New Construction for Similarity Measures between Intuitionistic Fuzzy Sets Based on Lower, Upper and Middle Fuzzy Sets

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Abstract

Similarity of intuitionistic fuzzy sets is an important measure to indicate the similarity degree between intuitionistic fuzzy sets. Based on the information carried by transforming intuitionistic fuzzy sets into their lower, upper and middle fuzzy sets, we propose a new construction for similarity measures between intuitionistic fuzzy sets. The proposed construction method on similarity measures between intuitionistic fuzzy sets satisfies the definition of a similarity between intuitionistic fuzzy sets. Moreover, we can reconstruct most existing similarity measures between intuitionistic fuzzy sets based on our new construction technique so that they can improve these original similarity measures to be better similarities. To demonstrate this construction efficiency, the comparisons of these reconstructed measures with original similarity measures between intuitionistic fuzzy sets are made. The comparison results show that our construction methods actually improve most existing similarity measures between intuitionistic fuzzy sets.

Keywords: Fuzzy set, intuitionistic fuzzy set, similarity measure, lower, upper and middle fuzzy sets.

1. Introduction

Fuzzy sets, first coined by Zadeh [1], give an approach for treating fuzziness. A fuzzy set is a generalization of an ordinary set in which it allows the membership value of an element in a fuzzy set to be between 0 and 1 so that it could convey partial memberships of belongingness described by a membership function. Although a membership function in a fuzzy set assigns a value between 0 and 1 as a partial membership of belongingness, the degree of non-belongingness in the fuzzy set is always defined as the complement to 1 of the membership degree. However, humans who express the degree of membership of a given element in a fuzzy set may not express a corresponding degree of non-membership as the complement to 1. Thus, Atanassov [2] introduced the concept of an intuitionistic fuzzy set (IFS) which is a generalization of a fuzzy set. Since an IFS can present the degrees of membership and non-membership with a degree of hesitancy, knowledge and semantic representation become more meaningful and applicable [3-5]. These IFSs have been widely studied and applied in various areas such as decision making [6], mathematical programming [7], medical diagnosis [8], clustering [9], metric space [10] and pattern recognition [11].

Similarity measures are an important tool for determining the degree of similarity between two objects. Kaufman and Rousseeuw [12] presented some examples to illustrate traditional similarity measure applications in hierarchical cluster analysis. Different similarity measures between fuzzy sets have been proposed and similarity measures between IFSs are also widely studied in the literature. Dengfeng and Chuntian [13] proposed some similarity measures between IFSs used in pattern recognition. Liang and Shi [14] proposed similarity measures between IFSs and also discussed the relationships between these measures with applications to pattern recognition. In Liang and Shi [14], they used numerical comparisons to show that Liang and Shi’s similarity measures are more reasonable than those of Dengfeng and Chuntian. Mitchell [15] interpreted IFSs as ensembles of ordered fuzzy sets from the statistical point of view to modify Dengfeng and Chuntian’s methods. Hung and Yang [16] proposed several similarity measures between IFSs based on Hausdorff distance which are well used with linguistic variables. Xu and Chen [17] gave comprehensive overview and comparisons of distance and similarity measures between IFSs. In general, the information in IFSs can be carried by its lower, upper and middle fuzzy sets. In this paper, we propose a new construction for similarity...
measures between IFSs based on the lower, upper and middle fuzzy sets so that we can reconstruct these existing similarity measures between IFSs to be better similarities.

The remainder of this paper is organized as follows. In Section 2, we first briefly review IFSs. We then review the definition of a similarity between IFSs with several existing similarity measures. In Section 3, we propose a new construction for similarity measures between IFSs based on the lower, upper and middle fuzzy sets transformed from the IFSs. We claim that the constructed similarity measures between IFSs always satisfy the similarity definition. In Section 4, some examples are illustrated and the comparisons are made with those existing methods. Finally, conclusions are stated in Section 5.

2. Intuitionistic Fuzzy Set and Similarity Measures

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be the universe of discrete discourses. Consider two intuitionistic fuzzy sets (IFSs) \( \tilde{A} \) and \( \tilde{B} \) in \( X \). We first describe the aspects of IFSs discussed by Atanassov [2] as follows.

Definition 1 (Atanassov [2]): An intuitionistic fuzzy set (IFS) \( A \) in \( X \) is defined as \( A = \{(x, \mu_A(x), v_A(x))|x \in X\} \) where \( \mu_A: X \rightarrow [0,1] \) and \( v_A: X \rightarrow [0,1] \) with the condition \( 0 \leq \mu_A + v_A(x) \leq 1, \forall x \in X \). The numbers \( \mu_A(x) \) and \( v_A(x) \) denote the degree of membership and non-membership of \( x \) to \( A \), respectively.

For each IFS \( \tilde{A} \) in \( X \), the number \( \pi_A(x) = 1 - \mu_A(x) - v_A(x) \) denotes a hesitancy degree of \( x \) to \( \tilde{A} \). It is called the intuitionistic index of \( x \) in \( \tilde{A} \).

Obviously, \( 0 \leq \pi_A(x) \leq 1 \) for each \( x \in X \). In this paper, we use IFSs(X) to denote the class of all IFSs of \( X \).

