

July 2014

Rethinking Portfolio Optimization

Harry Markowitz's Mean-Variance (MV) framework is a foundation of Modern Portfolio Theory (MPT) and perhaps the best-known optimization framework. Consequently, our analysis focuses primarily there, but the framework is applicable to optimization models in general. Although MPT has attracted many followers since its introduction in 1952, it has also drawn criticism. Today, even though it is still accepted as a primary theoretical framework for portfolio construction, its usage by investment professionals is not as ubiquitous as some textbooks might have suggested. In this paper, we argue that full-scale mean variance optimization—a portfolio construction process that relies on the classic mean-variance approach—is not optimal and tends to produce fragile portfolios. In particular, typical usage of return distribution assumptions and the fact that such optimizations are tuned toward specific forecasts form a shaky foundation for MV optimization, in our opinion. We believe that MV optimization can only be used in combination with fundamental judgments or within partial portfolio constructions. At PNC, we rely on our four pillars of investment management to develop robust strategic and tactical asset allocations that best match clients' objectives and risk budgets while avoiding fragile solutions. PNC's four pillars of investment management are discussed in more detail later in this paper.

We believe many modern adoptions of the MV model have missed some of the crucial points in Markowitz's paper "Portfolio Selection," originally published in *The Journal of Finance* in 1952. In the paper's first paragraph, Markowitz stated that the MV model is only the second stage of portfolio construction, and "the first stage starts with observation and experience that end with beliefs about the future performances of available securities." Further, he stated, "to use the E-V (Expected returns-Variance of returns) rule in the selection of securities we must have procedures for finding reasonable μ_i and σ_{ij} . These procedures, I believe, should combine statistical techniques and the judgment of practical men. Our feeling is that the statistical computations should be used to arrive at a tentative set of μ_i and σ_{ij} . Judgment should then be used in increasing or decreasing some of these μ_i and σ_{ij} on the basis of factors or nuances not taken into account by the formal computations." To us, it seemed that Markowitz was aware of the danger of using the mean-variance approach without fundamental judgment and the importance of reliable model inputs.

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The Mean Variance Optimization Process Illustrated

With MPT, portfolio performance is measured as expected portfolio return, and risk is measured as portfolio volatility. The objective of the MV model is to maximize performance with constraints of risk or to minimize risk with

constraints of performance:

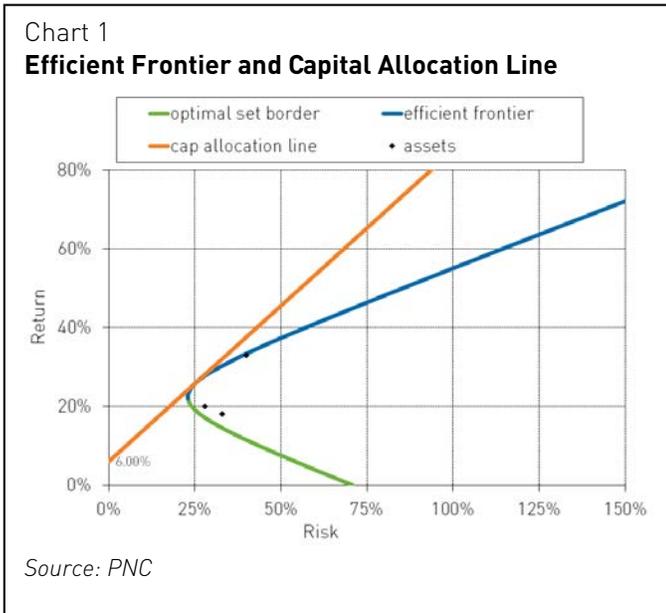
$$\min x^T V x \quad \text{or} \quad \max \mu^T x$$

with $\mu^T x \geq \bar{\mu}$ with $x^T V x \leq \bar{\sigma}^2$

where

- x is the set of asset weights;
- V is the variance-covariance matrix of the returns;
- μ is the set of discounted expected nominal returns of the assets;
- $\bar{\mu}$ is targeted portfolio return; and
- $\bar{\sigma}^2$ is the targeted portfolio variance.

