

1 INTRODUCTION

Local mesh refinement is needed for the numerical solution of problems which has stress concentration to capture any sharp gradients in the solution without having to use a very dense mesh throughout the domain.

There are various options to do refinement of mesh in finite element method, for example in h- refinement you increase the no of elements by subdividing them, where in p- refinement you choose higher order polynomial for an element. On the other side finite element analysis can be done using background mesh where the geometry is immersed in the mesh. As this mesh does not there to the geometry, it is easy to generate background mesh with uniform elements. While both FEM and meshless methods use nodes on the boundary to represent the geometry of the analysis, if a background mesh is used for the analysis, the mesh itself cannot be used to reasonably approximate the geometry. Therefore, a background mesh-based approaches represent geometry independent of the mesh by using equations of curves/surfaces to represent the boundary. Structured mesh has uniform, regular shaped or undistorted elements therefore they are easy to generate automatically compared to conforming mesh.

However, if higher resolution is needed in some areas to capture localized details of the solution like stress concentrations some local refinement scheme is needed. A technique used here for local refinement for structured background mesh is best suited and illustrated using the immersed boundary finite element method (IBFEM) where boundary conditions are applied using step boundary method. A key challenge is to maintain compatibility and continuity between adjacent elements that may be of different size, polynomial degree or approximation type. In this approach, the geometry is defined independently and not approximated by the mesh. The elements on the boundary are subdivided into triangles or tetrahedrons for accurately integrating over the region that represents the intersection of the solid model and the element.

Section 2 provides a brief summary of the implicit boundary method for applying boundary conditions. The solution structure for grid refinement and the corresponding modified weak form are described in section 3 and 4. Numerical implementation details are provided in section 5. The refinement technique is validated by application to 2D and 3D static problems in section 6 followed by conclusions and discussion in section 7.

2 IMPLICIT BOUNDARY FINITE ELEMENT METHOD

Implicit boundary method is one way to apply essential boundary conditions when there are no nodes on the boundary. It uses approximate step functions of the boundaries $\phi(x) = 0$ to construct solution structures for the trial and test functions such that the essential boundary conditions are guaranteed to be satisfied at these boundaries. Let $u_i(x)$ be a trial function defined over $\Omega \subset R^2$ or R^3 that must satisfy the boundary condition $u_i(x) = a_i(x)$ along Γ_a which is a subset of the boundary of Ω , then for the i th component of the displacement field $u_i(x)$ the trial function can be defined as

$$u_i(x) = \phi(x)u_i^h(x) + a_i(x) \quad (1)$$

Note that this weighting function, referred to here as the Dirichlet function or H-function, is not zero along the entire boundary of the domain. Instead it is assumed to be zero only at the boundaries with essential boundary conditions. In this equation, u_i^h is constructed by piecewise interpolation over the background mesh in which the boundary is immersed. Even though boundary conditions are satisfied for this trial solution regardless of the nature of approximation, basic for has several practical difficulties. Firstly, an appropriate function should be created for the boundary of interest, which may not be readily available. In addition, boundary value function $a_i(x)$ just be defined such that prescribed value is defined at the boundary Γ_a .

To avoid these difficulties, in the implicit boundary method approximate step functions are used to construct the implicit equation of the boundary [1]-[2]. It provides convergence rates comparable to traditional FEM and has been shown to be effective for a variety of element types including elements with B-spline shape functions [2]-[4]. Here we present the Step Boundary Method (SBM), which is a generalization of the implicit boundary method where approximate step functions are used to construct the trial solution within any element that contains a boundary with a specified boundary condition as follows.

$$u_i(x) = H_i u_i^h(x) + (1 - H_i) a_i \quad (2)$$

If the H-functions are constructed as step functions, then they can be defined locally within elements containing such boundaries. The solution structure for the trial function defined in Eq. (2) is guaranteed to satisfy the condition $u_i(x) = a_i(x)$

along these boundaries for any $u_i(x) : \Omega \rightarrow R$. The boundary value function, $a_i(x)$, must be defined such that at the boundaries Γ_a it has the prescribed values. If no boundary conditions are applied on the i^{th} component of displacement for a boundary within the element, then we set $H_i = 1$ at that boundary so that $u_i(x) = u_i^h(x)$ which is a piece-wise interpolation or approximation. Otherwise, H_i is an approximate step functions defined using signed distance functions $\phi(x)$, which is the distance from the boundary and is assumed to be positive inside the geometry. Since the distance $\phi(x) = 0$ at the boundary it serves as an implicit equation of the boundary, which can be used to define the approximate step functions as

$$H_i = \begin{cases} 1 & \phi \geq \delta \\ \frac{\phi}{\delta} \left(2 - \frac{\phi}{\delta} \right) & 0 \leq \phi \leq \delta \\ 0 & \phi < 0 \end{cases} \quad (3)$$