Definition 2: If \( \tilde{A} \) and \( \tilde{B} \) are two IFSs of \( X \), then

(i) \( \tilde{A} \subseteq \tilde{B} \) if and only if \( \forall x \in X, u_{\tilde{A}}(x) \leq u_{\tilde{B}}(x) \) and \( v_{\tilde{A}}(x) \geq v_{\tilde{B}}(x) \).

(ii) \( \tilde{A} = \tilde{B} \) if and only if \( \forall x \in X, u_{\tilde{A}}(x) = u_{\tilde{B}}(x) \) and \( v_{\tilde{A}}(x) = v_{\tilde{B}}(x) \).

Measuring a similarity between IFSs is important in IFSs researches. Some methods had been proposed to calculate the similarity degree between IFSs where Li et al. [18] introduced the following definition.

Definition 3 (Li et al. [18]): A mapping \( S: \text{IFSS}(X) \times \text{IFSS}(X) \rightarrow [0,1] \), \( S(\tilde{A}, \tilde{B}) \) is said to be the degree of similarity between \( \tilde{A} \) and \( \tilde{B} \) in IFSs(X) if \( S(\tilde{A}, \tilde{B}) \) satisfies the following properties:

(S1) \( 0 \leq S(\tilde{A}, \tilde{B}) \leq 1 \);

(S2) \( S(\tilde{A}, \tilde{B}) = 1 \) if \( \tilde{A} = \tilde{B} \);

(S3) \( S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A}) \);

(S4) \( S(\tilde{A}, \tilde{C}) \leq S(\tilde{A}, \tilde{B}) \) and \( S(\tilde{A}, \tilde{C}) \leq S(\tilde{B}, \tilde{C}) \) if \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \) in IFSs(X).

(S5) \( S(\tilde{A}, \tilde{B}) = 0 \) if \( \tilde{A} = \emptyset \) and \( \tilde{B} = \emptyset \) or \( \tilde{A} = \emptyset \) and \( \tilde{B} = X \).

Note that Mitchell [15] introduced (S2) by replacing the weak version “\( S(\tilde{A}, \tilde{B}) = 1 \) if \( \tilde{A} = \tilde{B} \)” that was originally proposed by Dengfeng and Chuntian [13].

Consider two IFSs \( \tilde{A} \) and \( \tilde{B} \) in IFSs(X), Dengfeng and Chuntian [13] proposed a similarity measure between them as follows:

\[
S_{DC}(\tilde{A}, \tilde{B}) = 1 - \frac{1}{n} \sum_{i=1}^{n} |m_{\tilde{A}}(i) - m_{\tilde{B}}(i)|^p
\]

where \( m_{\tilde{A}}(i) = (u_{\tilde{A}}(x_i) + 1 - v_{\tilde{A}}(x_i)) / 2 \) and \( 1 \leq p < \infty \) when \( p = 1 \).

Hong and King [20] and Fan and Zhangyan [21] proposed new similarity measures \( S_{D}(\tilde{A}, \tilde{B}) \), \( S_{L}(\tilde{A}, \tilde{B}) \), and \( S_{O}(\tilde{A}, \tilde{B}) \) (also see Li et al. [18]) as follows:

\[
S_{D}(\tilde{A}, \tilde{B}) = 1 - \frac{1}{n} \sum_{i=1}^{n} |(u_{\tilde{A}}(x_i) - v_{\tilde{A}}(x_i)) - (u_{\tilde{B}}(x_i) - v_{\tilde{B}}(x_i))| + |v_{\tilde{A}}(x_i) - v_{\tilde{B}}(x_i)|
\]

\[
S_{L}(\tilde{A}, \tilde{B}) = 1 - \left[ \frac{1}{n} \sum_{i=1}^{n} (u_{\tilde{A}}(x_i) - u_{\tilde{B}}(x_i))^2 + (v_{\tilde{A}}(x_i) - v_{\tilde{B}}(x_i))^2 \right]^{1/2}
\]

Liang and Shi [14] proposed a similarity measure between \( \tilde{A} \) and \( \tilde{B} \) in IFSs(X) as follows:

\[
S_{D}(\tilde{A}, \tilde{B}) = 1 - \frac{1}{n} \sum_{i=1}^{n} \left( \phi_{\tilde{A}}(i) + \phi_{\tilde{B}}(i) \right)^p
\]

where \( \phi_{\tilde{A}}(i) = |u_{\tilde{A}}(x_i) - u_{\tilde{B}}(x_i)| / 2 \) and \( \phi_{\tilde{B}}(i) = |1 - v_{\tilde{A}}(x_i) - (1 - v_{\tilde{B}}(x_i))| / 2 \) and \( 1 \leq p < \infty \). To get more information on IFSs, Liang and Shi [14] gave another similarity measure between \( \tilde{A} \) and \( \tilde{B} \) in IFSs(X) as follows:

\[
S_{D}(\tilde{A}, \tilde{B}) = 1 - \frac{1}{n} \sum_{i=1}^{n} \left( \phi_{\tilde{A}}(i) + \phi_{\tilde{B}}(i) \right)^p
\]

