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Similarity, inclusion and entropy measures between type-2 fuzzy sets based on the Sugeno integral

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1. Introduction

Type-2 fuzzy sets are fuzzy sets whose membership values are (type-1) fuzzy sets on the interval [0, 1]. This concept was proposed by Zadeh [1] as an extension of fuzzy sets. Type-2 fuzzy sets possess a great expressive power and are conceptually quite appealing. More studies on type-2 fuzzy sets were then sequentially explored by Mizumoto and Tanaka [2,3] and Yager [4], and so forth. The membership function of a type-2 fuzzy set provides additional degree of freedom that makes it possible to directly model uncertainties. However, type-2 fuzzy sets were relatively hard to understand and clarify as compared to fuzzy sets so that they had been ignored for a while. They have recently obtained more and more attentions and been analyzed and discussed in advance. A fuzzy set is two dimensional and a type-2 fuzzy set is three dimensional. Type-2 fuzzy sets can better improve certain kinds of inference than do fuzzy sets with increasing imprecision, uncertainty and fuzziness in information. Mendel and John [5] presented a new Representation Theorem for type-2 fuzzy sets and showed that it can be used to derive formula for the union, intersection and complement of type-2 fuzzy sets without using the Extension Principle.

Similarity is an important tool to provide the foundation for analogical reasoning between two fuzzy concepts and has widespread applications. Pappis and Karacapilidis [6] proposed three similarity measures for fuzzy sets. Based on these similarity measures for fuzzy sets, many researches and applications of similarity measures for fuzzy sets had been given,
such as [7–9], etc. Although Hung and Yang [10] proposed a similarity measure between type-2 fuzzy sets based on the fuzzy Hausdorff distance, the similarity between type-2 fuzzy sets is not yet widespread studied. Inclusion measures between fuzzy sets are the degrees to which a fuzzy set is a subset of another fuzzy set. Likewise, an inclusion measure between type-2 fuzzy sets is the degree to which a type-2 fuzzy set is a subset of another type-2 fuzzy set. The entropy of type-2 fuzzy sets is a measure of fuzziness between type-2 fuzzy sets so that the degree of uncertainty for a type-2 fuzzy set can be measured.

Although several similarity, inclusion and entropy measures for type-2 fuzzy sets have been proposed in previous studies, no one has considered the use of the Sugeno integral to define those for type-2 fuzzy sets. In this paper, we propose new similarity, inclusion and entropy measures between type-2 fuzzy sets based on the Sugeno integral. Some examples are used to compare the proposed measures with previous methods. Numerical results show that the proposed measures are more reasonable than those existing methods. We then combine the proposed similarity measure with a robust clustering algorithm and apply it in clustering for type-2 fuzzy data. These clustering results are compared with Hung and Yang’s [10] and Yang and Lin’s [11] results. The remainder of this paper is organized as follows. In Section 2, a brief review of type-2 fuzzy sets is first given. We then review and discuss similarity, inclusion and entropy measures between type-2 fuzzy sets. In Section 3, new similarity, inclusion and entropy measures between type-2 fuzzy sets are proposed and relevant properties are also considered. In Section 4, some examples and comparisons will be made with some existing methods. We then apply the proposed similarity measure for clustering the patterns of type-2 fuzzy sets. These results are compared with those of Hung and Yang [10] and Yang and Lin [11]. Finally, conclusions are stated in Section 5.

2. Type-2 fuzzy set with its different measure definitions

In this section, we first give a brief review of type-2 fuzzy sets. We then review and discuss similarity, inclusion and entropy measures between type-2 fuzzy sets.

2.1. Type-2 fuzzy sets

A type-2 fuzzy set was first proposed by Zadeh [1] as an extension of a fuzzy set. Up to date, type-2 fuzzy sets have been widely applied to areas, such as decision theory [4], signal processing [12], speech recognition [13], transport scheduling [14], pattern recognition [15], correlation coefficient [16], forecasting of time series [17], fuzzy equation systems [18]. Mendel and John [5] presented a Representation Theorem for type-2 fuzzy sets and showed that it can be used to derive formulas for the union, intersection and complement of type-2 fuzzy sets without using the Extension Principle. We give a brief review of type-2 fuzzy sets as follows.

Definition 1 (Zadeh [1]). A type-2 fuzzy set is a fuzzy set whose membership values are type-1 fuzzy sets on [0, 1].

Definition 2 (Mizumoto and Tanaka [2]). A type-2 fuzzy set $A$ in a set $X$ is the fuzzy set which is characterized by a fuzzy membership function $\mu_A$ as

$$\mu_A : X \rightarrow [0, 1]^\prime$$

with the value $\mu_A(x)$ being called a fuzzy grade and being a fuzzy set in $[0, 1]$ (or in the subset $J$ of $[0, 1]$) ($x \in X$).

Definition 3 (Mendel and John [5]). A type-2 fuzzy set, denoted $\tilde{A}$, is characterized by a type-2 membership function $\mu_{\tilde{A}}(x, u)$, where $x \in X$ and $u \in J_x \subseteq [0, 1]$, i.e.,

$$\tilde{A} = \{(x, u, \mu_{\tilde{A}}(x, u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\}$$

in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$. $\tilde{A}$ can be also expressed as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u), \quad J_x \subseteq [0, 1]$$

where $\int \int$ denotes union over all admissible $x$ and $u$. For a universe of discrete discourses, $\int$ is replaced by $\Sigma$.

Definition 4 (Mendel and John [5]). At each value of $x$, say $x = x'$, the 2-D plane whose axes are $u$ and $\mu_{\tilde{A}}(x', u)$ is called a vertical slice of $\mu_{\tilde{A}}(x, u)$. A secondary membership function is a vertical slice of $\mu_{\tilde{A}}(x, u)$. It is $\mu_{\tilde{A}}(x = x', u)$ for $x' \in X$ and $\forall u \in J'_x \subseteq [0, 1]$, i.e.,

$$\mu_{\tilde{A}}(x = x', u) = \mu_{\tilde{A}}(x, u') = \int_{u \in J'_x} f_x(u) / u, \quad J'_x \subseteq [0, 1]$$

in which $0 \leq f_x(u) \leq 1$. $\tilde{A}$ can be also re-expressed as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid \forall x \in X\}$$

or, as $\tilde{A} = \int_{x \in X} \mu_{\tilde{A}}(x) / x = \int_{x \in X} \left[ \int_{J_x} f_x(u) / u \right] / x, J_x \subseteq [0, 1]$. 
Definition 5 (Mendel and John [5]). The domain of a secondary membership function is called the primary membership of $x$. In the representation of $\tilde{A}, J_x$ is the primary membership of $x$, where $J_x \subseteq [0, 1], \forall x \in X$.

