\
  1 & 2 & 3 & 5 & 6 & 4
\end{array}
\]

\[p(h_1)=p(h_2)=p(h_3)=4; \quad p(h_5)=p(h_6)=3; \quad p(h_4)=2\]

\((p,x^*_H)\) is a competitive equilibrium price.
Proof that the TTC algorithm yields a core allocation. Consider some coalition $S$. We will show that, if the agents in $S$ exchange their houses only among them, at least one of these agents will not be better off.

Suppose, by way of contradiction, that all agents in $S$ can be better off by exchanging their houses only among them. Take the first cycle formed by the TTC algorithm that contains some agent in $S$, say agent $j$. Then, at the step in which this cycle was formed, the houses of everyone in $S$ were in the market. Now observe that at the allocation produced by the algorithm, agent $j$ is receiving his favorite house among all houses that were in the market at the step in which his cycle was formed. Therefore, agent $j$ prefers the house obtained with the algorithm to the house of any agent in $S$, so $S$ cannot block the allocation produced by the TTC algorithm. ■
Example 1. Consider the market $M(H,C)$ where the set of agents is $N=\{1,2\}$.

The agents preferences on the houses are given by:

$P_{H}(1)$: $h_1, h_2$  
$P_{H}(2)$: $h_1, h_2$

The agents preferences on the cars are given by:

$P_{C}(1)$: $c_2, c_1$  
$P_{C}(2)$: $c_1, c_2$

The agents preferences on the pairs $(h_j,c_k)$ are given by:

$P_{HC}(1)$: $(h_1, c_2)$, $(h_2, c_2)$, $(h_1, c_1)$, $(h_2, c_1)$

$P_{HC}(2)$: $(h_1, c_1)$, $(h_1, c_2)$, $(h_2, c_1)$, $(h_2, c_2)$

$\mathbf{x}$ is not a cooperative equilibrium but it is in the core and in the strong core.

However, $x_C$ is in the core of $M(C)$ but $x_H$ is not in the core of $M(H)$. 
Notice that, if the rules of the market of Example 1 were rigid, in the sense that the trades of an agent in one of the markets were not independent of his trades in the other market, the members of a deviating coalition should break all their current agreements in both markets and make all their agreements only among them.

In this case, the stable allocations would be precisely the core allocations. Then, allocation $x$ would be stable, since if player 1 broke his current agreement with player 2 in $M(H)$, then he would have to break the whole agreement, which would not be interesting for him.
Both markets, the one with rigid agreements and the one with independent agreements, have the same set of feasible allocations and the same function $V$, since this function does not inform the nature of the trades. Then, they are indistinguishable under their representation in the characteristic function form.

Therefore it is not possible to identifying if allocation $x$ is or is not a cooperative equilibrium if the market is described in the characteristic function form.
Our economy can be fully represented in the “deviation function form”, introduced in Sotomayor (2015). Instead of the sets \( V(S) \), this model provides, for each feasible allocation \( x \) and each coalition \( S \), the set \( \phi_x(S) \) of all feasible allocations obtained when the players of \( S \) act against \( x \) without violating the rules of the game. The allocations in \( \phi_x(S) \) are called feasible deviations from \( x \) via \( S \). Clearly, \( V(S) \subseteq \phi_x(S) \), for all \( x \in X \).

Thus, unlike function \( V \), functions \( \phi_x \) capture the nature of the agreements, flexible or rigid, that could be made by agents 1 and 2 in the situation described in Example 1. (\( x^* \) is a feasible deviation from \( x \) via \( S=\{1\} \) when the agreements are flexible and it is not so when the agreements are rigid)
By using the deviation function form we can say that:

an allocation $x$ is **stable** if it is feasible and there is no coalition $S$ and no allocation $y \in \phi_x(S)$ such that every player in $S$ prefers $y$ to $x$. 
Example 1 (contin.). Consider the market $M(H,C)$ where the set of agents is $N=\{1,2\}$.

The agents preferences on the pairs $(h_j,c_k)$ are given by:

$P_{HC}(1): (h_1, c_2), (h_2, c_2), (h_1, c_1), (h_2, c_1)$

$P_{HC}(2): (h_1, c_1), (h_1, c_2), (h_2, c_1), (h_2, c_2)$

Consider the price vector $p$ such that $p(h_1)=2$, $p(h_2)=1$, $p(c_1)=1$ e $p(c_2)=2$. The price vector $p$ is not competitive in $M(C)$; both agents demand $c_1$.

We have that $p(h_1c_1)=p(h_2c_2)=3$.