where \( \phi_{\tilde{A}}(i) = |m_{\tilde{A}}(x_i) - m_{\tilde{B}}(x_i)| / 2 \) and \( \phi_{\tilde{B}}(i) = |m_{\tilde{A}}(x_i) - m_{\tilde{B}}(x_i)| / 2 \) with

\[
m_{\tilde{A}}(i) = (u_{\tilde{A}}(x_i) + m_{\tilde{A}}(i)) / 2, \quad m_{\tilde{B}}(i) = (m_{\tilde{A}}(i) + 1 - v_{\tilde{A}}(x_i)) / 2,
\]
\[ m_{\tilde{A}}(i) = \frac{u_{\tilde{A}}(x_i) + m_{\tilde{B}}(i)}{2} \quad \text{and} \quad m_{\tilde{C}}(i) = \frac{m_{\tilde{B}}(i) + 1 - v_{\tilde{B}}(x_i)}{2}. \]

Liang and Shi [14] also gave a similarity measure between \( \tilde{A} \) and \( \tilde{B} \) in IFSSs(\( X \)) as follows:
\[ S^p_{\tilde{A},\tilde{B}}(1) = \frac{1}{\sqrt{q^n}} \left[ \sum_{i=1}^{n} \left( \frac{1}{q^n} \sum_{i=1}^{n} \phi_i(x)_i - \phi_i(x)_i \right)^p \right]^{\frac{1}{p}}, \]
where \( \phi_i(x)_i = \phi_1(x)_i + \phi_2(x)_i \), \( \phi_1(x)_i = m_{\tilde{B}}(i) - m_{\tilde{B}}(i) \), \( \phi_2(x)_i = \max(\ell_1(x)_i, \ell_2(x)_i) - \min(\ell_1(x)_i, \ell_2(x)_i) \) with \( \ell_1(x)_i = (1 - v_1(x)) - u_1(x)/2 \) and \( \ell_2(x)_i = u_2(x) - (1 - v_2(x))/2 \).

Mitchell [15] interpreted IFSSs as ensembles of ordered fuzzy sets from statistical viewpoint to modify Li and Cheng’s methods and proposed a similarity measure between \( \tilde{A} \) and \( \tilde{B} \) in IFSSs(\( X \)) as follows:
\[ S_{\tilde{A},\tilde{B}} = \frac{1}{2} (\rho_1(\tilde{A},\tilde{B}) + \rho_2(\tilde{A},\tilde{B})) \]
where \( \rho_1(\tilde{A},\tilde{B}) = \frac{1}{\sqrt{q^n}} \left[ \sum_{i=1}^{n} u_{\tilde{A}}(x)_i - u_{\tilde{B}}(x)_i \right]^p \) and \( \rho_2(\tilde{A},\tilde{B}) = \frac{1}{\sqrt{q^n}} \left[ \sum_{i=1}^{n} v_{\tilde{A}}(x)_i - v_{\tilde{B}}(x)_i \right]^p \).

Hung and Yang [16] proposed several similarity measures of IFSSs based on Hausdorff distance. For two IFSSs \( \tilde{A} \) and \( \tilde{B} \) in IFSSs(\( X \)), they first defined \( I_1(x)_i = [u_{\tilde{B}}(x)_i, 1 - v_{\tilde{B}}(x)_i] \) and \( I_2(x)_i = [u_{\tilde{B}}(x)_i, 1 - v_{\tilde{B}}(x)_i] \), \( i = 1, ..., n \). The Hausdorff distance \( H(I_1(x)_i, I_2(x)_i) \) between \( I_1(x)_i \) and \( I_2(x)_i \) was then defined as follows:
\[ H(I_1(x)_i, I_2(x)_i) = \max \left\{ |u_{\tilde{A}}(x)_i - u_{\tilde{B}}(x)_i|, |1 - v_{\tilde{A}}(x)_i - (1 - v_{\tilde{B}}(x)_i)| \right\} \]
They defined the distance of \( d_H(\tilde{A},\tilde{B}) \) between \( \tilde{A} \) and \( \tilde{B} \) with \( d_H(\tilde{A},\tilde{B}) = \frac{1}{n} \sum_{i=1}^{n} H(I_1(x)_i, I_2(x)_i) \). In Hung and Yang [16], they proposed three similarity measures of \( \tilde{A} \) and \( \tilde{B} \) as follows:
\[ S_{\tilde{A},\tilde{B}}^1 = 1 - d_H(\tilde{A},\tilde{B}), \]
\[ S_{\tilde{A},\tilde{B}}^2 = \frac{e^{-d_H(\tilde{A},\tilde{B})} - e^{-1}}{1 - e^{-1}}, \]
\[ S_{\tilde{A},\tilde{B}}^3 = 1 - \frac{d_H(\tilde{A},\tilde{B})}{1 + d_H(\tilde{A},\tilde{B})}. \]

Note that the above discussed similarity measures will be used to compare with our proposed similarity measure in Section 4.