Definition 6 (Mendel and John [5]). Uncertainty in the primary memberships of a type-2 fuzzy set $\tilde{A}$, consists of a bounded region that we call the footprint of uncertainty (FOU). It is the union of all primary memberships, i.e.,

$$\text{FOU}(\tilde{A}) = \bigcup_{x \in X} J_x.$$ 

2.2. Similarity, inclusion and entropy measures between type-2 fuzzy sets

A similarity measure between fuzzy sets is an important way to measure the degree of similarity between two fuzzy concepts. Zwick et al. [19] reviewed and compared 19 similarity measures between fuzzy sets based on both geometric and set-theoretic. Pappis and Karacapilidis [6] proposed three similarity measures for fuzzy sets. Afterwards, many researches and applications of similarity measures for fuzzy sets were given (see [7–9]).

Throughout this paper, the following notations are used. $F_1(X)$ is the class of all fuzzy sets of $X$; $F_2(X)$ is the class of all type-2 fuzzy sets of $X$; for any two fuzzy sets $A$ and $B$ in $F_1(X)$, Xuenergy [8] gave the axioms for a mapping $S : F_1(X) \times F_1(X) \rightarrow [0, 1]$ to be a similarity measure between $A$ and $B$ in $F_1(X)$. Based on the similar concept of Xuenergy’s [8], we give a definition of a similarity measure for type-2 fuzzy sets as follows.

Definition 7. A real function $S : F_2(X) \times F_2(X) \rightarrow [0, 1]$ is called a similarity measure for type-2 fuzzy sets, if $S$ satisfies the following axioms:

(S1) $S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A}), \forall \tilde{A}, \tilde{B} \in F_2(X)$.
(S2) $S(D, D^c) = 0, \forall D \in P(X)$ (the power set of $X$);
(S3) $S(\tilde{E}, \tilde{E}) = \max_{\tilde{A}, \tilde{B} \in F_2(X)} S(\tilde{A}, \tilde{B}), \forall \tilde{E} \in F_2(X)$.
(S4) For any $\tilde{A}, \tilde{B}, \tilde{C} \in F_2(X)$, if $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $S(\tilde{A}, \tilde{B}) \geq S(\tilde{A}, \tilde{C})$ and $S(\tilde{B}, \tilde{C}) \geq S(\tilde{A}, \tilde{C})$.

Note that, Mizumoto and Tanaka [2] had defined $\tilde{A} \subseteq \tilde{B}$ for any $\tilde{A}, \tilde{B} \in F_2(X)$. For $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$ in (S4) of Definition 7, we use the definition of Mizumoto and Tanaka [2]. For two fuzzy sets $A$ and $B$ in the finite set $X = \{x_1, x_2, \ldots, x_n\}$, Pappis and Karacapilidis [6] proposed a similarity measure $S(A, B) = \sum_{x \in X} \min[\mu_A(x), \mu_B(x)] / \sum_{x \in X} \max[\mu_A(x), \mu_B(x)]$. Based on the similar idea of Pappis and Karacapilidis [6], Yang and Lin [11] proposed a similarity measure between type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$ as

$$S_{\text{YL}}(\tilde{A}, \tilde{B}) = \int_{\Delta} \frac{1}{\sum_{x \in X} f_{\mu_{\tilde{A}}(u)}(x) \cdot f_{\mu_{\tilde{B}}(u)}(x) \cdot g_{\mu_{\tilde{C}}(u)}(x) \cdot g_{\mu_{\tilde{D}}(u)}(x) \cdot dx \cdot du}$$

where the notation $\int$ in the similarity $S_{\text{YL}}(\tilde{A}, \tilde{B})$ is an integral. For discrete universes of discourse, $\int$ is replaced by the summation $\sum$.

In 1965, Zadeh [20] gave a definition of fuzzy inclusion for any two fuzzy sets $A$ and $B$ in $F_1(X)$ with $A \subseteq B \iff \mu_A(x) \leq \mu_B(x), \forall x \in X$. Afterward, Sinha and Dogherty [21] considered an indicator $I(A, B)$ for an inclusion measure between two fuzzy sets $A$ and $B$ and then gave several axioms that $I(A, B)$ needs to satisfy (also see [22]). Zeng and Li [23] gave the axioms for the mapping $I : F_1(X) \times F_1(X) \rightarrow [0, 1]$ to be an inclusion measure of fuzzy sets $A$ and $B$. Based on Zeng and Li’s [23] axioms for an inclusion measure $I(A, B)$, we give a definition of an inclusion measure for type-2 fuzzy sets as follows.

Definition 8. A real function $I : F_2(X) \times F_2(X) \rightarrow [0, 1]$ is called an inclusion measure for type-2 fuzzy sets, if $I$ satisfies the following axioms:

(11) $I(\tilde{A}, \tilde{A}) = 1$
(12) $\tilde{A} \subseteq \tilde{B}$ if and only if $I(\tilde{A}, \tilde{B}) = 1$
(13) For any $\tilde{A}, \tilde{B}, \tilde{C} \in F_2(X)$, if $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $I(\tilde{C}, \tilde{A}) \leq I(\tilde{B}, \tilde{A})$ and $I(\tilde{C}, \tilde{A}) \leq I(\tilde{C}, \tilde{B})$.

Based on information from the primary similarity and secondary membership functions, Yang and Lin [11] proposed an inclusion measure between type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$ as

$$I_{\text{YL}}(\tilde{A}, \tilde{B}) = \frac{1}{\sum_{x \in X} f_{\mu_{\tilde{A}}(u)}(x) \cdot f_{\mu_{\tilde{B}}(u)}(x) \cdot g_{\mu_{\tilde{C}}(u)}(x) \cdot g_{\mu_{\tilde{D}}(u)}(x) \cdot dx \cdot du}$$

Measures of fuzziness are to indicate the degree of fuzziness of a fuzzy set. The entropy of a fuzzy set is used as a fuzziness measure for a fuzzy set. de Luca and Termini [24] introduced the axiom construction of entropy of fuzzy sets by using the similar concept of Shannon’s probability entropy in which they gave an axiom definition of entropy of fuzzy sets. There are several entropies of fuzzy sets proposed in the literature. However, there is less entropy definition for type-2 fuzzy sets in previous studies. Based on the de Luca and Termini [24] axioms for the entropy of fuzzy sets, we give an entropy definition for type-2 fuzzy sets as follows.
Definition 9. A real function $E : F_2(X) \rightarrow \mathbb{R}^+$ is called an entropy on $F_2(X)$, if $E$ has the following properties:

(E1) $E(D) = 0$ if $D$ is a crisp set.