Similar functions have been used by Hollig [25] for constructing solution structure for B-spline finite element method where large values of δ is used so that this weighting function transitions from zero to one over a wide band along the boundary. In the step boundary method, the limit of $\delta \rightarrow 0$ is used so that the H-function approximates the Heaviside step function which has a unit value inside the domain where $\phi > 0$ and is zero at the boundary and outside ($\phi \leq 0$). Approximate step functions have been used for level set method [30] to represent moving boundaries. The use of an approximate step functions as the H-functions in implicit boundary method ensures that only the boundary elements are affected by this function and all internal elements are similar to the traditional finite elements. Furthermore, this allows the H-function to be defined locally within the boundary elements which contains boundaries with specified boundary conditions. All the internal elements have identical shape and size if the grid is uniform and therefore all these elements will have the same stiffness matrix. The variable part of the solution structure, $u_i(x)$, can be defined using finite element like piece-wise interpolation or using smoother B-spline or other approximations. Using finite element shape functions yields a piece-wise interpolation which is C^0 continuous, whereas B-spline basis can provide a high degree of continuity depending on the degree of the polynomials used (eg. linear, quadratic, etc). Uniform B-spline elements applicable to structured grids have been studied [27] to demonstrate that the implicit boundary method can be used to apply boundary conditions even when the approximation used does not have Kronecker's delta properties. In other words, even though the nodal values are not equal to the B-spline approximation values at the nodes, the solution structure in Eq. (1) guarantees that the essential boundary conditions are imposed.

3 SOLUTION STRUCTURE FOR GRID REFINEMENT

In order to refine the grid at required locations, a solution is constructed using approximate step functions such that it blends the solutions from the coarse grid and refined grid while maintaining the continuity of the solution. The overall solution structure must also satisfy the boundary conditions, therefore it is written as

$$u = Hu^g + u^a = u^s + u^a \quad (4)$$

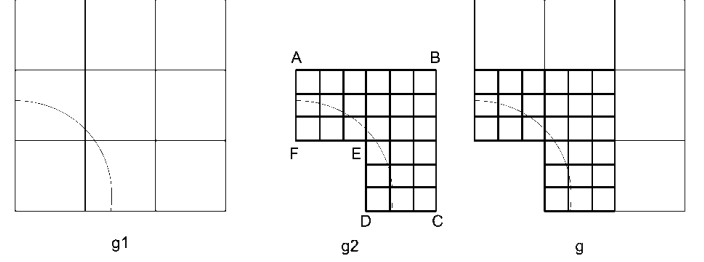


Figure 1 Grid Refinement

In this solution structure $u, u^s, u^a : \Omega \rightarrow R^m$ and m is the dimension of the problem. u is the displacement vector, u^s is a grid variable which is defined by piece-wise interpolation or approximation over multiple grids as explained later and $u^s = Hu^g$ is the variable part of the solution. H is a diagonal matrix whose components are $H_{ii} = H(\phi_i)$ where H is the Dirichlet function defined in Eq. (3) and $\phi_i(x) = 0$ is the implicit equation of the boundary on which the i^{th} component of displacement is specified. The boundary value function u^a is defined such that its value at the boundary is equal to the specified boundary condition. The grid variable u^g is the unknown to be computed and when the grid is refined its value is determined by the nodal values of multiple grids. Figure 1 shows the coarse grid represented by g1 and the locally refined grid is represented by g2. If u^{g1} and u^{g2} are parts of the solutions represented over the grids g1 and g2, then the grid variable will be defined as blend between these two parts of the solution as follows:

$$u^g = (1 - H)u^{g1} + Hu^{g2} \quad (5)$$

In the above expression, H is the step function for refinement constructed such that it vanishes outside the refined grid and transitions from zero to unity within a thin band inside the boundary of the refinement region. Even though the solution structure for refinement in Eq. (4) only considers one level of refinement, the structure can be applied repeatedly to further refine regions that were previously refined to achieve many levels of hierarchical refinement.

4 MODIFIED WEAK FORM

The weak form of the elasto-static boundary value problem is the principle of virtual work which can be written in the following form:

$$\int_{\Omega} \delta \varepsilon^T \sigma d\Omega = \int_{\Gamma_t} \delta u^T t d\Gamma + \int_{\Omega} \delta u^T b d\Omega \quad (6)$$

Using the solution structure in Eq. (4) the stresses and strains can also be decomposed into a homogeneous part and a boundary value part.

$$\sigma = C\varepsilon = C(\varepsilon^s + \varepsilon^a) = \sigma^s + \sigma^a \quad (7)$$

$$\text{where, } \varepsilon_{ij}^s = \frac{1}{2} \left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i} \right), \quad u_i^s = H_i u_i^g \quad \text{and}$$

$$\varepsilon_{ij}^a = \frac{1}{2} \left(\frac{\partial u_i^a}{\partial x_j} + \frac{\partial u_j^a}{\partial x_i} \right).$$

Substituting the preceding equations into the weak form, the following relation is obtained, where the known quantities have been moved to the right-hand side of the equation.

$$\int_{\Omega} \delta \varepsilon^T C \varepsilon^s d\Omega = - \int_{\Omega} \delta \varepsilon^T \sigma^a d\Omega + \int_{\Gamma_t} \delta u^T t d\Gamma + \int_{\Omega} \delta u^T b d\Omega \quad (8)$$