In our opinion, the major contributions of the MV model are its demonstration of the power of portfolio diversification and its introduction of the **Efficient Frontier (EF)**. In Chart 1, the EF is marked in blue. The orange



line above it is a **Capital Allocation Line (CAL)**, which shows possible combinations of a risky portfolio and the risk-free asset (each risky portfolio has its own CAL). Using Tobin's Separation Theorem, the investor can first draw the EF and then find the CAL that is tangent to the EF. Since the slope of the CAL equals its risky portfolio's Sharpe ratio¹, the tangent portfolio would have the largest Sharpe ratio. In theory, the tangent portfolio is the optimal portfolio.

Our analysis of the MV framework would be incomplete without a discussion about several major shortfalls of the process that we believe render it unsuitable for use as the sole input in building portfolios.

Shortfalls of the MV Model

Unrealistic Assumptions

We define four significant unrealistic assumptions about

the MV model:

Efficient frontier is the positively sloped portion of the minimum-variance frontier. It represents a set of portfolios that maximize expected return at each level of portfolio risk.

- The classic MV model assumes that asset returns are normally distributed. Although normality is an assumption needed to prove the optimality of the mean-variance approach, many practitioners have questioned this assumption. Numerous empirical studies have shown that asset returns are rarely normally distributed. In fact, distributions of historical returns tend to have fatter tails; that is, extreme events tend to happen more often than what the normal distribution suggests. Therefore, the normality assumption is unrealistic.
- The time horizon of the MV model is not well defined. A typical MV model assumes investors care only about one investment period, holding every parameter static. Missing considerations of changes of the discount rate or correlations of the assets, this assumption makes the pure MV model myopic, in our view.

¹ Developed by Nobel laureate William F. Sharpe to measure risk-adjusted performance, the Sharpe ratio is calculated by subtracting the risk-free rate from portfolio return and then dividing the difference by the standard deviation of the portfolio return.

- Focusing only on mean and variance, the MV model does not focus on tail events and understates their probability. That is, it does not give enough weight to events that are unlikely but extreme. This mindset has been challenged by many in the investment field; for example, Nassim Taleb analyzed the danger of neglecting tail events in his books *The Black Swan: The Impact of the Highly Improbable* and *Antifragile: Things That Gain from Disorder*. We believe investors should be attuned to the risk of ruin from tail events and that this risk can be increased by using a fully optimized mean variance framework, especially if the normality assumption is employed.
- Another fundamental assumption imbedded in the MV model is that variance is an adequate measure of risk. However, first and foremost, variance treats upside variations and downside variations equally, which is not accurate from a risk standpoint; upside variations are gains on the portfolio. In addition, the simple measure of return variance misses many other pieces that would make a portfolio or asset risky.

Capital allocation line is the frontier graph showing all feasible risk-return combinations of a risky and risk-free asset. The portfolio of tangency between the CAL and the EF represents the best risky portfolio because, when combined with the risk-free asset, it provides the highest Sharpe ratio.

Sensitivity to Model Inputs

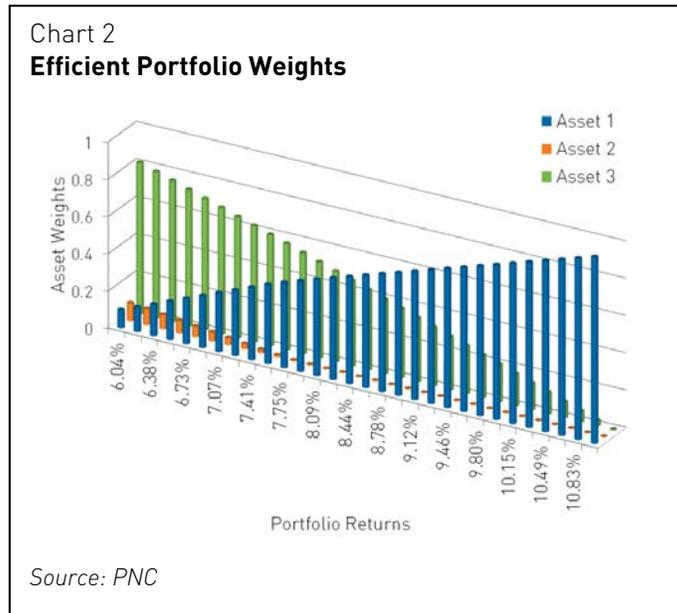
One of the most salient drawbacks of the MV framework, in our opinion, is its high sensitivity to model inputs. Consider this hypothetical example. Suppose we have a simple portfolio of three stocks with expected returns of 11%, 10%, and 5%; variances of 25%, 25%, and 9%; and covariances of Cov(1,2)=22.5%, Cov(1,3)=4.5%, and Cov(2,3)=4.5%. In other words, we have:

$$Expected\ return = \begin{bmatrix} 0.11 \\ 0.10 \\ 0.05 \end{bmatrix}$$

$$Variance\ Covariance\ Matrix = \begin{bmatrix} 0.25 & 0.225 & 0.045 \\ 0.225 & 0.25 & 0.045 \\ 0.045 & 0.045 & 0.09 \end{bmatrix}$$

In Chart 2, we show the composition of efficient frontier portfolios, with the vertical axis representing asset weights and the horizontal axis representing expected portfolio returns.

What we demonstrate in this hypothetical example is that although assets 1 and 2 are similar to each other (that is, same variances, same covariances with asset 3, similar expected returns), they are treated dramatically differently by the EF model. In most of the frontier portfolios, asset 2 is completely eliminated because its expected return is 1% less than that of asset 1. This can be dangerous because the model is too sensitive to the small differences in assets’ returns and variance inputs, making it unstable. This 1% difference could be due to estimation errors—maybe asset 2 actually has a higher expected return. If we make any amendments, putting a little more weight on the expected return of asset 2, we would likely end up with dramatically different optimal portfolios, with all asset 2



but no asset 1. We believe most investors should not be comfortable with this.

Following the same logic, in some cases the EF may pick only one asset or put the most weight on one asset instead of constructing a well-diversified portfolio. This is a predicament called the “corner solution problem.” In fact, if we rely heavily on a single asset, we are typically exposed to more idiosyncratic risk than we could have avoided. A more diversified and what we would call an almost efficient portfolio is likely more practical.

In another sense, the real values of future returns, variances, and covariances are unknown and must be forecast. One common practice is to use historical return averages as a proxy of expected returns. Yet such statistical forecasts derived from historical sample estimates change over time and are generally imbedded with estimation errors. They are at best expected returns plus an average error—a part that could not have been anticipated. Yet in the optimization process, the errors are treated as if they are part of the expectation.

Using statistical inference, the best estimator is not one that has the smallest bias, but rather one that has the lowest mean squared error (MSE)—the average of the squares of the estimation errors. Therefore, although the historical return average is an unbiased estimator, other estimators such as James-Stein Shrinkage estimators that are slightly biased might be a better choice. The high input sensitivity of the MV model makes choosing a statistical estimator particularly tricky.

Higher Risk Does Not Always Lead to Higher Returns

In our opinion, another issue with the MV model is the belief that higher risk, defined as higher volatility and higher expected return, always go hand in hand, which is unlikely. Indeed, we think it is quite different to say “to obtain higher return one may have to bear more risk” than to say “higher risk guarantees higher return.” For example, during extreme economic times, volatility risks across asset classes tend to be higher altogether, yet investors rarely see a big jump in their risky portfolios’ returns. The majority of empirical studies² have shown null or negative correlation between realized portfolio return and volatility. For example, in his book *The New Finance: The Case Against Efficient Markets*, Robert A. Haugen cited a study that showed among stocks traded on the New York Stock Exchange, a portfolio of lowest volatility stocks outperformed one with highest volatility stocks and the S&P 500[®] over a period of 64 years (1928–92). The study constructed those two portfolios using trailing 24-month volatilities, and the portfolios were recalibrated on a quarterly basis.