Hence, the set of bundles that agent 1 can pay= the set of bundles that agent 2 can pay =${(h_1 c_1), (h_2 c_2), (h_2 c_1)}$

$D_1(p)=\{(h_2,c_2)\}$ e $D_2(p)=\{(h_1,c_1)\}$.

Then $(p,x)$ is a competitive equilibrium.

The price taker behavior leads to an allocation in which the cooperative behavior of player 1 is not optimal: $x$ is a competitive allocation but it is not a cooperative equilibrium.
In this example the nature of the agreements is not a relevant detail of the rules of the game if we want to identify the core allocations: the agents need not know the nature of the agreements in order to object an allocation “by threatening to form an autonomous sub-economy”. Also, the price taker behavior of an agent is not affected by the nature of the agreements in the cooperative maket. However, the cooperative optimal behavior of the agents is closely related to the coalitional interactions that they are allowed to enter by the rules of the game. The independence of the agreements allow coalitional interactions that are not possible when the agreements are rigid. Thus, the knowledge of the nature of the agreements is crucial for a correct cooperative equilibrium analysis.
x ∈ strong core ⊆ core, x is competitive and x is not stable.

x* ∈ \{ competitive allocations \} ⊆ strong core (Teorema 4).

{strongly stable allocations} = \{ x* \} or \ø (Teorema 5).

x* is stable, strongly stable and competitive.
Consider the market $M(H,C)$ where the set of agents is $N = \{1, 2, 3, 4\}$. The agents preferences on houses are given by:

PH(1): $h_3, h_2, h_1, \ldots$

PH(2): $h_1, h_4, h_3, \ldots$

PH(3): $h_4, h_1, h_2, \ldots$

PH(4): $h_3, h_2, h_4, \ldots$

The agents preferences on cars are given by:

PC(1): $c_4, c_3, \ldots$

PC(2): $c_1, c_2, \ldots$

PC(3): $c_4, c_3, \ldots$

PC(4): $c_1, c_2, \ldots$

The agents preferences on pairs $(h_j, c_k)$ are given by:

PHC(1): $(h_3, c_4), (h_3, c_2), (h_2, c_4), (h_2, c_3), \ldots$

PHC(2): $(h_1, c_4), (h_4, c_4), (h_1, c_3), (h_4, c_3), \ldots$

PHC(3): $(h_4, c_1), (h_1, c_1), (h_4, c_2), (h_1, c_2), \ldots$

PHC(4): $(h_3, c_1), (h_2, c_1), (h_3, c_2), (h_2, c_2), \ldots$

$x^* = (x^*_H, x^*_C)$:

\[
\begin{array}{cccc}
  h_2 & c_4 & h_1 & c_3 & h_4 & c_2 & h_3 & c_1 \\
  1 & 2 & 3 & 4 & \\
\end{array}
\]

$x^*$ is not in the core of $M(H,C)$. y: $(h_3, c_3) (h_1, c_1)$ dominates $x^*$ via $S = \{1, 3\}$.

Then, $x^*$ is not stable, so it is not strongly stable.

$p^* = ((p^*_H, p^*_C), x^*)$ is not a competitive equilibrium for $M(H,C)$. In fact, agent 1 prefers $(h_3, c_3)$ to $x^*(1) = (h_2, c_4)$ and he can pay the price of $(h_3, c_3)$. By Theorem 4, every competitive allocation is in the strong core and $x^*$ is not in the strong core. Hence, $x^*$ is not a competitive allocation.

By Theorem 5, if $x \neq x^*$ then $x$ is not strongly stable. Then, the set of strongly stable allocations is empty.
Unlike the market of Shapley and Scarf the set of strongly stable allocations may be empty.
MAIN RESULTS FOR M(H,C)

THEOREM 1. The set of the stable allocations (respect. strongly stable allocations) is a subset of the core (respect. strong core) and may be a proper subset of the core (respect. strong core).

THEOREM 2: The set of strongly stable allocations for M(H,C) may be empty.

THEOREM 3: A competitive equilibrium allocation is not necessarily a cooperative equilibrium.

THEOREM 4. Every competitive equilibrium allocation for M(H,C) is a strong core allocation of M(H,C).

THEOREM 5: Suppose the set of strongly stable allocations is non-empty. Then this set is a singleton and $x^*$ is the only strongly stable allocation.
Theorem 3 implies that the prediction that the price taker behavior of the agents is always associated to an optimal cooperative behavior is not correct.