3. New Construction Method on Similarity Measures between IFSSs

For any IFSS \( \tilde{A} = \{x, u_{\tilde{A}}(x), v_{\tilde{A}}(x) : x \in X\} \), we first define the lower fuzzy set \( A^- \) and the upper fuzzy set \( A^+ \) to the IFSS \( \tilde{A} \) according to Grzegorzewski [22] as follows:
\[ A^- = \{x, u_{\tilde{A}}(x) : x \in X\}, \quad u_{A^-}(x) = u_{\tilde{A}}(x) \]
\[ A^+ = \{x, u_{\tilde{A}}(x) : x \in X\}, \quad u_{A^+}(x) = u_{\tilde{A}}(x) + 1 - v_{\tilde{A}}(x) \]

Additionally, we define a middle fuzzy set \( A^m \) to the IFSS \( \tilde{A} \) as follows:
\[ A^m = \{x, u_{\tilde{A}}(x) : x \in X\}, \quad u_{A^-}(x) = \frac{u_{\tilde{A}}(x) + 1 - v_{\tilde{A}}(x)}{2} \]

Assume that there is a similarity measure \( S(\tilde{A},\tilde{B}) \) between any two IFSSs \( \tilde{A} \) and \( \tilde{B} \) that satisfies the conditions of Definition 3 in Section 2. We can (OH.5) a new similarity measure \( S_{\tilde{A},\tilde{B}} \) based on the defined lower, upper and middle (lum) fuzzy sets as follows:
\[ S_{\tilde{A},\tilde{B}} = \frac{1}{3} \left( S((A^-)^+, (B^-)^+) + S(A^+, B^+) + S(A^+, B^+) \right) \]

We next prove that the proposed \( S_{\tilde{A},\tilde{B}} \) is a similarity measure if \( S(\tilde{A},\tilde{B}) \) is a similarity measure between two IFSSs \( \tilde{A} \) and \( \tilde{B} \). For all following propositions, we assume that \( S(\tilde{A},\tilde{B}) \) is a similarity measure between \( \tilde{A} \) and \( \tilde{B} \).

Proposition 1: \( 0 \leq S_{\tilde{A},\tilde{B}} \leq 1 \).

Proof: Because \( 0 \leq S((A^-)^+, (B^-)^+) \leq 1 \), \( 0 \leq S(A^+, B^+) \leq 1 \) and \( 0 \leq S(A^+, B^+) \leq 1 \), it is obviously that \( 0 \leq S_{\tilde{A},\tilde{B}} \leq 1 \).

Proposition 2: \( S_{\tilde{A},\tilde{B}} = 1 \) if and only if \( \tilde{A} = \tilde{B} \).

Proof: Since \( S(\tilde{A},\tilde{B}) = 1 \) is a similarity measure between \( \tilde{A} \) and \( \tilde{B} \), by (S2) in Definition 3, we have that \( S(\tilde{A},\tilde{B}) = 1 \) if \( \tilde{A} = \tilde{B} \). Thus, \( S_{\tilde{A},\tilde{B}} = 1 \) if \( S((A^-)^+, (B^-)^+) = 1 \), \( S(A^+, B^+) = 1 \) and \( S(A^+, B^+) = 1 \) if \( (A^-)^+ = (B^-)^+ \), \( A^+ = B^+ \) and \( A^+ = B^+ \) if \( u_{A^-}(x) = u_{B^-}(x) \) and \( v_{A^-}(x) = v_{B^-}(x) \) if \( \tilde{A} = \tilde{B} \).}

Proposition 3: \( S_{\tilde{A},\tilde{B}} = S_{\tilde{B},\tilde{A}} \).

Proof: Since \( S(A^+, B^+) = S(B^+, A^+) \), \( S(A^+, B^+) = S(B^+, A^+) \) and \( S(A^+, B^+) = S(B^+, A^+) \), we have that
\[ S_{\tilde{A},\tilde{B}} = \frac{1}{3} \left( S((A^-)^+, (B^-)^+) + S(A^+, B^+) + S(A^+, B^+) \right) \]
\[ = \frac{1}{3} \left( S((B^-)^+), (A^-)^+ + S(B^+, A^+) + S(B^+, A^+) \right) \]
\[ = S_{\tilde{B},\tilde{A}} \]

Proposition 4: \( S_{\tilde{A},\tilde{B}} \leq S_{\tilde{A},\tilde{B}} \) and \( S_{\tilde{A},\tilde{B}} \leq S_{\tilde{B},\tilde{C}} \) if \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \).

Proof: If \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \), then for all \( x \), we have \( 1 - u_{\tilde{B}}(x) \geq 1 - u_{\tilde{A}}(x) \geq 1 - u_{\tilde{C}}(x) \), \( 1 - v_{\tilde{B}}(x) \leq 1 - v_{\tilde{A}}(x) \leq 1 - v_{\tilde{C}}(x) \).
and 
\[ \frac{1 + u_j(x) - v_j(x)}{2} \leq \frac{1 + u_k(x) - v_k(x)}{2} \leq \frac{1 + u_k(x) - v_j(x)}{2}. \]
It follows that 
\( (A')' \supseteq (B')' \supseteq (C')' \) and 
\( A' \subseteq B' \subseteq C' \) and 
\( A'' \subseteq B'' \subseteq C'' \). Thus, 
\( S(A', C') \leq S((A')', (B')', (C')') \) and 
\( S(A'', C'') \leq S(A', B') \). Hence, 
\( \text{Sim}_{\text{sa}} (A, \tilde{C}) \leq \text{Sim}_{\text{sa}} (A, \tilde{B}) \).
Similarly, we have that 
\( \text{Sim}_{\text{sa}} (\tilde{A}, \tilde{C}) \leq \text{Sim}_{\text{sa}} (\tilde{A}, \tilde{B}) \). □

**Proposition 5:** 
\( \text{Sim}_{\text{sa}} (\tilde{A}, \tilde{B}) = 0 \) if \( \tilde{A} = X \) and \( \tilde{B} = \emptyset \) or \( \tilde{A} = \emptyset \) and \( \tilde{B} = X \).