Moreover, even if the definition of risk is extended to the capital asset pricing model (CAPM) beta or to the Fama French 3 factor model, the positive relationship of risk and return assumed in MV generally does not hold. In the same book, Haugen quoted a study that shows negative correlation between beta and annualized return across all ten size groupings

² Thomas C. Chang and Shuh-Chyi Doong, “Empirical Analysis of Stock Returns and Volatility: Evidence from Seven Asian Stock Markets Based on TAR-GARCH Model,” *Review of Quantitative Finance and Accounting* 17: 301–318, 2001.

(group stocks based on market capitalization). The data are actually derived from a 1992 paper by Fama and French.³ Also, in their paper “High Idiosyncratic Volatility and Low Returns: International and Further U.S. Evidence,” Ang, Hodrick, Xing, and Zhang examined cross-sectional stocks’ returns and idiosyncratic risks of the G7 countries and a group of 23 developed countries. In both tests, they found that after sectioning stocks into quintiles based on their idiosyncratic risks, the higher the risk, the lower the grouped return (Chart 3). Their study uses monthly data from September 1980 to December 2003. (Please refer to Appendix A, page 14, for more information about their calculation of idiosyncratic risk.)

Oversimplified Definition of Risk

The MV model, in our view, lacks sufficiency in terms of its definition of risk. Although it is true that risk is relevant to unexpected deviation from anticipated future returns, historical volatility alone is typically not good enough in understanding risk. It can be misleading especially if the underlying distribution is not Normal. When the underlying distribution is not Normal, skewness and excess kurtosis present additional risks.

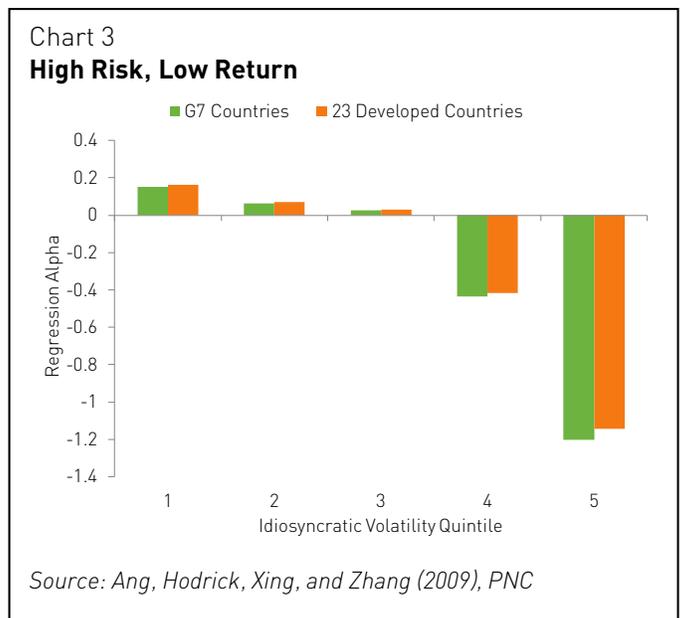
Within the financial industry, a few risk factors are commonly used, including:

- market risk factors, such as equity risk, interest rate risk, currency risk, and commodity risk;
- operational risk factors, such as credit risk, liquidity risk, and disasters; and
- model risk factors, which can be attributed to unrealistic assumptions, missing factors, calibration error, and parameter uncertainty.

We think it is obvious that volatility cannot capture all of these factors. At PNC, we estimate risk from both quantitative and qualitative perspectives. We employ a comprehensive manager selection process along with ongoing monitoring efforts. Please refer to our January 2014 white paper *Selecting the Managers: Research and Due Diligence Process* for more information.

In addition, volatility considers both upside and downside variations when upside variations should not be counted as risk. Harry Markowitz himself also noticed this and mentioned it in his 1959 treatise “Portfolio Selection—Efficient Diversification Investment,” that analyses based on semi-deviation tend to produce better portfolios than those based on variance, though both measures have their own pros and cons.