Overall, the results of this paper suggest that, while competitive equilibria may not be cooperative equilibria, the set inclusion relation between the set of competitive equilibrium allocations and the core, which persists in the economy treated here, may be even more fundamental than previous results have suggested. We refined this result by showing that the competitive equilibrium allocations are in the strong core.
We also analyze the special case in which each agent has the same preference in all $k$ markets. In this case it is not surprising that all results for the Shapley and Scarf market are preserved. However, the proofs for those results are not so straightforward. The equivalence between the strong core, the set of strongly stable allocations and the set of competitive equilibrium allocations follows from the fact that these sets are non-empty, the last two sets are contained in the strong core, which is a singleton. This last result uses a key theorem that asserts that every strong core allocation has the *equal treatment property*. According to this property, at every strong core allocation, everyone gets the bundle of initial endowments of some agent.

\[
\begin{align*}
  h_2c_2 \quad h_1c_1 \quad h_3c_3 \\
  h_t > j \quad h_K \leftrightarrow c_t > j c_k \\
  x^*: \quad 1 \quad 2 \quad 3
\end{align*}
\]

is the only strong core allocation.
Example 3. Let $E = M(H)$ be the market of houses where $N = (1,2,3)$ and the preferences are given by:

$P(1) = h_2, h_1, h_3 \quad P(2) = h_3, h_1, h_2 \quad P(3) = h_2, h_1, h_3$

Consider the allocation $x$:

$$
\begin{array}{ccc}
\hline
& h_2 & h_3 & h_1 \\
\hline
1 & 2 & 3 \\
\hline
\end{array}
$$

$x$ is in the core of $E$ but it is not in the strong core ($S = \{2,3\}$ weakly blocks $x$). Hence $x$ is not competitive. ($p(h_2) \leq p(h_1)$ because $x(1) = h_2; p(h_2) > p(h_3)$, so $p(h_3) < p(h_1)$) For each positive integer $k$ let $kE$ be the economy derived from $E$ by making $k$ copies of each agent in $E$. There are $k$ houses identical to $h_j$, for all $j \in N$. It can easily be verified that the allocation $kx$, in which each copy of $j$ receives the house $x(j)$, $\forall j = 1,2,3$, is in the core of $kE$, for all positive integer $k$. Nevertheless, $kx$ is not competitive in $kE$, for every $k$ ...
The point is that the importance of the theory of the core to the exchange economies lies on the fact that non-core allocations cannot be yielded in a competitive environment where the agents behave as price takers and they cannot be yielded in a cooperative environment where the agents behave optimally. In other words, the allocation yielded in a competitive environment where the agents behave as price takers cannot be objected in a cooperative environment by any coalition. Such objection consists in threatening to form an autonomous sub-economy that benefits all members of the coalition. However there are other ways of objecting an allocation that is yielded when the agents negotiate at fixed prices, and our work shows that.
A principal contribuição do presente trabalho é mostrar que a crença de que as economias de troca podem ser completamente representadas na forma de função característica para fins de servir de veículo para uma análise de equilíbrio cooperativo do jogo, nem sempre é verdadeira. Em algumas economias, os membros de uma coalizão podem fazer mais do que meramente interagir entre si.

De um modo geral, tais ações têm a ver com o nível de flexibilidade da natureza das transações realizadas entre os agentes.
Nos modelos tradicionais, a representação da economia na forma de função característica pressupõe que os acordos entre os agentes são rígidos, no sentido de que se um agente negocia mais de um bem com um grupo de agentes e posteriormente quebra o acordo referente a um desses bens, os acordos referentes aos demais bens negociados com esse grupo também deverão ser quebrados. Isto é, para que uma coalizão se desvie de uma alocação, os seus membros devem quebrar todos os acordos correntes e fazer novos acordos somente entre si.

Nessas economias é natural basear a análise cooperativa do jogo na relação de dominação. Então, as alocações do núcleo são precisamente as alocações de equilíbrio cooperativo.
Exemplo 1. \( m=2, \, N=\{1,2,3,4\}, \, w_1=(2, \, 1), \, w_2=(1, \, 2), \, w_3=(0, \, 3), \, w_4=(3, \, 0) \).

Os acordos entre os agentes são rígidos.

Considere a alocação \( x \) onde \( x_1=(1, \, 2), \, x_2=(2, \, 2), \, x_3=(2, \, 1), \, x_4=(1, \, 1) \).