(E2) $E(\tilde{A}) = E(\tilde{A'}), \forall \tilde{A} \in F_2(X)$.

(E3) $E(\tilde{A})$ assumes a unique maximum if $\tilde{A} = \{\frac{1}{2}\}$.

(E4) $E(\tilde{A}) \leq E(\tilde{B})$ if $\tilde{A}$ is crispier than $\tilde{B}$.

Note that $\tilde{A}$ is said to be crispier than $\tilde{B}$ if $\tilde{A}$ and $\tilde{B}$ are with

$$\tilde{A} = \int_{x \in X} \left[ \int_{u \in J_k} f_k(u)/u \right] \, \frac{1}{x} \text{ and } \tilde{B} = \int_{x \in X} \left[ \int_{u \in J_k} f_k(u)/u \right] \, \frac{1}{x},$$

then we have that $0 < f_k(u) \leq g_k(u) \leq \frac{1}{2}$ for $0 < g_k(u) \leq \frac{1}{2}$ and $\frac{1}{2} \leq g_k(u) \leq f_k(u) \leq 1$ for $\frac{1}{2} \leq g_k(u) \leq 1$. For any $\tilde{A} \in F_2(X)$ with $\tilde{A} = \int_{x \in X} \left[ \int_{u \in J_k} f_k(u)/u \right] \, \frac{1}{x}$, $\tilde{A'}$ is defined as $\tilde{A'} = \int_{x \in X} \left[ \int_{u \in J_k} f_k(u)/(1 - u) \right] \, \frac{1}{x}$. Furthermore, $\tilde{A} = \{\frac{1}{2}\}$ means that $\tilde{A} = \int_{x \in X} \left[ \int_{u \in J_k} f_k(u)/u \right] \, \frac{1}{x}$ with $f_k(u) = \frac{1}{2}, u \in J_k$ and $J_k = [0, 1]_X$.

3. New similarity, inclusion and entropy measures between type-2 fuzzy sets

In this section, we first review fuzzy measures and the Sugeno integral (see Murofushi and Sugeno [25]). Let $X$ be a nonempty set and let $\mathcal{F}$ be a $\sigma$-field of subsets of $X$. Let $m : \mathcal{F} \rightarrow [0, 1]$ be a non-negative and real-valued set function defined on $\mathcal{F}$.

Definition 10. The measure $m$ is called a fuzzy measure on $(X, \mathcal{F})$ if it satisfies the following conditions:

(FM1) $m(\emptyset) = 0$ (vanishing at $\emptyset$).

(FM2) $E \in \mathcal{F}, F \in \mathcal{F}$ and $E \subset F$ imply $m(E) \leq m(F)$ (monotonicity).

(FM3) $\{E_n\} \in \mathcal{F}, E_1 \subset E_2 \subset \cdots$, and $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$ imply

$$\lim_{n} m(E_n) = m \left( \bigcup_{n=1}^{\infty} E_n \right) \text{ (continuity from below).}$$

(FM4) $\{E_n\} \in \mathcal{F}, E_1 \supset E_2 \supset \cdots$, $\mu(E_1) < \infty$, and $\bigcap_{n=1}^{\infty} E_n \in \mathcal{F}$ imply

$$\lim_{n} m(E_n) = m \left( \bigcap_{n=1}^{\infty} E_n \right) \text{ (continuity from above).}$$

On the other hand, $m$ is called a lower or upper semi-continuous fuzzy measure if it satisfies the above conditions (FM1), (FM2), and (FM3) or (FM1), (FM2), and (FM4), respectively. Both of them are simply called the semi-continuous fuzzy measure. Furthermore, we say that the fuzzy measure or semi-continuous fuzzy measure $m$ is regular if and only if $X \in \mathcal{F}$ and $m(X) = 1$. We call $(X, \mathcal{F}, m)$ a fuzzy measure space if $m$ is a fuzzy measure on a measurable space $(X, \mathcal{F})$. Sometimes, we also refer to fuzzy measures and semi-continuous fuzzy measures as non-additive measures. Let $F$ be the class of all finite non-negative measurable functions defined on $(X, \mathcal{F})$. For any given $f \in F$, we write $f_\alpha = \{x|f(x) \geq \alpha\}$ and $f_{\alpha+} = \{x|f(x) > \alpha\}$, where $\alpha \in [0, \infty]$. The sets $f_\alpha$ and $f_{\alpha+}$ are usually called $\alpha$-cut and strict $\alpha$-cut sets, respectively. For simplicity, we consider the range of the function with the closed interval $[0, 1]$.

Definition 11. Let $A \in \mathcal{F}$ and $f \in F$. A fuzzy integral of $f$ on $A$ with respect to $m$, denoted by $\int_A f dm$, is defined by

$$\int_A f dm = \sup_{\alpha \in [0, 1]} [\alpha \wedge m(A \cap f_\alpha)].$$

When $A = X$, the fuzzy integral may also be denoted by $\int f dm$. Sometimes, the fuzzy integral is also called the Sugeno integral, which can provide an “expected value” like operation.

Proposition 1. If $A \subseteq B$, then $\int_A f dm \leq \int_B f dm$.

Proof. By the Sugeno integral, we have

$$\int_A f du = \sup_{0 \leq \beta \leq 1} \beta \wedge m(A \cap f_\beta) \leq \sup_{0 \leq \beta \leq 1} \beta \wedge m(B \cap f_\beta) = \int_B f dm. \quad \square$$

Proposition 2. If $f \leq g$, then $\int_A f dm \leq \int_A g dm$. 