The boundary value function $\mathbf{u}^a$ must be constructed such that at the boundary where Dirichlet boundary conditions are applied it must have value equal to the specified boundary condition. When Dirichlet boundary conditions are specified at multiple boundaries and the assigned values are not the same for all boundaries, the boundary value function must take on appropriate values at respective boundaries and transition smoothly in between. To accomplish this, the boundary value function is constructed by piecewise interpolation or approximation using the shape functions N_i of the elements through which the boundary is passing. A displacement component u^a is defined within an element as $u^a = \sum_i N_i u_i^a$,

where u_i^a are the nodal values of the displacement component. The test function, $\delta \mathbf{u}$ (or the virtual displacement), must be constructed such that it vanishes on boundaries where Dirichlet boundary conditions are specified and it should blend the contributions from the coarse and refined grids in a manner similar to the trial function. The solution structure for the test function $\delta \mathbf{u}$ and the homogenous part $\mathbf{u}^s$ of the trial function can be written as:

$$\delta u = (1-H)\delta u^{g1} + H\delta u^{g2} \quad (9)$$

$$u^s = (1-H)u^{g1} + Hu^{g2} \quad (10)$$

The refinement step function $H(\phi)$ are both constructed such that they transition from 0 at the boundary $\phi(\mathbf{x})=0$ to 1 at $\phi(\mathbf{x})=\delta$. The width of the band over which the step functions transition is controlled by the range parameter δ . If this band is very broad, then it becomes necessary to keep track of the elements over which the parameter δ is varying which would make the stiffness computation and implementation very cumbersome. In the limit when $\delta \rightarrow 0$ $H(\phi)$ become approximate step functions and it can be assumed that these functions are constants for elements that do not contain the boundary $\phi(\mathbf{x})=0$.

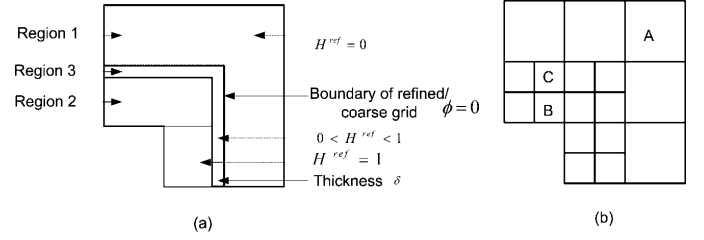


Figure 2 Refinement Regions

Figure 2 shows the regions in a typical grid. Region 1 is the region with coarse (or unrefined) elements, Region 2 is the refined elements and Region 3 is the transition region where the step function for refinement varies from 0 to 1. In the limit $\delta \rightarrow 0$, $H = 0$ in region 1 and $H = 1$ in region 2 while region 3 reduces to a very narrow band along the boundary between regions 1 and 2. In the next section, integration method is described for computing the stiffness matrix for elements along these boundaries as well as in region 3 where the step functions transition sharply from 0 to 1.

The displacements within elements that are in region 3 are dependent on the nodal values of the overlapping coarse element and the refined elements. For a general element therefore the displacement vector within an element can be expressed as:

$$\{\mathbf{u}\} = [\bar{\mathbf{N}}]\{\mathbf{X}_e\} = \begin{bmatrix} [\bar{\mathbf{N}}_1] & [\bar{\mathbf{N}}_2] \end{bmatrix} \begin{Bmatrix} \{\mathbf{X}_{g1}\} \\ \{\mathbf{X}_{g2}\} \end{Bmatrix} \quad (11)$$

where,

$\{\mathbf{X}_e\}$ = The nodal displacement vector for the element

$\{\mathbf{X}_{g1}\}$ = The nodal values corresponding to coarse grid 1

$\{\mathbf{X}_{g2}\}$ = The nodal values corresponding to refined grid 2

The matrices $[\bar{\mathbf{N}}_i]$ can be expressed as:

$$[\bar{\mathbf{N}}] = [\mathbf{H}][\mathbf{N}] \quad (12)$$

$$[\bar{\mathbf{N}}_2]^T = ([\mathbf{I}] - [\mathbf{H}])[\mathbf{N}] \quad (13)$$

$$[\mathbf{N}] = [[\mathbf{N}_1] \quad [\mathbf{N}_2] \quad \dots \quad [\mathbf{N}_n]] \quad (14)$$

$$[\mathbf{N}_I] = \begin{bmatrix} N_I & 0 \\ 0 & N_I \end{bmatrix} \quad (15)$$

Similarly, the virtual displacements can be expressed as $\{\delta u\} = [\bar{\mathbf{N}}]\{\delta X_e\}$ (16)

The strains for 2D models can be computed for each element using the following expression:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \end{Bmatrix} = [[\bar{\mathbf{B}}_1] \quad [\bar{\mathbf{B}}_2]] \begin{Bmatrix} \{X_{g1}\} \\ \{X_{g2}\} \end{Bmatrix} = [\bar{\mathbf{B}}]\{X_e\} \quad (17)$$

$$[\bar{\mathbf{B}}] = [[\bar{\mathbf{B}}_1] \quad [\bar{\mathbf{B}}_2]] \quad (18)$$

Combining the strain-displacement matrices of the two grids, the modified strain- displacement matrix is given as:

$$[\bar{\mathbf{B}}_1] = \nabla_s ([\bar{\mathbf{N}}_1]) \quad (19)$$

$$[\bar{\mathbf{B}}_2] = \nabla_s ([\bar{\mathbf{N}}_2]) \quad (20)$$

5 MANUAL VS AUTOMATIC REFINEMENT

This refinement method can be implemented manually by changing input file of model by specifying which elements to be refined. Reference file which gives closer values of Von Mises Stress as compared to exact solution is used to compute the relative error. It needs accurate knowledge and judgment which and where exactly local refinement is needed. on the other hand, automatic refinement works for the elements who has total error greater than 5%. It means only elements which has error greater than 5% are refined using the same principle which is mentioned above. Sometimes it refines unnecessary elements hence manual refinement is better in such case. For all the examples automatic refinement is used by changing the error percentage value.