Nesta alocação, são formadas as coalizões \( \{1, \, 2\} \) e \( \{3, \, 4\} \) no mercado do bem 1 e \( \{1, \, 3, \, 4\} \) e \( \{2\} \) no mercado do bem 2.
Consider \( M(H, C) \) where \( N = \{1, 2, 3, 4\} \).

\[ \begin{array}{c}
\text{PH(1)}: \quad (h_2, c_4), & (h_3, c_2), & (h_4, c_3), & (h_1, c_4), & (h_1, c_2) \\
\text{PH(2)}: \quad (h_3, c_1), & (h_3, c_2), & (h_4, c_4), & (h_1, c_3) \\
\text{PH(3)}: \quad (h_4, c_4), & (h_4, c_1), & (h_4, c_3), & (h_4, c_2), & (h_2, c_4), & (h_2, c_1), & (h_2, c_3), & (h_2, c_2) \\
\text{PH(4)}: \quad (h_1, c_4), & (h_3, c_1), & (h_3, c_3), & (h_3, c_2) \\
\end{array} \]

\[ \begin{array}{c}
\text{PC(1)}: \quad (c_4), & (c_2), & (c_4), & (c_2) \\
\text{PC(2)}: \quad (c_1), & (c_2), & (c_3), & (c_4), & (c_1), & (c_2), & (c_3) \\
\text{PC(3)}: \quad (c_4), & (c_1), & (c_3), & (c_2), & (c_4), & (c_1), & (c_3), & (c_2) \\
\end{array} \]

PHC(1): \( (h_2, c_4), \ (h_3, c_2), \ (h_1, c_4), \ (h_1, c_2), \ ... \)

PHC(2): \( (h_3, c_1), \ (h_3, c_2), \ (h_3, c_3), \ (h_2, c_1), \ (h_2, c_2), \ (h_2, c_3), \ ... \)

PHC(3): \( (h_4, c_4), \ (h_4, c_1), \ (h_4, c_3), \ (h_4, c_2), \ (h_2, c_4), \ (h_2, c_1), \ (h_2, c_3), \ ... \)

PHC(4): \( (h_1, c_1), \ (h_1, c_3), \ ... \)

\( x^*: \quad (h_2, c_4), \ (h_3, c_2), \ (h_4, c_3), \ (h_1, c_1) \)

\( x^* \) is strongly stable (\( \rightarrow \) it is in the strong core). Consider the allocation

\[ \begin{array}{c}
\text{x:} \quad (h_2, c_2), & (h_3, c_1), & (h_4, c_4), & (h_1, c_3) \\
\end{array} \]

We have that \( x_H = x^*_H \), but \( x_C \neq x^*_C \), so \( x_C \) is not in the strong core of \( M(C) \). Nevertheless \( x \) is in the core of \( M(H,C) \). In fact, 2 e 3 have their first choice. Any profitable deviation for 4 must give to him \( h_1, c_1 \), which involves agent 1. However, any profitable deviation for 1 involves 2, which cannot be part of any blocking coalition. Indeed \( x \) is in the strong core. Therefore, the strong core is not a singleton.
We also have that $x$ is unstable. In fact, if 1 and 4 keep their houses given at $x$ and exchange their cars among them, all of them are better than at $x$.

Hence, $x$ is not strongly stable but it is in the strong core.

$x$: (h$_2$, c$_2$) (h$_3$, c$_1$) (h$_4$, c$_4$) (h$_1$, c$_3$) 

   1    2    3    4

$y$: (h$_2$, c$_4$) (h$_1$, c$_1$)