Proof. By the Sugeno integral, we have
\[
\int_A f \, dm = \sup_{0 \leq \beta \leq 1} \beta \wedge m(A \cap f_\beta) \leq \sup_{0 \leq \beta \leq 1} \beta \wedge m(A \cap g_\beta) = \int_B g \, dm. \quad \square
\]

Proposition 3. If \( f \leq g \), then \( \int_0^1 (\int_{f_x} f \, dm) \, dx \leq \int_0^1 (\int_{g_x} g \, dm) \, dx \).

Proof. The proof is obvious. \( \square \)

Based on the Sugeno integral, we propose a new inclusion measure as follows:
\[
l(\tilde{A}, \tilde{B}) = \frac{1}{\int_{x \in X} \, dx} \int_{x \in X} \min \left\{ \frac{\int_{f_0} f \, dm, \int_{g_0} g \, dm}{\int_{f_0} f \, dm} \right\} \, dx
\]
where two type-2 fuzzy sets \( \tilde{A} \) and \( \tilde{B} \) are \( \tilde{A} = \int_{x \in X} [\min(f_{\tilde{A}}, f_x)]/x, \tilde{B} = \int_{x \in X} [\min(f_{\tilde{B}}, g_x)]/x \) and \( f_{x_0} \) and \( g_{x_0} \) are 0-cuts of \( f_x \) and \( g_x \), respectively. Note that if the universe of discourses is discrete, then the integral \( \frac{1}{\int_{x \in X} \, dx} \int_{x \in X} \) shall be replaced with \( \frac{1}{n} \sum_{x \in X} \). Based on the proposed inclusion measure, new similarity and entropy measures are proposed as follows:
\[
S(\tilde{A}, \tilde{B}) = \min(l(\tilde{A}, \tilde{B}), l(\tilde{B}, \tilde{A}))
\]
\[
E(\tilde{A}) = S(\tilde{A}, (\tilde{A})^c).
\]

Next, we claim that these measures of \( l(\tilde{A}, \tilde{B}), S(\tilde{A}, \tilde{B}) \) and \( E(\tilde{A}) \) are satisfied the conditions of inclusion, similarity and entropy measures, respectively.

Proposition 4. \( l(\tilde{A}, \tilde{B}) \) is an inclusion measure.

Proof. (11) \( l(\tilde{A}, \tilde{A}) = \frac{1}{\int_{x \in X} \, dx} \int_{x \in X} \min(\int_{f_0} f \, dm, \int_{g_0} g \, dm) \, dx = 1 \).

(12) If \( \tilde{A} \subseteq \tilde{B} \), then \( 0 \leq f_x(u) \leq g_x(u) \leq 1 \), \( \forall x \in X, \forall u \in J_x \subseteq [0, 1] \).
We have that
\[
l(\tilde{A}, \tilde{B}) = \frac{1}{\int_{x \in X} \, dx} \int_{x \in X} \min\left\{ \frac{\int_{f_0} f \, dm, \int_{g_0} g \, dm}{\int_{f_0} f \, dm} \right\} \, dx = \frac{1}{\int_{x \in X} \, dx} \int_{x \in X} \frac{\int_{f_0} f \, dm}{\int_{f_0} f \, dm} \, dx = 1.
\]

(13) If \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \), then \( 0 \leq f_x(u) \leq g_x(u) \leq h_x(u) \leq 1 \), \( \forall x \in X, \forall u \in J_x \subseteq [0, 1] \).
We have that
\[
l(\tilde{C}, \tilde{A}) = \frac{1}{\int_{x \in X} \, dx} \int_{x \in X} \min\left\{ \frac{\int_{f_0} f \, dm, \int_{h_0} h \, dm}{\int_{f_0} f \, dm} \right\} \, dx = \frac{1}{\int_{x \in X} \, dx} \int_{x \in X} \frac{\int_{f_0} f \, dm}{\int_{f_0} f \, dm} \, dx \\
\leq \frac{1}{\int_{x \in X} \, dx} \int_{x \in X} \frac{\int_{f_0} f \, dm}{\int_{g_0} g \, dm} \, dx = \frac{1}{\int_{x \in X} \, dx} \int_{x \in X} \frac{\min\{\int_{g_0} g \, dm, \int_{f_0} f \, dm\}}{\int_{g_0} g \, dm} \, dx = l(\tilde{B}, \tilde{A}).
\]

Thus, we prove that \( l(\tilde{C}, \tilde{A}) \leq l(\tilde{B}, \tilde{A}) \).
Similarly, we can prove that \( l(\tilde{C}, \tilde{A}) \leq l(\tilde{B}, \tilde{A}) \). \( \square \)

Proposition 5. \( S(\tilde{A}, \tilde{B}) \) is a similarity measure.

Proof. (S1) \( S(\tilde{A}, \tilde{B}) = \min(l(\tilde{A}, \tilde{B}), l(\tilde{B}, \tilde{A})) = \min(l(\tilde{B}, \tilde{A}), l(\tilde{A}, \tilde{B})) = S(\tilde{B}, \tilde{A}) \).
(S2) \( S(D, D^c) = \min(l(D, D^c), l(D^c, D)) = 0 \) if \( D \) is a crisp set.
(S3) Since \( S(\tilde{E}, \tilde{E}) = 1 \) and \( 0 \leq I(\tilde{A}, \tilde{B}) \leq 1, 0 \leq |I(\tilde{B}, \tilde{A})| \leq 1, \)
\[
S(\tilde{E}, \tilde{E}) = \max_{\tilde{A}, \tilde{B} \in \mathcal{F}_2} S(\tilde{A}, \tilde{B}) \quad \forall \tilde{E} \in \mathcal{F}_2(X).
\]

(S4) For any \( \tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}_2(X), \) if \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}, \) then \( I(\tilde{A}, \tilde{B}) = 1, I(\tilde{A}, \tilde{C}) = 1. \)
We have that \( S(\tilde{A}, \tilde{C}) = \min |I(\tilde{A}, \tilde{C}), I(\tilde{C}, \tilde{A})| = I(\tilde{C}, \tilde{A}) \) and \( S(\tilde{A}, \tilde{B}) = \min |I(\tilde{A}, \tilde{B}), I(\tilde{B}, \tilde{A})| = I(\tilde{B}, \tilde{A}). \)
Thus, if \( I(\tilde{C}, \tilde{A}) \leq I(\tilde{B}, \tilde{A}), \) then \( S(\tilde{A}, \tilde{C}) \leq S(\tilde{A}, \tilde{B}). \)
Similarly, we can obtain that \( S(\tilde{A}, \tilde{C}) \leq S(\tilde{B}, \tilde{C}). \)

Proposition 6. \( E(\tilde{A}) \) is an entropy measure.