**Proof:** If \( \tilde{A} = X \) and \( \tilde{B} = \emptyset \), then \( A' = A'' = X \) and \( B' = B'' = \emptyset \). By (S5) in Definition 3, since \( X \) and \( \emptyset \) are crisp sets, we have that 
\( S((A')', (B')') = 0 \), 
\( S(A', B') = 0 \), and 
\( S(A'', B'') = 0 \).

Thus,
\[ \text{Sim}_{\text{sa}} (\tilde{A}, \tilde{B}) = \frac{1}{3} \left( S((A')', (B')') + S(A', B') + S(A'', B'') \right) = 0. \]
Similarly, if \( \tilde{A} = \emptyset \) and \( \tilde{B} = X \), then \( \text{Sim}_{\text{sa}} (\tilde{A}, \tilde{B}) = 0 \). On the other hand, if \( \text{Sim}_{\text{sa}} (\tilde{A}, \tilde{B}) = 0 \), then \( S((A')', (B')') = 0 \), 
\( S(A', B') = 0 \), and 
\( S(A'', B'') = 0 \).

This implies that \( A' = (B')' = A'' = X \) and 
\( B' = (A')' = B'' = \emptyset \) or \( A' = (B')' = A'' = \emptyset \) and 
\( B' = (A')' = B'' = X \). Thus, we get that \( \tilde{A} = X \) and \( \tilde{B} = \emptyset \) or \( \tilde{A} = \emptyset \) and \( \tilde{B} = X \). □

Obviously, we obtain the following Theorem 1 from Propositions 1 to 5.

**Theorem 1:** 
\( \text{Sim}_{\text{sa}} (\tilde{A}, \tilde{B}) \) is a similarity measure if 
\( \text{Sim}(\tilde{A}, \tilde{B}) \) is a similarity measure between IFSs \( \tilde{A} \) and \( \tilde{B} \).

We next make the extensions to most existing similarity measures between two IFSs \( \tilde{A} \) and \( \tilde{B} \) based on our construction method as follows:

\[ S_{\text{sa}} (\tilde{A}, \tilde{B}) = -\frac{1}{2n} \sum_{i=1}^{n} \left| u_j(x_i) - v_j(x_i) \right| \]

is extended to be
\[ S_{\text{sim}} (\tilde{A}, \tilde{B}) = -\frac{1}{3n} \sum_{i=1}^{n} \left| u_j(x_i) - u_k(x_i) \right| + \left| v_j(x_i) - v_k(x_i) \right| \]
\[ + \frac{1}{2} \left| u_j(x_i) - u_k(x_i) + v_j(x_i) - v_k(x_i) \right| \]
\[ S_{\text{dc}} (\tilde{A}, \tilde{B}) = -\frac{1}{2n} \sum_{i=1}^{n} \left( u_j(x_i) - v_j(x_i) \right)^p \]
\[ + \frac{1}{2n} \sum_{i=1}^{n} \left( v_j(x_i) - u_j(x_i) \right)^p \]

is extended to be
\[ S_{\text{sim}} (\tilde{A}, \tilde{B}) = -\frac{1}{3n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) \right)^p + \left( v_j(x_i) - v_k(x_i) \right)^p \]
\[ + \frac{1}{2n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) + v_j(x_i) - v_k(x_i) \right)^p \]
\[ S_{\text{dc}} (\tilde{A}, \tilde{B}) = -\frac{1}{2n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) \right)^p \]
\[ + \frac{1}{2n} \sum_{i=1}^{n} \left( v_j(x_i) - v_k(x_i) \right)^p \]

is extended to be
\[ S_{\text{sim}} (\tilde{A}, \tilde{B}) = -\frac{1}{3n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) \right)^p + \left( v_j(x_i) - v_k(x_i) \right)^p \]
\[ + \frac{1}{2n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) + v_j(x_i) - v_k(x_i) \right)^p \]
\[ S_{\text{dc}} (\tilde{A}, \tilde{B}) = -\frac{1}{2n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) \right)^p \]
\[ + \frac{1}{2n} \sum_{i=1}^{n} \left( v_j(x_i) - v_k(x_i) \right)^p \]