6 RESULTS AND DISCUSSION

This approach explained in previous session was implemented in a program to study and validate the solution structure for uniform and adaptive refinement. In uniform refinement mesh

density is increased to get accurate results whether in adaptive refinement local refinement of element is done for elements.

1) L Shaped Plate

In this example, the refinement technique developed in this paper is demonstrated by studying stress concentrations at a sharp corner. [Figure 3](#) shows the model of a square plate with a square hole where only a quarter of the plate is modeled, and symmetry boundary conditions are used. Unit traction was applied on the right edge as shown in the [Figure 3](#).Material properties can be given as below.

Youngs Modulus	2.07E5 MPa
Poisson's Ratio	0.3

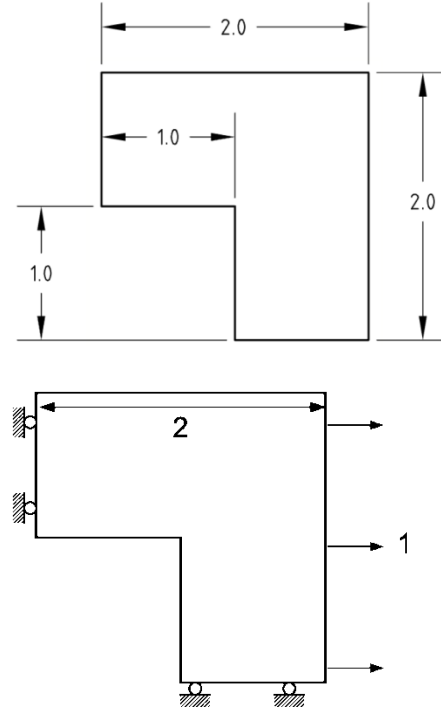
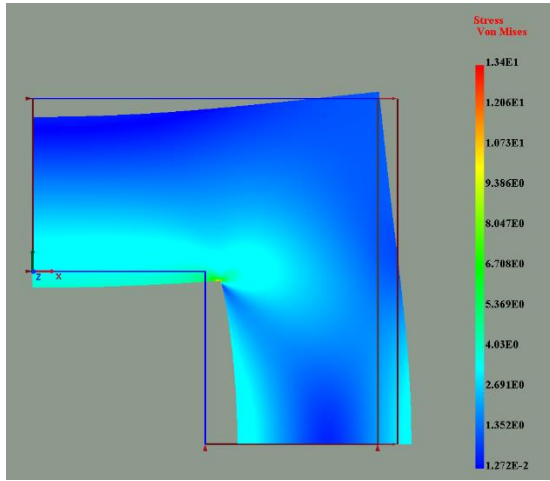
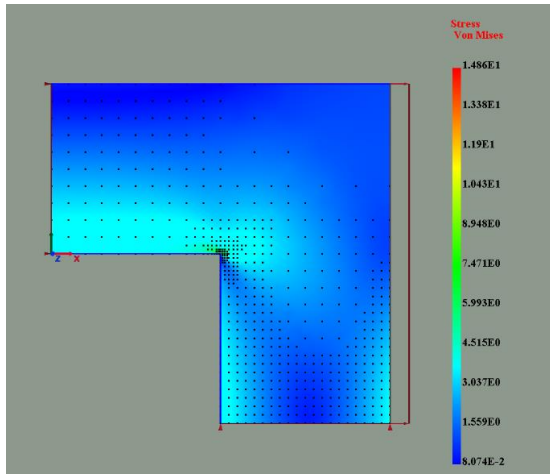


Figure 3 L Shaped Plate

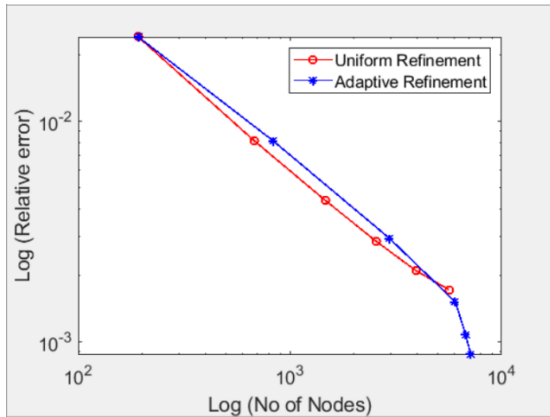
Firstly, to get the von mises stress closer to exact solution we use uniform mesh density. The deformed configuration, von Mises stresses and mesh resulting at the last step of adaptive refinement is shown in [figure 4](#). Uniform refinement is done using linear interpolation. Mesh density is reset to original and then automatic refinement is done. Graph of relative error Vs no of degree of freedom is plotted to compare uniform and adaptive refinement. It can be seen that for same number of degree of freedom relative error is more for adaptive refinement. If we choose quadratic B-spline or cubic B-spline interpolation type result will be much better. Technically adaptive refinement should give better result but for this example adaptive refinement is done for 6 times hence number of degrees of freedom is greater than compared to uniform element.