Proof. (E1) \( E(D) = S(D, D^c) = 0 \) if \( D \) is a crisp set.
(E2) \( E(\tilde{A}) = S(\tilde{A}, (\tilde{A})^c) = \min |I(\tilde{A}, (\tilde{A})^c), I((\tilde{A})^c, \tilde{A})| = \min |I((\tilde{A})^c, \tilde{A}), I(\tilde{A}, (\tilde{A})^c)| = E((\tilde{A})^c). \)
(E3) \( E(1/2, 1/2) = 1. \)
(E4) If \( f_x(u) \leq g_x(u) \leq \frac{1}{2}, \forall x \in X, \) then
\[
E(\tilde{A}) = S(\tilde{A}, (\tilde{A})^c)
\begin{align*}
&= \frac{1}{\int_{x \in X} dx} \min \left( \int_{x \in X} \frac{\min \left( \int_{f(x, 0)} f_x dm, \int_{f(x, 0)} f_x dm \right)}{\int_{f(x, 0)} f_x dm} dx, \int_{x \in X} \frac{\min \left( \int_{f(x, 0)} f_x dm, \int_{f(x, 0)} f_x dm \right)}{\int_{f(x, 0)} f_x dm} dx \right) \\
&= \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{f(x, 0)} f_x dm}{\int_{f(x, 0)} g_x dm} dx \leq \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{g(x, 0)} g_x dm}{\int_{g(x, 0)} g_x dm} dx \\
&= \frac{1}{\int_{x \in X} dx} \min \left( \int_{x \in X} \frac{\min \left( \int_{g(x, 0)} g_x dm, \int_{g(x, 0)} g_x dm \right)}{\int_{g(x, 0)} g_x dm} dx, \int_{x \in X} \frac{\min \left( \int_{g(x, 0)} g_x dm, \int_{g(x, 0)} g_x dm \right)}{\int_{g(x, 0)} g_x dm} dx \right) \\
&= E(\tilde{B}).
\end{align*}
\]
Thus, if \( f_x(u) \leq g_x(u) \leq \frac{1}{2}, \forall x \in X, \) we have that \( E(\tilde{A}) \leq E(\tilde{B}). \)
Similarly, if \( f_x(u) \geq g_x(u) \geq \frac{1}{2}, \forall x \in X, \) we can obtain that \( E(\tilde{A}) \leq E(\tilde{B}). \)

4. Examples and application to clustering

In this section, we use several examples to demonstrate the proposed similarity, inclusion and entropy measures between type-2 fuzzy sets. We also combine the proposed similarity measure with Yang and Shih’s [26] method to create a hierarchical clustering for type-2 fuzzy data. We then compare these results with [10,11].

Example 1. Assume that there are two patterns denoted with type-2 fuzzy sets in \( X = \{x_1, x_2, x_3\}. \) The two patterns are denoted as follows:
\[
\tilde{A} = \{(x_i, u_{\tilde{A}}(x_i)|x_i \in X)\} \quad \text{and} \quad \tilde{B} = \{(x_i, u_{\tilde{B}}(x_i)|x_i \in X)\} \quad \text{where}
\]
\[
u_{\tilde{A}}(x_1) = \{(0.1, 0.04), (0.3, 0.1), (0.5, 0.5), (0.7, 0.8)\},
\nu_{\tilde{A}}(x_2) = \{(0.2, 0.2), (0.4, 0.3), (0.6, 0.7), (0.8, 0.4)\},
\nu_{\tilde{A}}(x_3) = \{(0.1, 0.3), (0.3, 0.5), (0.6, 0.9), (0.8, 0.5)\},
\nu_{\tilde{B}}(x_1) = \{(0.1, 0.04), (0.3, 0.2), (0.5, 0.2), (0.7, 0.4)\},
\nu_{\tilde{B}}(x_2) = \{(0.2, 0.1), (0.4, 0.2), (0.6, 0.6), (0.8, 0.3)\},
\nu_{\tilde{B}}(x_3) = \{(0.1, 0.2), (0.3, 0.5), (0.6, 0.5), (0.8, 0.3)\}.
\]
Assume that a datum \( \tilde{C} = \{(x_i, u_{\tilde{C}}(x_i)|x_i \in X)\} \) is given with
\[
u_{\tilde{C}}(x_1) = \{(0.1, 0.2), (0.3, 0.2), (0.5, 0.2), (0.7, 0.5)\},
\nu_{\tilde{C}}(x_2) = \{(0.2, 0.3), (0.4, 0.3), (0.6, 0.4), (0.8, 0.2)\},
\nu_{\tilde{C}}(x_3) = \{(0.1, 0.2), (0.3, 0.4), (0.6, 0.4), (0.8, 0.4)\}.\]
According to Eq. (1), since the universe of discourses is discrete, the integral \( \int_{\mathcal{X}} \frac{1}{m} \sum_{x \in \mathcal{X}} \left( \int_{\mathcal{X}} f_{\mu_{A}}(x) g_{\nu}(x) \, dx \right) \) shall be replaced with \( \frac{1}{n} \sum_{x \in \mathcal{X}} \). Thus, the inclusion measure between type-2 fuzzy sets \( \tilde{A} \) and \( \tilde{B} \) is given by

\[
I(\tilde{A}, \tilde{B}) = \frac{1}{n} \sum_{1 \leq i \leq n} \min \left\{ \int_{\mathcal{X}} f_{\mu_{A}}(x) g_{\nu}(x) \, dx, \int_{\mathcal{X}} f_{\mu_{B}}(x) g_{\nu}(x) \, dx \right\} = \frac{1}{3} \left( \frac{0.25}{0.5} + \frac{0.3}{0.4} + \frac{0.5}{0.5} \right) = 0.75
\]

\[
I(\tilde{B}, \tilde{A}) = \frac{1}{3} \left( \frac{0.25}{0.25} + \frac{0.3}{0.3} + \frac{0.5}{0.5} \right) = 1.
\]

