Precise Geometrical Correspondence to the Perfect 5th Tuned to the 432Hz Frequency

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Abstract

In this paper, we show how the 12 notes of the octave have inherent decimal references that correspond precisely to the internal angles of regular polygonal shapes that exhibit symmetries found abundantly in nature. This exact correspondence manifests itself only when the standard pitch tuning is set to 432Hz instead of the modern 440Hz, which is one indicator on the importance of this tuning, from a mathematical perspective at least.

I. INTRODUCTION

In their attempt to understand the nature of music, scientists and philosophers tried to formulate the experience within a mathematical or numerical context.

Most famous of these attempts is that of the Greek Philosopher Pythagoras[1] where one legend goes that while passing by a hammersmith shop, Pythagoras realized that specific hammers produced pleasant sounds when struck together while others didn't. He observed that for a weight ratio of 2:1, the two hammers produced a consonance note, same as for ratios of 4:3 and 3:2, etc. while for other ratios, the notes were dissonance.

Whether this legend is true or not, it doesn't matter as what matters is that the key concept is correct; specific ratios do create pleasant notes while others don't[2]. And thus, came the concept of the octave with ratios of (2:1), the perfect 4th (4:3), the perfect 5th (3:2), etc.

The figure below illustrates the main notes listed as string ratios for one octave.

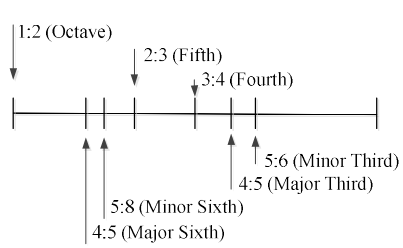

Fig.1: The various notes ratios for a string with their respective names.

Studying individual sound tones can be achieved by studying their individual physical qualities[3], such as frequency, wavelength, amplitude, etc.

For music, however, the task is more complicated as we need to study the phenomenon as a whole and to do so while taking time as a relevant variable, as music is not a momentary experience; it can only be perceived over time.

While putting forth a mathematical model to describe some aspects of music is feasible, as we saw earlier, still, formalizing the whole experience by the simple ratios of physical qualities or quantities is almost an impossible task. There are so much involved in the music experience that adds more variables and dimensions to its theory.

In this paper, we investigate one of these dimensions by illustrating the connection the musical notes have with the geometry of certain regular polygons.

II. GEOMETRICAL CORRESPONDENCE TO THE MUSICAL OCTAVE

Music is composed of individual sound notes superimposed over time[4]. One of the most effective ways to quantify these notes is by measuring their frequencies; the rate of one complete vibration per second, measured in units of inverse second or Hertz (Hz). These notes repeat within a harmonic ratio called *the octave* calculated by doubling or halving any Hertz value.

There is a total of 12 notes in the Western musical scale with the 13^{th} one being equivalent to the first, only an octave higher[5].

For example, notes A_4 = 216 Hz, A_5 = 432 Hz, and A_6 = 864 Hz, are all different octaves or pitches of the same A note, and where the subscripts' numbers are referring to their respective octave ranges.

Notice that we used the tuning pitch of A_5 = 432Hz instead of the modern one of 440Hz. The 432Hz is necessary for us in order to see the direct connections between geometry and sound.

Below is a figure where two whole octaves are tuned to the 432Hz pitch. Notice how moving up the note scale requires progressively smaller distances, like guitar frets.

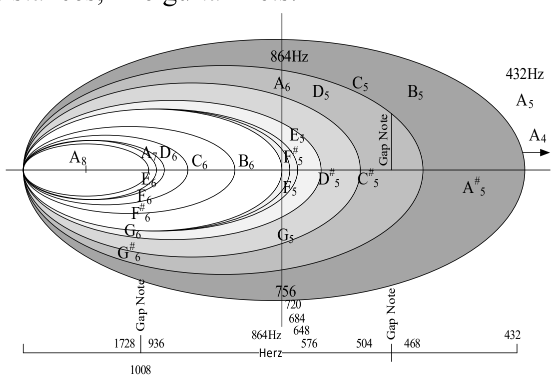

Fig.2: Two whole octaves and their relative notes. Notice how the distances between the notes get smaller as we go form the 5^{th} to the 6^{th} octave.

Connecting sound with shapes is not a new idea. It is well established that certain frequencies of sound create specific geometrical patterns, such as in Chladni figures[6] (shown below) in reference to the German physicist and musician Ernst Chladni (1756 - 1827), who generated geometrical shapes using violin bow drawn over a piece of metal covered with sand. They can also be produced through an instrument called the Harmonograph[7].

Today, these shapes are commonly known as Cymatics[8].

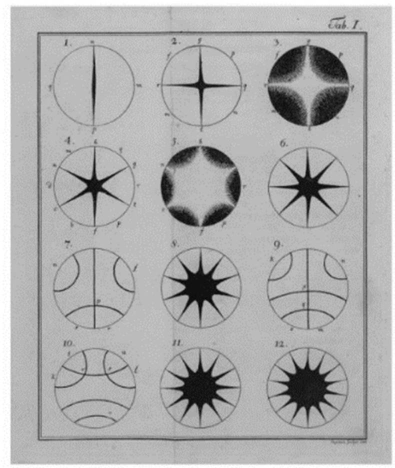

Fig.3: A sample of Chladni figures. Notice the perfect symmetry of the geometry.