There has not been a consensus on how to measure risk. Since the 1950s, new risk measures have been proposed, such as Conditional Value-at-Risk (CVaR) and Stochastic discount factors. New varieties of risk measures have prompted the development of the Coherent Risk Measure (CRM) guideline



³ Fama, E. and K. French, “The Cross-Section of Expected Stock Returns.” *The Journal of Finance*, June, 1992, pp. 427-466.

introduced in the late 1990s. The CRM sets axioms for properties that a coherent risk measure should satisfy, which include:

- monotonicity;
- translation invariance;
- positive homogeneity; and
- sub-additivity (see Appendix B, page 14, for definitions of the axioms).

Variance and volatility are not coherent risk measures primarily because they fail to hold up to the monotonicity axiom and the translation invariance

axiom. On the contrary, CVaR is a coherent risk measurement (Chart 4). We have used Return-to-CVaR ratios in our portfolio construction of alternative investments; please refer to our August 2009 white paper *The Science of Alternative Investments* for more details.

Even with the CRM guideline, we note that there is no way to accurately measure all the aspects of risk on a purely quantitative basis.

Constraints

In the original MV model, the only constraints applied are a long-only constraint, a broad investment constraint, and a risk or return target. In the real world, however, asset allocation typically encounters much more complicated limitations. For example, an investor may have specific concerns over taxation, require a certain level of liquidity, and may have to comply with various legal requirements. Needless to say, many of these constraints are incredibly

difficult to incorporate in the MV model. Even if they can be represented mathematically in the MV model, they could possibly destroy the simple linear-quadratic framework. In addition, with too many constraints, the MV model may not be able to produce any feasible solution due to its simple structure.

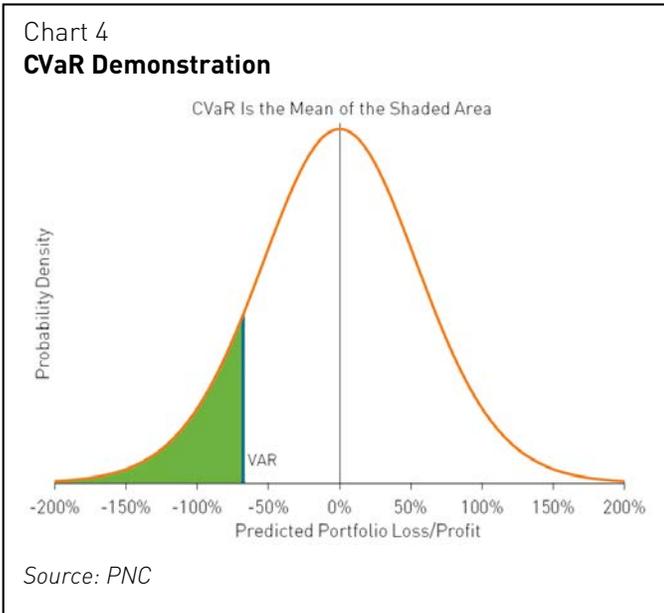
Alternatives of the MV Model

A variety of portfolio construction methodologies have been proposed. Following are a few examples.

Portfolio Optimization with Downside Risk Measures: Conditional Value-at-Risk

A CVaR at X% confidence level is the expected value of a portfolio's worst (100-X) % possible outcomes. CVaR is a coherent risk measure and has been used in the portfolio optimization process. For example, an optimization process might want to maximize expected return $\mu^T x$ while minimizing CVaR at level α %:

$$\begin{aligned} \max \quad & \mu^T x \\ \min \quad & CVaR_\alpha(L(x)) \end{aligned}$$



where

- μ is the vector of assets' expected returns;
- x is the vector of asset weights;
- $CVaR_\alpha$ is the CVaR at confidence level of α %; and
- $L(x)$ is the loss function.

It has been proved that if we define

$$F_\beta(x, \alpha) = \alpha + (1 - \beta)^{-1} \int_{f(x,y) > \alpha} (f(x, y) - \alpha)p(y)dy$$

where

- x is the vector of asset weights;
- α is the level of confidence;
- $\beta = \int_{f(x,y) \leq \alpha} p(y)dy$;
- $f(x, y)$ is the loss function associated with the decision vector x ;
- Y is the market parameters that affect loss; and
- $p(y)$ is the probability distribution of y .