is extended to be
\[ S_{\text{sim}} (\tilde{A}, \tilde{B}) = -\frac{1}{3n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) \right)^p + \left( v_j(x_i) - v_k(x_i) \right)^p \]
\[ + \frac{1}{2n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) + v_j(x_i) - v_k(x_i) \right)^p \]
\[ S_{\text{dc}} (\tilde{A}, \tilde{B}) = -\frac{1}{2n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) \right)^p \]
\[ + \frac{1}{2n} \sum_{i=1}^{n} \left( v_j(x_i) - v_k(x_i) \right)^p \]

is extended to be
\[ S_{\text{sim}} (\tilde{A}, \tilde{B}) = -\frac{1}{3n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) \right)^p + \left( v_j(x_i) - v_k(x_i) \right)^p \]
\[ + \frac{1}{2n} \sum_{i=1}^{n} \left( u_j(x_i) - u_k(x_i) + v_j(x_i) - v_k(x_i) \right)^p \]
where

\[ \phi_x(i) = \left\{ \begin{array}{ll}
\phi_x(i) + \phi_y(i), & |m_{x_1}(x_i) - m_{y_1}(x_i)| / 2 \\
\phi_x(i) - \phi_y(i), & |m_{x_2}(x_i) - m_{y_2}(x_i)| / 2 \\
\phi_y(i), & |m_{x_2}(x_i) + 1 - v_{y_2}(x_i)| / 2, \\
\phi_y(i) + m_{y_2}(x_i) + m_{y_2}(x_i), & |m_{x_1}(x_i) + v_{y_1}(x_i)| / 2 
\end{array} \right. \]

and

\[ \phi_y(i) = \max([-\ell_x(i), -\ell_y(i)]) - \min(\ell_x(i), \ell_y(i)) \]

with

\[ \omega = 1/3, \quad \ell_x(i) = (1 - v_{x_2}(x_i) - u_{x_2}(x_i)) / 2 \quad \text{and} \quad \ell_y(i) = (1 - v_{y_2}(x_i) - u_{y_2}(x_i)) / 2 \]

is extended to be

\[ S_{ext}^p(\tilde{A}, \tilde{B}) = 1 - \frac{1}{3} \left( \frac{1}{n} \sum_{i=1}^{n} |u_{x}(x_i) - u_{y}(x_i)| + |v_{x}(x_i) - v_{y}(x_i)| + \frac{1}{2} |u_{x}(x_i) - u_{y}(x_i) + v_{y}(x_i) - v_{y}(x_i)| \right) \]

(12) \quad S_{ext}^p(\tilde{A}, \tilde{B}) = 1 - \frac{d_{B}(\tilde{A}, \tilde{B})}{1 + d_{B}(\tilde{A}, \tilde{B})}

where \( d_{B}(\tilde{A}, \tilde{B}) = \frac{1}{n} \sum_{i=1}^{n} |H(I_{x}(x_i), I_{y}(x_i))| \) is extended to be

\[ S_{ext}^p(\tilde{A}, \tilde{B}) = 1 - \frac{3n}{2n} \sum_{i=1}^{n} |u_{x}(x_i) - u_{y}(x_i)| + \frac{2n}{3n} |v_{x}(x_i) - v_{y}(x_i)| + \frac{n}{3n} |u_{x}(x_i) - u_{y}(x_i) + v_{y}(x_i) - v_{y}(x_i)| \]

From the previous construction, we extended the existing 12 similarity measures between IFSs. In next section, we will use some examples to demonstrate the efficiency of our construction.

4. Comparisons and Application to Pattern Recognition

In this section, we give examples to compare the proposed construction method on similarity measures with these existing similarity measures between IFSs.

Example 1: Li et al [18] compared and analyzed 12 similarity measures (1)-(12) between IFSs as listed in Section 3. Based on the comparison and analysis, they provided similarity measure expressions and counterintuitive cases, and a demonstration table of counterintuitive cases (bold italic) as Table 1.

From the above similarity measure expressions and counterintuitive cases, it is obvious that \( S_{I}^p \) is the only one with no counterintuitive cases. We find that \( S_{I}^p(\tilde{A}, \tilde{B}) \) does not satisfy the definition of the similarity measure between IFSs \( \tilde{A} \) and \( \tilde{B} \) as shown in Section 3. Furthermore, the similarity measure \( S_{II} \) proposed by Hong and Kim [20] has the counter cases of type I and type II as follows:

Type I: Suppose that \( \tilde{A} = \{(x, 0.3, 0.3)\}, \tilde{B} = \{(x, 0.4, 0.4)\}, \tilde{C} = \{(x, 0.3, 0.4)\} \) and \( \tilde{D} = \{(x, 0.4, 0.3)\} \).

We have \( S_{II}(\tilde{A}, \tilde{B}) = S_{II}(\tilde{C}, \tilde{D}) = 0.9 \) which is not intuitively consistent. The lower fuzzy set corresponding to the IFS \( \tilde{A} \) is given by \( A' = \langle \times, 0.3 \times x \in X \rangle \). The upper fuzzy set corresponding to the IFS \( \tilde{A} \) is given by \( A^\prime = \langle \times, 0.7 \times x \in X \rangle \). The middle fuzzy set corresponding to the IFS \( \tilde{B} \) is given by \( B' = \langle \times, 0.4 \times x \in X \rangle \). The upper fuzzy set corresponding to the IFS \( \tilde{B} \) is given by \( B'' = \langle \times, 0.6 \times x \in X \rangle \). The middle fuzzy set corresponding to the IFS \( \tilde{B} \) is given by \( B'' = \langle \times, 0.5 \times x \in X \rangle \). Thus, we have that \( S_{ext}(\tilde{A}, \tilde{B}) = 0.933 \).
Table 1. Demonstrations of counter-intuitive cases (bold italic).