Producing shapes due to sound may not seem very intriguing at first. After all, sound is basically a vibrational phenomenon, and when a piece of metal with sand beads on it vibrates, these beads will also vibrate and move around forming different shapes.

However, when the vibrations form perfectly symmetrical geometrical patterns, this is when the phenomena become much more interesting.

Even though producing these sound-generated shapes is an easy and uncomplicated process, however, putting forth a general mathematical formalism capable of fully explaining this phenomenon is very difficult and complicated[9]. This is because these shapes not only depend on the sound notes and their properties, they also depend on the material of the vibrating surface, the medium used to create the shapes, the ambient temperature, etc. among many other factors.

However, we will show that, on a very fundamental level, connecting music to geometry, numerically, can be achieved and with a high degree of precision.

We start by setting up the following table which lists the harmonic relationships between notes in the chromatic scale and their degrees references.

NOTE	HZ	DECIMAL (HZ/432)	LENGTH (UNIT)	DECIMAL	DEGREES	INTERNAL ANGLE	SUM OF ANGLES
A_5	432	1.00	12/12	1.00	360˚	0˚	0˚
$A\#_5$	450	1.0416	11/12	0.916	330˚	30˚	30˚
B_5	468	1.083	10/12	0.833	300˚	60˚	180˚
C_5	504	1.166	9/12	0.750	270˚	90˚	360˚
GAPI							540˚
$C\#_5$	540	1.250	8/12	0.666	240˚	120˚	720˚
D_5	576	1.333	7/12	0.583	225˚	135˚	1080˚
$D\#_5$	612	1.416	6.75/12	0.562	210˚	150˚	1800˚
E_5	648	1.500	6.48/12	0.540	195˚	165˚	3960˚
F_5	684	1.583	6.24/12	0.520	187.5˚	172.5˚	8280˚
$F\#_5$	720	1.666	6.18/12	0.517	186.32˚	173.68˚	9900˚
G_5	756	1.750	6.124/12	0.510	184.76˚	175.24˚	13140˚
$G\#_5$	792	1.833	6.062/12	0.505	182.4˚	177.6˚	26640˚
GAPII	828	1.916	6.039/12	0.503	181.2˚	178.8˚	53640˚
A_6	864	2.00	6/12	0.5	180˚	Next Octave	0

The *Length* field refers to the various notes' ratios for a string length of 12 units.

By taking the values of the *Length* field in *Decimal* and multiply them by 360, we get the *Degrees* reference for the notes. For example, 11/12 = 0.916 and 0.916×360˚ = 330˚, and so is for the rest of the notes.

The Internal Angle field is calculated by subtracting the *Degrees* field from that of the starting point, being the note A_5. Therefore, for the note B_5 (468hz), the internal angle will be A_5° – B_5° or 360°-300° = 60°, which corresponds to an equilateral triangle a shown in Figure (4) below.

Moving a "half-step" from A_5 to $A_{\#5}$, spans a 30° angle, a shape commonly used as a symbol for the compass.

The note C_5 produces an inscribed square since its internal angle of 90° is 1/4th of the 360°-unit circle. The note $C_{\#5}$ produces an inscribed hexagon since the angle is 120°.

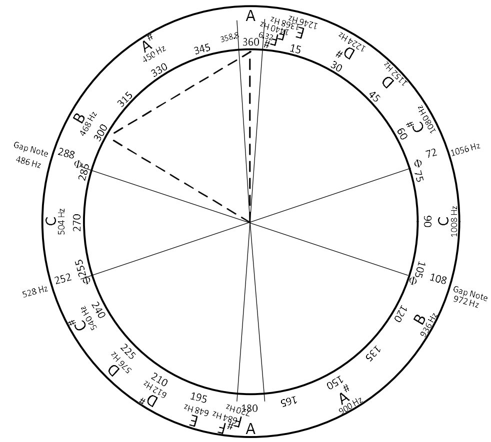

Fig.4: The tonal unit circle of the perfect 5th with note B_5 creating an equilateral triangle with the note A_5.

The pattern continues through the audible spectrum of sound and is calculated for one full octave as shown in the table below. The *Arc Length* field is calculated by measuring the distance around the circle from the starting point (note A_5). For example, the note $C_{\#5}$ produces an inscribed hexagon since the arc length of 60° is 1/6th of the unit circle, meaning there would be a total of 6 equal sides to the polygon, and so for the rest of the notes.