a) Mesh Density:250*250*1



b) Mesh Density:10*10*1



c) Relative Error Vs no of DOF

Figure 4. a) Von Mises Stress for Reference file
b) Von Mises Stress after Adaptive Refinement
c) Relative Error Vs No of Degrees of Freedom for uniform and adaptive refinement

2) Infinite Plate with Circular Hole

This is the standard benchmark problem of an infinite plate with circular hole subjected to uniform traction at infinity. Due to symmetry only $\frac{1}{4}$ th model is developed using On-Shape CAD modelling software. Analytical solution [5] is used to compute the reactions on the traction boundaries (right and top boundaries) and sliding boundary conditions are applied to bottom and left edges as shown in Figure 5. Material properties are listed in the table shown below. Exact solution for stresses is given by equation 21,22,23.

Youngs Modulus	1E5 MPa
Poisson's Ratio	0.33

$$\sigma_x = \sigma \left[1 - \frac{a^2}{r^2} \left(\frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3a^4}{2r^4} \cos 4\theta \right] \quad (21)$$

$$\sigma_{xy} = \sigma \left[-\frac{a^2}{r^2} \left(\frac{1}{2} \sin 2\theta + \sin 4\theta \right) + \frac{3a^4}{2r^4} \sin 4\theta \right] \quad (22)$$

$$\sigma_y = \sigma \left[-\frac{a^2}{r^2} \left(\frac{1}{2} \cos 2\theta - \cos 4\theta \right) - \frac{3a^4}{2r^4} \cos 4\theta \right] \quad (23)$$

Bi-linear Lagrange elements are used to study uniform and adaptive refinement for comparison of relative error. Plots are shown in Figure 6 shows Von mises stress for reference file which is considered as best solution here to compute the errors. It can be seen as exact solution for this example should be 3, and we are getting results for reference model by using uniform mesh density of 200*200*1. Figure 6.b) shows stress distribution after automatic refinement. It obvious from plot c) relative error is smaller for adaptive refinement than uniform refinement. Hence the method yields accurate results for this example.

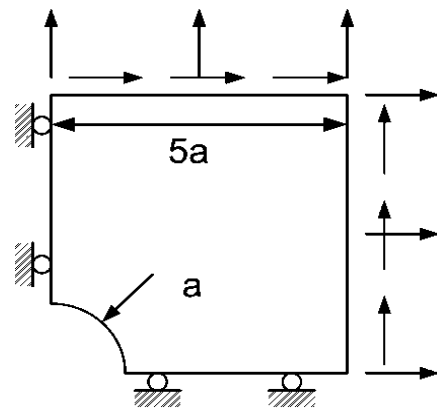
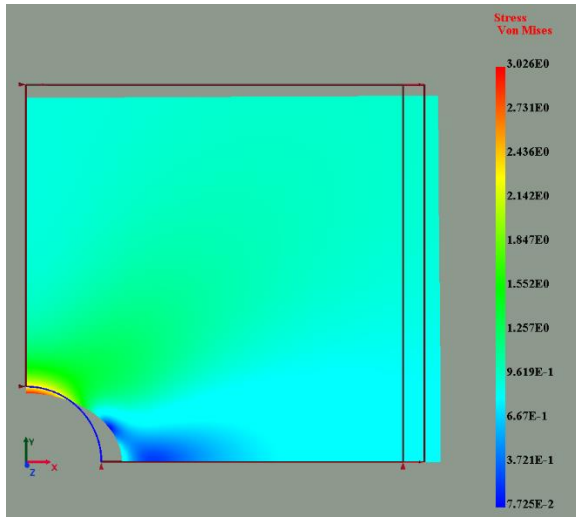
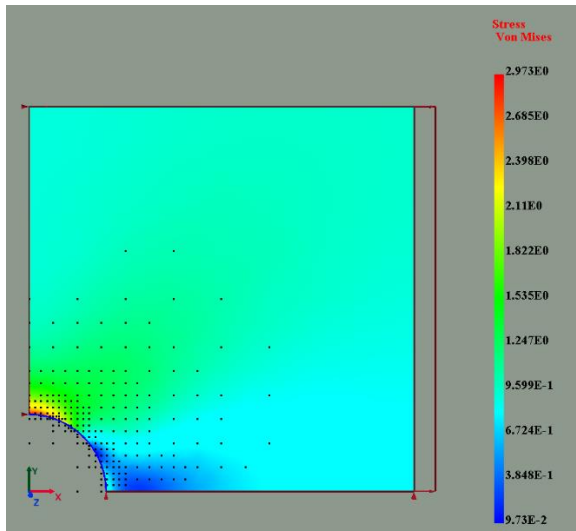