Then, $\min CVaR_\alpha = \min (F_\beta(x, \alpha))$.

Therefore, one can follow a portfolio construction process with:

$$\begin{aligned} & \max \mu^T x \\ & \min F_\beta(x, \alpha) \end{aligned}$$

While the CVaR efficient frontier does not solve the problem of input sensitivity, we see it as an improvement regarding its definition of risk. In addition, unlike some of the other risk measures such as semi-variance, the linear representation of CVaR gives the CVaR optimization computational advantages. As mentioned earlier, we employed CVaR as part of our alternative investment portfolio construction process. We believe this measure has referential value, but we would not use it on a full-scale portfolio optimization, especially as it does not fix the input sensitivity problem related to the MV optimization.

Incorporating Estimation Errors into Portfolio Selection

Black-Litterman

The Black-Litterman (BL) model, developed by Fischer Black and Robert Litterman, provides the flexibility to combine the market equilibrium with an investor's views on the markets, adjusted by degrees of confidence.

Within the BL model, an investor can generally translate various types of views into quantitative terms. For example, an investor may want to add 5% to Germany's equilibrium return (absolute views) or may believe that Germany's return is 4% higher than the market-cap weighted average returns of France and the United Kingdom (relative views).

In mathematical terms, the BL model can be demonstrated as follows. Suppose there are N assets in the market, and the investor has K different views. The equilibrium returns can be represented as:

$$\Pi = \delta \Sigma w_{eq}$$

The CAPM prior distribution for the expected returns is

$$\mu = \Pi + \varepsilon^e$$

The investor has K views on the market. They are independent of the CAPM prior and are independent of each other. They are expressed as

$$P\mu = Q + \varepsilon^v$$

Then the mean of the expected (posterior) returns is

$$\bar{\mu} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q]$$

where

- Π is the implied excess equilibrium return vector ($N \times 1$ column vector);
- δ is the risk aversion coefficient;
- Σ is the covariance matrix of the returns ($N \times N$ matrix);
- w_{eq} is the vector of market equilibrium portfolio's asset weights;
- μ is a vector of normally distributed random variables with mean $\bar{\mu}$;
- ε^e is the error term, normally distributed with mean zero and covariance $\tau\Sigma$;
- τ is a scalar that measures the uncertainty of the CAPM prior;
- P is a $K \times N$ matrix that identifies the assets involved in the views;
- Q is the view vector ($K \times 1$ column vector);
- ε^v is the error term, normally distributed with mean zero and covariance Ω ; and
- Ω is a diagonal covariance matrix of error terms from the expressed views that represent the uncertainty in each view.

We view the BL model as another attempt to help reduce estimation errors encountered in evaluating expected returns. In our 10-year capital market projection, we utilized the BL model in our forecasts, combining market equilibrium returns with our subjective views on equity and bond markets. (Please refer to the January 2014 *Capital Market Projections – 10-Year Forecast of Asset Class Performance* for more information.) Our estimations are based upon our best understanding of the capital markets. The BL model simply provides a set of referential ranges, but since it is almost impossible to quantify the confidence level of our views, let alone coming up with close-enough views, we do not rely on the BL model for the totality of portfolio construction.

Mean Variance Resampling

Richard and Robert Michaud (1998) are the pioneers of the resampled efficiency optimization methodology. The main idea of this methodology is to produce multiple efficient frontier estimations and then take an average of the results. To be more specific, one can start by simulating multiple time series of returns and variances to obtain a number of efficient frontiers, then take averages of the equally ranked efficient portfolios to get the resampled efficient portfolios.

The MV resampling technique is an attempt to improve model input sensitivity, although it has several limitations such as:

- It is computationally intensive, especially for large portfolios.
- One cannot put reasonable constraints.
- As a result of averaging, almost all assets end up with positive weights.