NOTE	INTERNAL ANGLE	ARC LENGTH	POLYGON
A_5	0◦	0˚	Line
$A_{\#5}$	30◦	30˚	Angle/Compass
B_5	60◦	60˚	Triangle
C_5	90◦	90˚	Square
$C_{\#5}$	120◦	60˚	Hexagon
D_5	135◦	45˚	Octagon
$D_{\#5}$	150◦	30˚	12-sided
E_5	165◦	15˚	24-sided
F_5	172.5◦	7.5˚	48-sided
$F_{\#5}$	173.68◦	6.32˚	57-sided
G_5	175.24◦	4.76˚	75-sided
$G_{\#5}$	177.6◦	2.4˚	150-sided

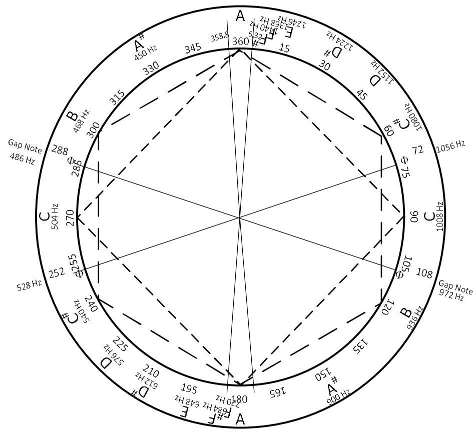

Fig.5: Note C and C# create a square and a hexagon respectively.

All further progressions of regular polygonal geometry follow with perfect symmetry as shown in figure (6) below. Notes F_5, $F_{\#5}$, G_5, and $G_{\#5}$ correlate to polygons (48-, 57-, 75-, and 150-sided polygons) that are not shown as they closely resemble the circle.

The $F_{\#5}$ note is particularly interesting with its length being 6.18/12. The number 6.18 is equivalent to $10/\Phi$, where Φ is the famous golden section[10,11]. Hence $F_{\#5}$ can be written as $\frac{10}{12\times\phi} = 0.8333\,\phi$. Also, the value of 6.18/12 = 0.515 is almost identical to Φ/π.

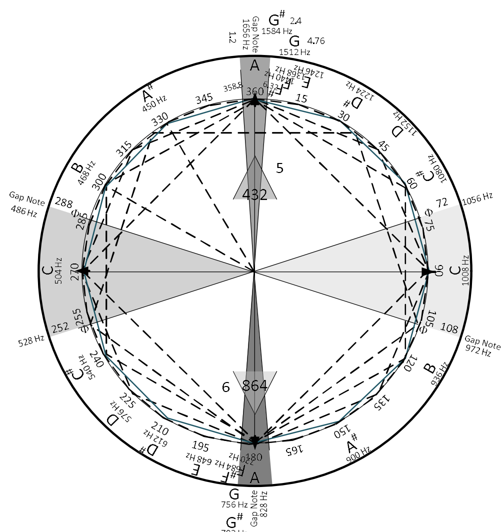

Fig. 6: All notes values plotted around a circle, connecting the musical scale to arc lengths, interior angles, and natural progressions of regular polygons.

The pentagon seems to relate to “dissonant” or enharmonic notes, or gap notes like $B_{\#5}$, which are not used in the classical chromatic scale of music and therefore this shape is missing from the list of the polygons.

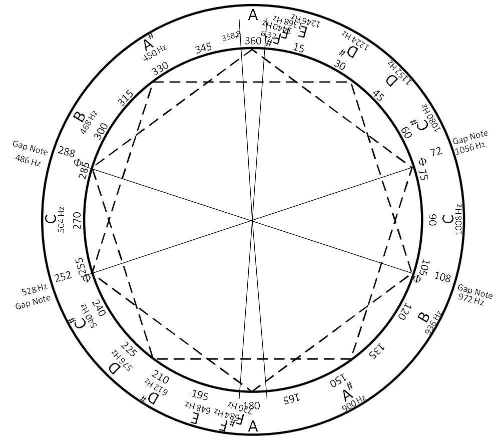

Fig. 7: Two pentagons, one in reverse to the other, denoting the dissonant or gaps on the musical scale.

Also missing from the list is the heptagon, the nonagon, and the hendecagon. Interestingly, these shapes correspond to symmetries that are not observed in nature[12].

(The decagon is also missing; however, we can think of the double pentagons of the gap notes as one decagon.)

Pentagons, however, are very much observed in nature[13] and on many levels. Thus, maybe we should reconsider those gap notes that correspond to pentagons and reevaluate their role. They may not be pleasant to our ears; however, they may play an essential role in nature in a way that we do not yet comprehend.

Interestingly, the two notes that are used to tune the musical scale, A_5 and C_5, correspond to the shapes of the circle and the square, which are two opposites that have the same sum of internal angles (360°). (These are the same two shapes that mathematician and philosophers have struggled to unify in what is generally known as *squaring the circle* problem[12].)

Thus, for the notes of the octave to have this exact correspondence with polygonal symmetries that are found abundantly in nature, while those that are not exhibited in nature have no musical parallel, this indicates a very profound link between geometry and music on a level that we may not fully understand at the moment. This topic definitely requires much more study and attention.

IV. CONCLUSION

By implementing the pitch tuning of 432 Hz for the A_5 note, we were able to establish a profound connection between the 12 notes of the octave and regular geometrical polygons of natural symmetries. The exactness of the correspondence provides us with a new perspective toward music, geometry, and the role they play in nature.

Moreover, the above correspondence is urging us to reconsider the standard pitch tuning value; as when working with the modern value of 440 Hz, the geometrical correspondence disappears completely.

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