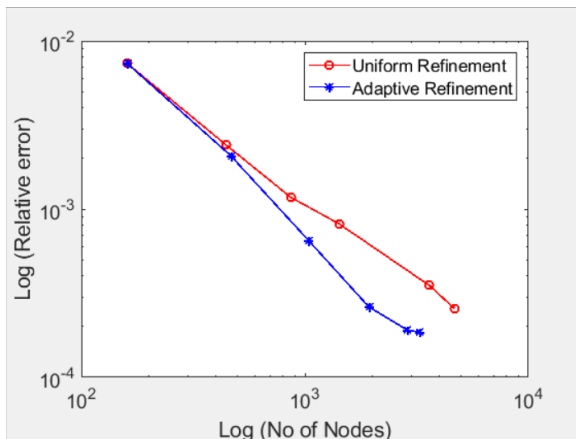
Figure 5. Infinite Plate with Circular Hole



a) Mesh Density:200*200*1



b) Mesh Density:8*8*1



c) Relative Error Vs No of DOF

3) Connecting Rod

In this example, we study a problem involving more complex 3D geometry, namely that of a connecting rod assembly shown in Figure 7. We note that a similar problem geometry was also studied in the context of weakly-coupled patches in [6]. A force $F = 1$ kN pointing downward is applied to the surface of the internal ring on the right side of the domain. The deformed configuration, von Mises stresses for reference file is shown in Figure 8. Meshes and stresses resulting at the last step of adaptive refinement are shown in Figure 8. As an analytical solution is not available, comparing results with the reference file. It can be seen that stress concentration is at top and bottom edge towards the geometry is fixed. As this more complex geometry only uniform refinement is done, results are shown in the Fig 8. In Fig 9 automatic refinement is done which gives poor results. As 8-node trilinear element is used results are poor. Quadratic B-spline or Cubic B-spline elements might yield better result. Computation time is higher for all the 3D examples. Material properties are shown in below table.

Youngs Modulus	2E11 N/m ²
Poisson's Ratio	0.3

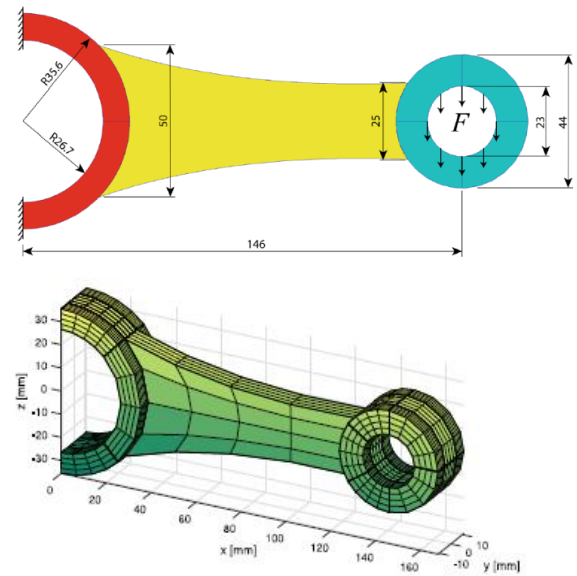


Figure 7. Connecting Rod Assembly

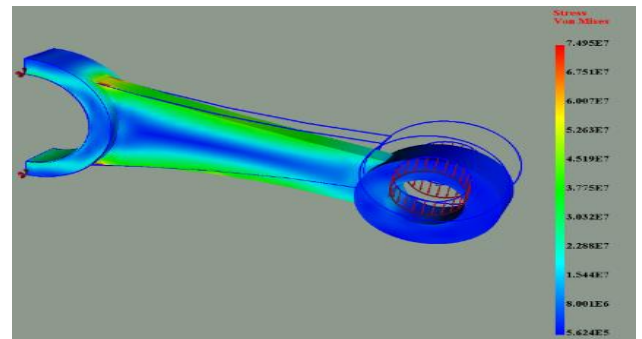


Figure 8. Mesh Density: 50*50*31

Figure 6. a) Von Mises Stress for Reference file
b) Von Mises Stress after Adaptive Refinement
c) Relative Error Vs No of Degrees of Freedom for uniform and adaptive refinement

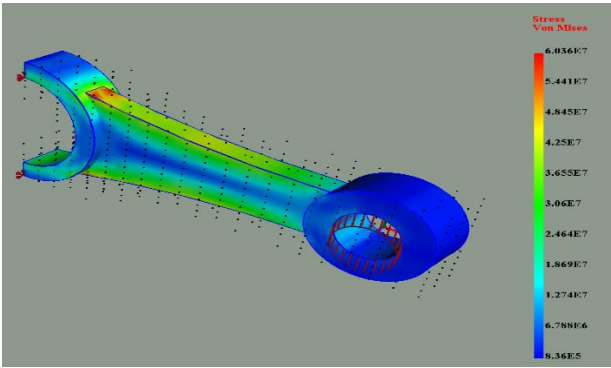
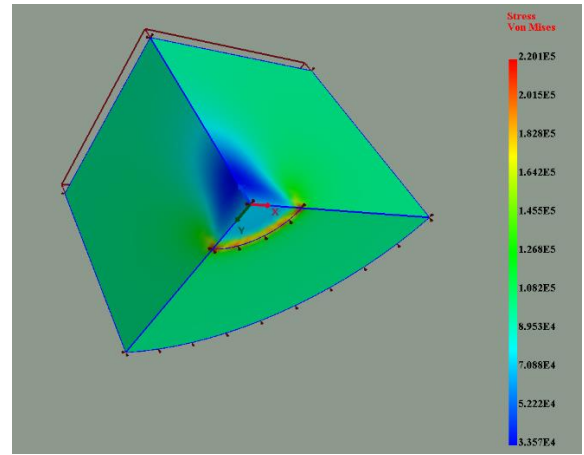
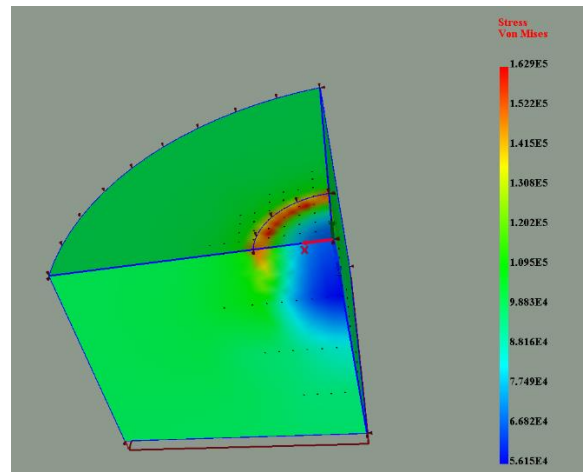