According to Eq. (2), we have that \( S(\tilde{A}, \tilde{B}) = \min(I(\tilde{A}, \tilde{B}), I(\tilde{B}, \tilde{A})) = \min(0.75, 1.0) = 0.75 \). Similarly, we have that \( I(\tilde{A}, \tilde{C}) = 0.683, I(\tilde{C}, \tilde{A}) = 1, I(\tilde{B}, \tilde{C}) = 0.933 \) and \( I(\tilde{C}, \tilde{B}) = 1 \). Hence, \( S(\tilde{A}, \tilde{C}) = \min(I(\tilde{A}, \tilde{C}), I(\tilde{C}, \tilde{A})) = 0.683 \) and \( S(\tilde{B}, \tilde{C}) = 0.933 \). According to the principle of the maximum degree of similarity, it is seen that the datum \( \tilde{C} \) is more closed to the pattern \( \tilde{B} \) than \( \tilde{A} \). This result well matches the structure of the data set \( \{\tilde{A}, \tilde{B}, \tilde{C}\} \). On the other hand, using Eq. (3), the entropy for the type-2 fuzzy set \( \tilde{A} \) is given with \( E(\tilde{A}) = S(\tilde{A}, (\tilde{A})^c) = 0.778 \). Similarly, we have \( E(\tilde{B}) = 0.587 \) and \( E(\tilde{C}) = 0.476 \).

Note that the Hausdorff distance is generally used to define a distance measure for subsets of a metric space. Let \( W^\lambda \) denote the operation of dilating the set \( W \) by the radius \( \lambda \) (i.e., \( W^\lambda \) is the set of all points within distance \( \lambda \) of \( W \)). For any two non-empty compact sets \( U \) and \( V \), let \( I(U, V) = \inf \{ \lambda \in [0, \infty) | U^\lambda \supset V \} \). The Hausdorff distance between \( U \) and \( V \) is defined by \( H(U, V) = \max \{ I(U, V), I(V, U) \} \). Based on the Hausdorff distance, Hung and Yang [10] proposed a similarity measure between type-2 fuzzy sets \( \tilde{A} \) and \( \tilde{B} \) as

\[
S_{HY}(\tilde{A}, \tilde{B}) = 1 - \frac{d(\tilde{A}, \tilde{B})}{n}
\]

where \( d(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} H_f(u_i^A(x_i), (u_i^B(x_i))) \). If \( u_i^A(x_i) \) and \( u_i^B(x_i) \) take only discrete membership values on \( t_1, \ldots, t_n \), then

\[
H_f(u_i^A(x_i), (u_i^B(x_i))) = \frac{\sum_{j=1}^{m} t_j H_f(u_i^A(x_i), (u_i^B(x_i)))}{\sum_{j=1}^{m} t_j}
\]

where \( H_f(u_i^A(x_i), (u_i^B(x_i))) \) is a Hausdorff distance between the two level-sets \( u_i^A(x_i) \) and \( u_i^B(x_i) \). If both \( u_i^A(x_i) \) and \( u_i^B(x_i) \) are continuous, then

\[
H_f(u_i^A(x_i), (u_i^B(x_i))) = \int_{0}^{1} t H_f(u_i^A(x_i), (u_i^B(x_i)), dt
\]

where \( H_f(u_i^A(x_i), (u_i^B(x_i))) \) is a Hausdorff distance between the two level-sets \( u_i^A(x_i) \) and \( u_i^B(x_i) \). To explore the drawback of the similarity measure between type-2 fuzzy sets proposed by Hung and Yang [10], we use the following example for explanation and comparison.

**Example 2.** Assume that there are two patterns denoted with type-2 fuzzy sets in \( X = \{x\} \) as follows:

\[\tilde{A} = \{(x, u^A(x)|x \in X)\}\] and \[\tilde{B} = \{(x, u^B(x)|x \in X)\}\]

where

\[u^A(x) = \{(0.8, 1.0), (0.7, 0.4), (0.6, 0.6)\},\]

\[u^B(x) = \{(0.8, 1.0), (0.7, 0.7), (0.6, 0.3)\}.\]

Assume that a sample pattern \( \tilde{C} = \{(x, u^C(x)|x \in X)\} \) is given, where

\[u^C(x) = \{(0.8, 1.0), (0.7, 0.8), (0.6, 0.5)\}.\]

To determine if the sample pattern \( \tilde{C} \) is more similar to \( \tilde{B} \) than \( \tilde{A} \), the similarity measures \( S_{HY}(\tilde{A}, \tilde{C}) \) and \( S_{HY}(\tilde{B}, \tilde{C}) \) proposed by Hung and Yang [10] are calculated as follows:

\[S_{HY}(\tilde{A}, \tilde{C}) = 0.97 \quad \text{and} \quad S_{HY}(\tilde{B}, \tilde{C}) = 0.97.\]

It is found that the similarity degree of \( \tilde{C} \) and \( \tilde{A} \) is the same as that of \( \tilde{C} \) and \( \tilde{B} \) from Hung and Yang’s method [10]. Using the proposed similarity measure, we have that \( S(\tilde{A}, \tilde{C}) = 0.6 \) and \( S(\tilde{B}, \tilde{C}) = 0.67 \). Thus, the sample pattern \( \tilde{C} \) is closer to the pattern \( \tilde{B} \) than \( \tilde{A} \) by our proposed similarity. According to the structure of the data set \( \{\tilde{A}, \tilde{B}, \tilde{C}\} \), it is obvious that the sample pattern \( \tilde{C} \) is actually more similar to the pattern \( \tilde{B} \) than \( \tilde{A} \).
In the next example, we compare our proposed similarity measure with Yang and Lin’s [11] method. Recall that Yang and Lin [11] proposed a similarity measure between type-2 fuzzy sets \( \tilde{A} \) and \( \tilde{B} \) as

\[
S_{IL}(\tilde{A}, \tilde{B}) = \frac{1}{\sum_{x \in X}} \int_{x \in X} \left| f_{\tilde{A}}(x) - f_{\tilde{B}}(x) \right| + \left( |g_{\tilde{A}}(x) - g_{\tilde{B}}(x)| \right) \mathrm{d}x.
\]

For the discrete universes of discourse, \( \int_{x \in X} \mathrm{d}x \) is replaced with \( \frac{1}{n} \sum_{x \in X} \).