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(x, 0.3, 0.3)</td>
<td>(x, 0.3, 0.4)</td>
<td>(x, 1, 0)</td>
<td>(x, 0.5, 0.5)</td>
<td>(x, 0.4, 0.2)</td>
<td>(x, 0.4, 0.2)</td>
</tr>
<tr>
<td>B</td>
<td>(x, 0.4, 0.4)</td>
<td>(x, 0.4, 0.3)</td>
<td>(x, 0, 0)</td>
<td>(x, 0, 0)</td>
<td>(x, 0.5, 0.3)</td>
<td>(x, 0.5, 0.2)</td>
</tr>
<tr>
<td>S_C</td>
<td>1</td>
<td>0.9</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>0.95</td>
</tr>
<tr>
<td>S_H</td>
<td>0.9</td>
<td>0.9</td>
<td>0.5</td>
<td>0.5</td>
<td>0.9</td>
<td>0.95</td>
</tr>
<tr>
<td>S_L</td>
<td>0.95</td>
<td>0.9</td>
<td>0.5</td>
<td>0.75</td>
<td>0.95</td>
<td>0.95</td>
</tr>
<tr>
<td>S_D</td>
<td>0.9</td>
<td>0.9</td>
<td>0.3</td>
<td>0.5</td>
<td>0.9</td>
<td>0.93</td>
</tr>
<tr>
<td>S_DC</td>
<td>1</td>
<td>0.9</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>0.95</td>
</tr>
<tr>
<td>S_HB</td>
<td>0.9</td>
<td>0.9</td>
<td>0.5</td>
<td>0.5</td>
<td>0.9</td>
<td>0.95</td>
</tr>
<tr>
<td>S_H</td>
<td>0.95</td>
<td>0.9</td>
<td>0.5</td>
<td>0.75</td>
<td>0.95</td>
<td>0.95</td>
</tr>
<tr>
<td>S_D</td>
<td>0.93</td>
<td>0.93</td>
<td>0.5</td>
<td>0.67</td>
<td>0.93</td>
<td>0.95</td>
</tr>
<tr>
<td>S_H</td>
<td>0.9</td>
<td>0.9</td>
<td>0</td>
<td>0.5</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>S_H</td>
<td>0.85</td>
<td>0.85</td>
<td>0</td>
<td>0.38</td>
<td>0.85</td>
<td>0.85</td>
</tr>
<tr>
<td>S_H</td>
<td>0.82</td>
<td>0.82</td>
<td>0</td>
<td>0.33</td>
<td>0.82</td>
<td>0.82</td>
</tr>
</tbody>
</table>

Similarly, we have $S_{\text{mod}}(\tilde{C},\tilde{D})=0.917$. Hence, this is intuitively consistent.

Type II: Suppose that $\hat{A} = \{(x,1,0,0.0)\}$, $\hat{B} = \{(x,0,0,0.0)\}$ and $\tilde{C} = \{(x,0.5,0.5)\}$. We have $S_H(\hat{A},\hat{B}) = S_H(\tilde{C},\tilde{B})=0.5$, which is not intuitively consistent. Furthermore, we have $S_{\text{mod}}(\hat{A},\tilde{B})=0.5$. Similarly, we have $S_{\text{mod}}(\hat{B},\tilde{C})=0.667$. Hence, this is intuitively consistent. We can similarly construct these intuitively inconsistent and consistent cases. We set up a new table as shown in Table 2 for our new constructed similarity measures based on lower, upper and middle fuzzy sets.

From Table 2, we find that these re-constructed similarity measures based on our construction method are all intuitively consistent. That is, we actually improve most existing similarity measures between IFSs based on our construction method.

**Example 2:** There are given two known patterns $A_1$ and $A_2$. The patterns in the given finite universe $X = \{x\}$ are represented as follows:

$A_1 = \{(x,0.3,0.3)\}$ and $A_2 = \{(x,0.4,0.4)\}$

Given another pattern $Q$ which is represented by the IFS:

$Q = \{(x,0.2,0.4)\}$

Our aim is to classify the pattern $Q$ in one of the classes $A_1$ and $A_2$. According to the recognition principle of maximum degree of similarity between IFSs, the process of assigning $Q$ to $A_k$ is described by

$$S_{x,y} = \arg \max_{i=1,2} S_{xy}(A_i, Q)$$

From the 12 formula, we can compute these similarity measures between $A_i, i=1,2$ and $Q$ as follows:

- $S_{x}(\tilde{A}_1, Q) = 1, S_{x}(\tilde{A}_2, Q) = 0.9$
- $S_{\text{mod}}(\tilde{A}_1, Q) = 0.933, S_{\text{mod}}(\tilde{A}_2, Q) = 0.9$
- $S_{H}(\tilde{A}_1, Q) = 0.93, S_{H}(\tilde{A}_2, Q) = 0.9$
- $S_{D}(\tilde{A}_1, Q) = 0.95, S_{D}(\tilde{A}_2, Q) = 0.9$
- $S_{DC}(\tilde{A}_1, Q) = 1, S_{DC}(\tilde{A}_2, Q) = 0.95$
- $S_{\text{modDC}}(\tilde{A}_1, Q) = 0.933, S_{\text{modDC}}(\tilde{A}_2, Q) = 0.9$
- $S_{H}(\tilde{A}_1, Q) = 0.9, S_{H}(\tilde{A}_2, Q) = 0.9$
- $S_{\text{modDC}}(\tilde{A}_1, Q) = 0.933, S_{\text{modDC}}(\tilde{A}_2, Q) = 0.9$
- $S_{D}(\tilde{A}_1, Q) = 0.93, S_{D}(\tilde{A}_2, Q) = 0.9$
- $S_{DC}(\tilde{A}_1, Q) = 1, S_{DC}(\tilde{A}_2, Q) = 0.95$
- $S_{\text{modDC}}(\tilde{A}_1, Q) = 0.933, S_{\text{modDC}}(\tilde{A}_2, Q) = 0.9$
- $S_{D}(\tilde{A}_1, Q) = 0.9, S_{D}(\tilde{A}_2, Q) = 0.9$
- $S_{DC}(\tilde{A}_1, Q) = 1, S_{DC}(\tilde{A}_2, Q) = 0.95$
- $S_{\text{modDC}}(\tilde{A}_1, Q) = 0.933, S_{\text{modDC}}(\tilde{A}_2, Q) = 0.9$
Table 2. Demonstrations from our construction.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{A}$</td>
<td>(x, 0.3, 0.3)</td>
<td>(x, 0.3, 0.4)</td>
<td>(x, 1.0)</td>
<td>(x, 0.5, 0.5)</td>
<td>(x, 0.4, 0.2)</td>
<td>(x, 0.4, 0.2)</td>
</tr>
<tr>
<td>$\mathbf{B}$</td>
<td>(x, 0.4, 0.4)</td>
<td>(x, 0.4, 0.3)</td>
<td>(x, 0.0)</td>
<td>(x, 0.0)</td>
<td>(x, 0.5, 0.3)</td>
<td>(x, 0.5, 0.2)</td>
</tr>
<tr>
<td>$S_{lumC}$</td>
<td>0.933</td>
<td>0.9</td>
<td>0.5</td>
<td>0.667</td>
<td>0.933</td>
<td>0.95</td>
</tr>
<tr>
<td>$S_{lumH}$</td>
<td>0.933</td>
<td>0.917</td>
<td>0.5</td>
<td>0.667</td>
<td>0.933</td>
<td>0.95</td>
</tr>
<tr>
<td>$S_{lumL}$</td>
<td>0.933</td>
<td>0.909</td>
<td>0.5</td>
<td>0.667</td>
<td>0.933</td>
<td>0.95</td>
</tr>
<tr>
<td>$S_{lumO}$</td>
<td>0.933</td>
<td>0.9</td>
<td>0.5</td>
<td>0.667</td>
<td>0.933</td>
<td>0.95</td>
</tr>
<tr>
<td>$S_{lumDC}$</td>
<td>0.933</td>
<td>0.9</td>
<td>0.5</td>
<td>0.667</td>
<td>0.933</td>
<td>0.95</td>
</tr>
<tr>
<td>$S_{lumHC}$</td>
<td>0.933</td>
<td>0.917</td>
<td>0.5</td>
<td>0.667</td>
<td>0.933</td>
<td>0.95</td>
</tr>
<tr>
<td>$S_{lumC1}$</td>
<td>0.933</td>
<td>0.917</td>
<td>0.5</td>
<td>0.667</td>
<td>0.933</td>
<td>0.95</td>
</tr>
<tr>
<td>$S_{lumC2}$</td>
<td>0.933</td>
<td>0.917</td>
<td>0.5</td>
<td>0.667</td>
<td>0.933</td>
<td>0.95</td>
</tr>
<tr>
<td>$S_{lumC3}$</td>
<td>0.933</td>
<td>0.917</td>
<td>0.5</td>
<td>0.667</td>
<td>0.933</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Intuitively, the pattern Q should be classified in $A$. According to the recognition principle of maximum degree of similarity between IFSs, most 12 similarity measures between IFSs have classify the pattern Q to $A$, except $H$, $H_2$, $e$, and $s$. We find that, based on our construction, we have improve these similarity measures $S_{lumC}$, $S_{lumH}$, $S_{lumL}$, and $S_{lumO}$ to have correct recognition.

5. Conclusions

Adopting the measures of lower, upper and middle fuzzy sets corresponding to IFSs, we give a new construction on most existing similarity measures between IFSs. We find that our new construction can improve and also correct their counterintuitive cases for most existing similarity measures between IFSs. As a whole, our proposed construction method for similarity measures between IFSs can provide a more effective way for measuring similarity degrees between IFSs. In the future, we will consider clustering objects in uncertain and ambiguous environments using our proposed new construction method.

References


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