Figure 9. Mesh Density: 10*10*2



a) Mesh Density:40*40*20



b) Mesh Density:5*5*4

4) Penny Shaped Crack

This is common example in fracture mechanics which is penny shaped crack. This allows us to analyze the behavior of the adaptive refinement scheme in the presence of a strong discontinuity. Due to symmetry, the geometry is modeled by considering 1/8 of the cracked domain. The geometry and material properties are shown in Fig. 10. The radius of the crack is $a = 1$, the radius of the considered domain is $a + b = 4$ and the height is $c = 4$. Symmetry boundary conditions are imposed on the bottom of the annular patch and the force acting at infinity is set as $\sigma = 10^4$. The exact solutions for the stress and displacement fields can be found in [7]. The von Mises stress field obtained for this example is shown in Fig.11(a) which is not accurate. A logarithmic scale is used to better show relative error shown in Fig. 11(c). The meshes obtained the last step of refinement for each polynomial degree, are shown in Fig.11(b). When adaptive refinement is used, the convergence rate is clearly higher. Errors calculated are based on stress we are getting in Fig.11(a), which is different from exact solution given in [7]. Results can be improved by using B spline elements or may be higher mesh density.

Youngs Modulus	1E5 MPa
Poisson's Ratio	0.3

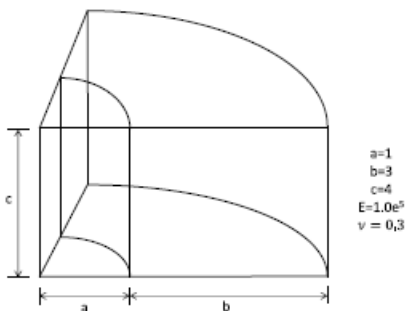
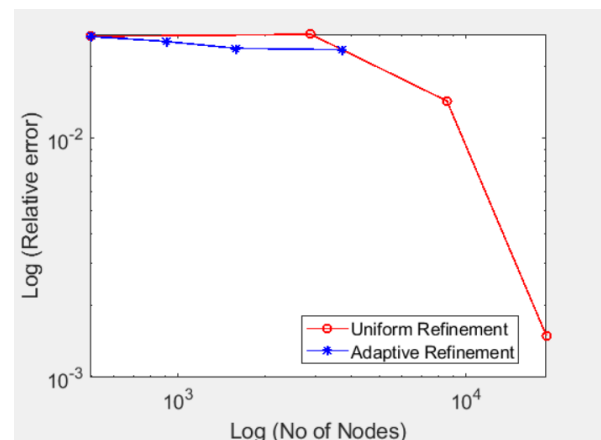


Figure 10. Penny Shaped Crack



c) Relative Error Vs no of DOF

Figure 11. a) Von Mises Stress for Reference file
b) Von Mises Stress after Adaptive Refinement
c) Relative Error Vs No of Degrees of Freedom for uniform and adaptive refinement

5) Spherical Hole inside a Cube

In this problem, we are considering a cube with an internal spherical hole in an infinite domain in R3 is shown in Fig 12. The exact stresses, given using spherical coordinates (r, φ, θ) by [8,9], are:

$$\begin{aligned}\sigma_{rr} &= S \cos^2 \theta + \frac{S}{7-5\nu} \left(\frac{a^3}{r^3} (6-5(5-\nu) \cos^2 \theta) + \frac{6a^5}{r^5} (3 \cos^2 \theta - 1) \right) \\ \sigma_{\phi\phi} &= \frac{3S}{2(7-5\nu)} \left(\frac{a^3}{r^3} (5\nu-2+5(1-2\nu) \cos^2 \theta) + \frac{a^5}{r^5} (1-5 \cos^2 \theta) \right) \\ \sigma_{\theta\theta} &= S \sin^2 \theta + \frac{S}{2(7-5\nu)} \left(\frac{a^3}{r^3} (4-5\nu+5(1-2\nu) \cos^2 \theta) + \frac{3a^5}{r^5} (3-7 \cos^2 \theta) \right) \\ \sigma_{r\theta} &= S \left(-1 + \frac{1}{7-5\nu} \left(-\frac{5a^3(1+\nu)}{r^3} + \frac{12a^5}{r^5} \right) \right) \sin \theta \cos \theta,\end{aligned}$$