**Example 3.** Assume that there are two patterns denoted with type-2 fuzzy sets in \( X = \{ x \} \) as follows:

\[
\tilde{A} = \{(x, u_\tilde{A}(x)|x \in X)\} \quad \text{and} \quad \tilde{B} = \{(x, u_\tilde{B}(x)|x \in X)\}
\]

where

\[
u_\tilde{A}(x) = \{(0.1, 0.5), (0.2, 0.6), (0.3, 0.4), (0.6, 0.8), (0.8, 0.3)\}
\]

\[
u_\tilde{B}(x) = \{(0.1, 0.6), (0.2, 0.7), (0.3, 0.5), (0.6, 0.7), (0.8, 0.3)\}.
\]

Assume that a sample pattern \( \tilde{C} = \{(x, u_\tilde{C}(x)|x \in X)\} \) is given with

\[
u_\tilde{C}(x) = \{(0.1, 0.7), (0.2, 0.7), (0.3, 0.5), (0.6, 0.8), (0.8, 0.3)\}.
\]

To determine if the sample pattern \( \tilde{C} \) is more similar to the pattern \( \tilde{A} \) than \( \tilde{B} \), the similarity measures \( S_{IL}(\tilde{A}, \tilde{C}) \) and \( S_{IL}(\tilde{C}, \tilde{B}) \) proposed by Yang and Lin [11] are calculated as follows:

\[
S_{IL}(\tilde{A}, \tilde{C}) = 0.935 \quad \text{and} \quad S_{IL}(\tilde{B}, \tilde{C}) = 0.935.
\]

It is found that the similarity degree of \( \tilde{A} \) and \( \tilde{C} \) is the same as that of \( \tilde{C} \) and \( \tilde{B} \) based on Yang and Lin’s [11] similarity \( S_{IL} \). However, it is obvious that the pattern \( \tilde{C} \) should be more similar to the pattern \( \tilde{B} \) than \( \tilde{A} \) according to the structure of the data set \( \{\tilde{A}, \tilde{B}, \tilde{C}\} \). Using the proposed similarity measure, we have that \( S(\tilde{A}, \tilde{C}) = 0.714 \) and \( S(\tilde{B}, \tilde{C}) = 0.857 \). Thus, the sample pattern \( \tilde{C} \) is closer to the pattern \( \tilde{B} \) than \( \tilde{A} \) that actually matches the behavior of the data structure.

In the following example, we combine the proposed similarity with Yang and Shih’s [26] algorithm such that it can be a clustering algorithm for type-2 fuzzy data. These clustering results will be compared to Hung and Yang [10] results.

**Example 4.** Consider the Gaussian type-2 fuzzy data set \( \{\tilde{A}_j, j = 1, 2, 3, 4, 5, 6\} \) on a discrete domain consisting of only three points with \( x_1 = 1, x_2 = 3 \) and \( x_3 = 5 \). Suppose that

\[
m(x_1) = 0.1, \quad m(x_2) = 0.8, \quad m(x_3) = 0.6 \quad \text{and}
\]

\[
u_\tilde{A}_j(x) = \int_{u|j \leq u < 0.1} \frac{\exp \left(-\frac{(u-m(x_j))^2}{2\sigma_i(x_j)^2}\right)}{u}, \quad i = 1, 2, 3, j = 1, 2, 3, 4, 5, 6
\]

where \( \sigma_1(x_1) = 1 m(x_1), \sigma_2(x_2) = 2 m(x_2), \sigma_3(x_3) = 0.1 m(x_1), \sigma_2(x_1) = 0.2 m(x_1), \sigma_3(x_2) = 0.01 m(x_1) \) and \( \sigma_3(x_3) = 0.02 m(x_1) \), for \( i = 1, 2, 3 \). We want to cluster \( \{\tilde{A}_j, j = 1, 2, 3, 4, 5, 6\} \) based on the proposed similarity measure. The similarity degree between the type-2 fuzzy sets \( \tilde{A}_1 \) and \( \tilde{A}_2 \) is calculated as follows.

\[
l(\tilde{A}_1, \tilde{A}_2) = \frac{1}{3} \left( \min \left\{ \int_{[0,1]} e^{-\frac{(u-0.1)^2}{2\sigma_1(x_1)^2}} du, \int_{[0,1]} e^{-\frac{(u-0.1)^2}{2\sigma_2(x_2)^2}} du \right\} + \min \left\{ \int_{[0,1]} e^{-\frac{(u-0.1)^2}{2\sigma_3(x_3)^2}} du, \int_{[0,1]} e^{-\frac{(u-0.1)^2}{2\sigma_2(x_2)^2}} du \right\} + \min \left\{ \int_{[0,1]} e^{-\frac{(u-0.1)^2}{2\sigma_3(x_3)^2}} du, \int_{[0,1]} e^{-\frac{(u-0.1)^2}{2\sigma_1(x_1)^2}} du \right\} \right)
\]

\[
= \frac{1}{3} \left( \frac{0.48}{0.57} + \frac{0.8}{0.87} + \frac{0.84}{0.9} \right) = 0.898
\]

\[
l(\tilde{A}_2, \tilde{A}_1) = \frac{1}{3} \left( \frac{0.48}{0.57} + \frac{0.8}{0.87} + \frac{0.84}{0.9} \right) = 1.
\]

Thus, we have that \( S(\tilde{A}_1, \tilde{A}_2) = \min(\rho(\tilde{A}_1, \tilde{A}_2), \rho(\tilde{A}_2, \tilde{A}_1)) = 0.898 \). By similar calculation, we obtain the other similarities as shown in Table 1.

Based on Yang and Shih’s [26] method for the similarities of Table 1, we obtain the hierarchical clustering results for the Gaussian type-2 fuzzy data \( \{\tilde{A}_j, j = 1, 2, 3, 4, 5, 6\} \) as follows:

\[
0 < \alpha \leq 0.274 \Rightarrow \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5, \tilde{A}_6\}
\]

\[
0.274 < \alpha \leq 0.35 \Rightarrow \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5, \tilde{A}_6\}
\]
Table 1
Similarities of $S(\tilde{A}, \tilde{B})$ between the type-2 fuzzy data $[\tilde{A}_j, j = 1, 2, 3, 4, 5, 6]$.