   1    4

**PHC(1):** (h$_2$, c$_4$), (h$_2$, c$_2$), (h$_1$, c$_4$), (h$_1$, c$_2$), ...

**PHC(4):** (h$_1$, c$_1$), (h$_1$, c$_3$), ...

Now consider $p_H$=(1,4,1,2) and $p_C$=(4,1,3,2). Every agent can pay the bundle that is allocated to him at $x$. Also, if any agent prefers some bundle to that one which is allocated to him at $x$, then he cannot pay the given bundle. Then, $(p,x)$ is a competitive equilibrium for $M(H,C)$, although $x$ is not a cooperative equilibrium allocation; $x_H$ is a competitive allocation ($x_H=x^*_H$), but $p_H$ is not a competitive price in $M(H)$, since agent 1 cannot pay the price of house $h_2$. 
This example also illustrates that the set of competitive equilibrium allocations is not necessarily a singleton. In fact, \((p^*,x^*)\), where \(p^*_H=(3, 3, 3, 3)\) and \(p^*_C=(3, 1, 1, 3)\), is a competitive equilibrium for \(M(H,C)\), so \(x^*\) is a competitive equilibrium allocation. On the other hand, \(p^*\) is not compatible with \(x\), which shows that an equilibrium price is not necessarily compatible with any competitive equilibrium allocation. ■
Na economia de troca que apresentamos aqui, as regras do mercado permitem que os acordos sejam **flexíveis** no seguinte sentido. **Os acordos firmados com relação a quaisquer dois bens são independentes.** Assim, uma alocação factível deve ser individualmente racional em cada mercado separadamente. Essa independência permite que membros de uma coalizão desviante mantenham alguns de seus acordos correntes com jogadores fora da coalizão. Ocorre que tal flexibilidade não é capturada pela função \( V \). No entanto ela afeta a estrutura cooperativa da economia e a relação entre esta estrutura e a sua estrutura competitiva. Um outro modelo de jogo para representar o mercado faz-se então necessário. O mais abrangente é a **forma de função desvio**, introduzida em Sotomayor (2015), que é adequada para representar jogos em que **uma alocação factível é descrita por uma estrutura de coalizões junto com o conjunto dos acordos alcançados dentro de cada coalizão.**
Exemplo 2. \( m=2, N=\{1,2,3,4\}, \ w_1=(2, 1), \ w_2=(1, 2), \ w_3=(0, 3), \ w_4=(3, 0) \). As negociações em cada mercado são independentes.

Considere a alocação \( x \) onde \( x_1=(1, 2), \ x_2=(2, 2), \ x_3=(2, 1), \ x_4=(1, 1) \).

Nesta alocação, são formadas as coalizões \( \{1, 2\} \) e \( \{3, 4\} \) no mercado do bem 1 e \( \{1, 3, 4\} \) e \( \{2\} \) no mercado do bem 2.

A alocação \( x \) não é um equilíbrio cooperativo, desde que a coalizão \( S=\{1, 4\} \) pode se desviar lucrativamente de \( x \) agindo de acordo com as regras do jogo. De fato, os jogadores 1 e 4 poderiam quebrar seus acordos com os jogadores 2 e 3, respectivamente, realizadas no mercado do bem 1, trocar entre si as suas unidades do primeiro bem e, como as negociações em cada mercado são independentes, poderiam manter as transações com o jogador 3 para o segundo bem. A alocação \( y \) poderia então ser obtida, onde \( y_1=(2.5, 2), \ y_2=(1, 2), \ y_3=(0, 1), \ y_4=(2.5, 1) \). A alocação \( y \) é claramente preferida a \( x \) por 1 e 4 (relação de preferências não decrescente).
Exemplo 2. \( m=2, N=\{1,2,3,4\}, \ w_1=(2, 1), \ w_2=(1, 2), \ w_3=(0, 3), \ w_4=(3, 0) \). As negociações em cada mercado são independentes.

Considere a alocação \( x \) onde \( x_1=(1, 2), \ x_2=(1, 2), \ x_3=(3, 1), \ x_4=(1, 1) \).

Nesta alocação, são formadas as coalizões \( \{1, 3, 4\} \) e \( \{2\} \) no mercado do bem 1 e \( \{1, 3, 4\} \) e \( \{2\} \) no mercado do bem 2.

A alocação \( x \) **não é um equilíbrio cooperativo**, desde que a coalizão \( S=\{1, 4\} \) pode se desviar lucrativamente de \( x \) agindo de acordo com as regras do jogo. De fato, os jogadores 1 e 4 poderiam quebrar seus acordos com o jogador 3, realizadas no mercado do bem 1, trocar entre si as suas unidades do primeiro bem e, como as negociações em cada mercado são independentes, poderiam manter as transações com o jogador 3 para o segundo bem. A alocação \( y \) poderia então ser obtida, onde \( y_1=(2.5, 2), \ y_2=(1, 2), \ y_3=(0, 1), \ y_4=(2.5, 1) \). A alocação \( y \) é claramente preferida a \( x \) por 1 e 4 (relação de preferências não decrescente).
Proof that the TTC algorithm yields a core allocation. Consider some coalition $S$. We will show that, if the agents in $S$ exchange their houses only among them, at least one of these agents will not be better off.

Take the first cycle formed by the TTC algorithm that contains some agent in $S$, say agent $j$. Then, at the step in which this cycle was formed, the houses of everyone in $S$ were in the market. Now observe that at the allocation produced by the algorithm, agent $j$ is receiving his favorite house among all houses that were in the market at the step in which his cycle was formed. Therefore, agent $j$ cannot be better off by receiving the house of some agent in $S$, so $S$ cannot block the allocation produced by the TTC algorithm. ■