where ‘a’ denotes the radius of the sphere, S is the uniaxial tension applied at infinity, μ is the shear modulus, and ν is Poisson’s ratio. Because of the symmetry, 1/8 of the problem domain in Fig 13 (a) is modeled by using On-shape software. The stress field in the x direction obtained for a fine mesh 40*40*20 and the refined meshes are plotted in Fig 13 (b). We note that the maximum stress of 2.21MPa is higher than stress 2.05MPa which corresponds to the theoretical maximum stress. It is observed that starting from coarse mesh of 5*5*4 defining in 4 steps gives exact value of Von Mises Stress which is 2.05Mpa. The relative error Versus degree of number of nodes is shown in Fig 13 (c). It can be seen that adaptive refinement gives better result as compared to uniform refinement. Material properties are given in table below.

Youngs Modulus	1E5 MPa
Poisson's Ratio	0.3

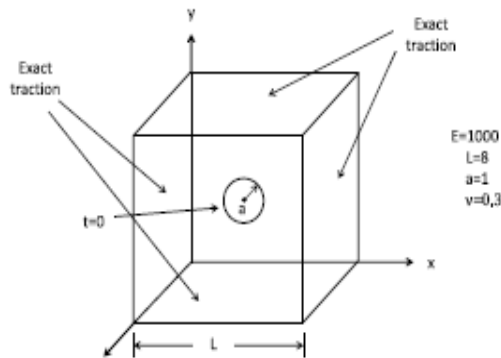
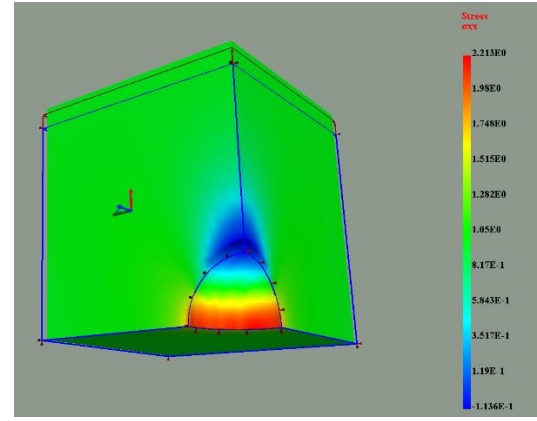
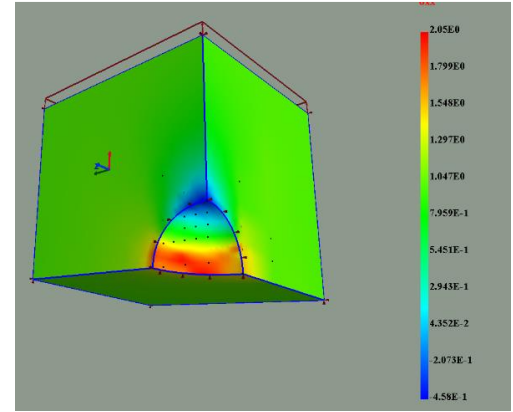


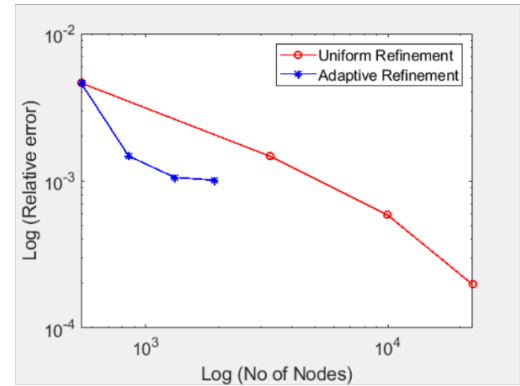
Figure 12. Spherical Hole in a Cube



a) Mesh Density:40*40*20



b) Mesh Density:5*5*4



c)Relative Error Vs No of DOF

Figure 13. a) Stress in X direction for Reference file
b) Stress in X direction after Adaptive Refinement
c) Relative Error Vs No of Degrees of Freedom for uniform and adaptive refinement

7 CONCLUSIONS

A method for local refinement of structured background mesh for immersed boundary FEM is presented in this paper that allows the use of a variety of elements for refinement. A solution structure is constructed such that the compatibility condition is automatically imposed between grids with different size elements while essential boundary conditions are satisfied using the step boundary method. Grids can be refined manually or automatically. Validation for the automatic refinement is done by performing it on 2D as well as 3D examples. Through results we can conclude that for 2D examples automatic refinement gives better results. But for 3D depending upon how complex the geometry is stresses are obtained. For example, connecting rod assembly does not give accurate results when adaptive refinement is done. Advantage of using this method that conforming mesh to the geometry is not required as geometry is immersed in the background mesh. Instead of using uniform refinement it is always beneficial to use adaptive refinement to save computation time and to get better results. There is future scope of improving the method to estimate error. Also, there are some bugs in automatic refinement which can be removed for 3D examples.

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