<table>
<thead>
<tr>
<th>$\tilde{A}_1$</th>
<th>$\tilde{A}_2$</th>
<th>$\tilde{A}_3$</th>
<th>$\tilde{A}_4$</th>
<th>$\tilde{A}_5$</th>
<th>$\tilde{A}_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.898</td>
<td>0.612</td>
<td>0.719</td>
<td>0.303</td>
<td>0.388</td>
</tr>
<tr>
<td>0.898</td>
<td>1</td>
<td>0.551</td>
<td>0.648</td>
<td>0.274</td>
<td>0.350</td>
</tr>
<tr>
<td>0.612</td>
<td>0.552</td>
<td>1</td>
<td>0.850</td>
<td>0.494</td>
<td>0.632</td>
</tr>
<tr>
<td>0.719</td>
<td>0.648</td>
<td>0.850</td>
<td>1</td>
<td>0.420</td>
<td>0.551</td>
</tr>
<tr>
<td>0.303</td>
<td>0.274</td>
<td>0.494</td>
<td>0.420</td>
<td>1</td>
<td>0.780</td>
</tr>
<tr>
<td>0.388</td>
<td>0.350</td>
<td>0.632</td>
<td>0.551</td>
<td>0.780</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2
Similarities of $S_{HY}(\tilde{A}, \tilde{B})$ between the type-2 fuzzy data $[\tilde{A}_j, j = 1, 2, 3, 4, 5, 6]$.

<table>
<thead>
<tr>
<th>$\tilde{A}_1$</th>
<th>$\tilde{A}_2$</th>
<th>$\tilde{A}_3$</th>
<th>$\tilde{A}_4$</th>
<th>$\tilde{A}_5$</th>
<th>$\tilde{A}_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.758</td>
<td>0.599</td>
<td>0.838</td>
<td>0.472</td>
<td>0.496</td>
</tr>
<tr>
<td>0.758</td>
<td>1</td>
<td>0.357</td>
<td>0.434</td>
<td>0.229</td>
<td>0.254</td>
</tr>
<tr>
<td>0.599</td>
<td>0.357</td>
<td>1</td>
<td>0.923</td>
<td>0.873</td>
<td>0.898</td>
</tr>
<tr>
<td>0.838</td>
<td>0.434</td>
<td>0.923</td>
<td>1</td>
<td>0.796</td>
<td>0.821</td>
</tr>
<tr>
<td>0.472</td>
<td>0.229</td>
<td>0.873</td>
<td>0.796</td>
<td>1</td>
<td>0.976</td>
</tr>
<tr>
<td>0.496</td>
<td>0.254</td>
<td>0.898</td>
<td>0.821</td>
<td>0.976</td>
<td>1</td>
</tr>
</tbody>
</table>

Based on the similarity measure $S_{HY}$ proposed by Hung and Yang [10], we can obtain the similarities for Gaussian type-2 fuzzy data $[\tilde{A}_j, j = 1, 2, 3, 4, 5, 6]$ as shown in Table 2.

Based on Yang and Shih’s [26] method for the similarities of Table 2, we obtain the hierarchical clustering results for the Gaussian type-2 fuzzy data $[\tilde{A}_j, j = 1, 2, 3, 4, 5, 6]$ as follows:

$0 < \alpha \leq 0.229 \Rightarrow [\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5, \tilde{A}_6]$  
$0.229 < \alpha \leq 0.472 \Rightarrow [\tilde{A}_1, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5, \tilde{A}_6, \{\tilde{A}_2\}]$  
$0.472 < \alpha \leq 0.758 \Rightarrow [\tilde{A}_1, \tilde{A}_4, \tilde{A}_5, \tilde{A}_6, \{\tilde{A}_2, \tilde{A}_3\}]$  
$0.758 < \alpha \leq 0.796 \Rightarrow [\tilde{A}_1, \tilde{A}_3, \tilde{A}_5, \tilde{A}_6, \{\tilde{A}_2, \tilde{A}_4\}]$  
$0.796 < \alpha \leq 0.838 \Rightarrow [\tilde{A}_1, \tilde{A}_5, \tilde{A}_6, \{\tilde{A}_2, \tilde{A}_3, \tilde{A}_4\}]$  
$0.838 < \alpha \leq 0.873 \Rightarrow [\tilde{A}_1, \tilde{A}_5, \tilde{A}_6, \{\tilde{A}_2, \tilde{A}_4, \tilde{A}_3\}]$  
$0.873 < \alpha \leq 0.923 \Rightarrow [\tilde{A}_5, \tilde{A}_6, \{\tilde{A}_1, \tilde{A}_4, \tilde{A}_2\}]$  
$0.923 < \alpha \leq 0.976 \Rightarrow [\tilde{A}_5, \tilde{A}_6, \{\tilde{A}_1, \tilde{A}_4, \tilde{A}_2\}]$  
$0.976 < \alpha \leq 1 \Rightarrow \{\tilde{A}_5, \tilde{A}_6\}$

Based on the above two clustering results, our proposed similarity measure first separates the data point $\tilde{A}_6$ from the other data points when the $\alpha$ level is with $0.274 < \alpha \leq 0.35$ and then groups $\{\tilde{A}_5, \tilde{A}_6\}$ with $\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4\}$ as another class, and so forth. However, Hung and Yang [10] separates the data point $\tilde{A}_2$ from the other data points when the $\alpha$ level is with $0.229 < \alpha \leq 0.472$ and then groups $\{\tilde{A}_1, \tilde{A}_2\}$ with $\{\tilde{A}_3, \tilde{A}_4, \tilde{A}_5, \tilde{A}_6\}$ as another class. From the structure of this Gaussian type-2 fuzzy data set $[\tilde{A}_j, j = 1, 2, 3, 4, 5, 6]$, we find that these clustering results based on the proposed similarity $S(\tilde{A}, \tilde{B})$ is logically more expressed than these results with Hung and Yang’s [10] method in a hierarchical tree structure according to different $\alpha$-levels. In summary, our proposed method actually shows more rational clustering results, compared with the results from [10].
5. Conclusions

In this paper, we propose new similarity, inclusion and entropy measures between type-2 fuzzy sets based on the Sugeno integral. By using examples, we present the calculations of these measures between type-2 fuzzy sets. We also make comparisons of the proposed measures with some existing methods, such as [10, 11]. In a practical example, we embed the proposed similarity measure with Yang and Shih’s [26] algorithm and then apply it for clustering type-2 fuzzy data. The comparisons between the proposed method and Hung and Yang’s [10] method are made. The comparison results show that the proposed similarity measure presents better clustering results than those of [10]. Overall, the proposed method is recommended to combined with Yang and Shih’s [26] algorithm as a clustering method for type-2 fuzzy data patterns.

References