Model Fragility

In the paper “Mathematical Definition, Mapping, and Detection of (Anti) Fragility,” Raphael Douady and Nassim Taleb proposed a mathematical definition of model fragility as *left tail-Vega sensitivity* (sensitivity of the left-tail shortfall below a certain threshold). More specifically, they categorized such fragilities as either intrinsic fragility or inherited fragility. Intrinsic fragility is the fragility that comes from the shape of the variable’s distribution and its sensitivity of a model parameter that controls the left side of its own distribution. Inherited fragility, on the other hand, comes from the changes in the underlying sources of stress. The authors then expanded these definitions with a series of transformation functions that compare models’ intrinsic and inherited fragilities. For example, they proved that if we have $\varphi: R \rightarrow R$ be a twice differentiable function, Ω be a constant such that $\varphi(\Omega) = \Omega$, and for any $x < \Omega$, $\frac{d\varphi}{dx}(x) > 0$, then the random variable $Y = \varphi(X)$ is more fragile at level $L = \varphi(K)$ and pdf g_λ than X at level K and pdf f_λ if, and only if, one has:

$$\int_{-\infty}^{\Omega} H_\lambda^K(x) \frac{d^2\varphi}{dx^2}(x) dx < 0$$

where

$$H_\lambda^K(x) = \frac{dP_\lambda^K}{d\lambda}(x) / \frac{dP_\lambda^K}{d\lambda}(\Omega) - \frac{dP_\lambda}{d\lambda}(x) / \frac{dP_\lambda}{d\lambda}(\Omega)$$

and

$$P_\lambda(x) = \int_{-\infty}^x F_\lambda(t) dt \quad F_\lambda(t) = \int_{-\infty}^t f_\lambda(t) dt$$

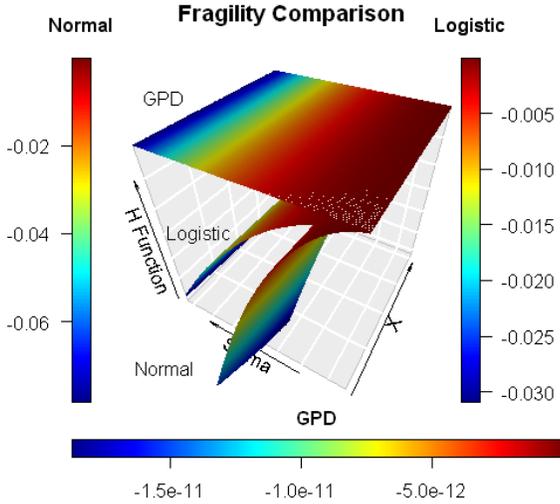
$$P_\lambda^K(x) = \int_{-\infty}^x F_\lambda^K(t) dt = \int_{-\infty}^x F_\lambda(\min(x, K)) dt$$

$$= \int_{-\infty}^x \min(F_\lambda(x), F_\lambda(K)) dt \quad \text{Due to monotonicity}$$

In the plot in Chart 5 (page 10), we compare the fragility integrant $H_\lambda^K(x)$ of a Normal Distribution, a logistic distribution, and a Generalized Pareto distribution (GPD) using a set of μ , k , and Ω definitions. We use the logistic distribution as a comparison because it has a similar shape as the Normal distribution, but comes with much heavier tails. We pick the GPD as a comparison because it has been used by practitioners in their extreme value models.

From the Fragility Comparison plot (Chart 5), we can see that with the given definitions of μ , k , and Ω , Normal distribution has the lowest H function values across combinations of Sigma (σ) and X values. Especially, its scale of negativity is the highest among the three distributions. Therefore, if we have an asset that is a convex function of an underlying asset that is defined

Chart 5
Fragility Comparison Plot



Source: PNC

on any of the three distributions, the Normal distribution would be the most fragile one. (Please refer to Appendix C on page 15 for more detailed discussions of the derivation and results.)

Moreover, we further studied the Fragility Integration $\int_{-\infty}^{\Omega} H_{\lambda}^K(x) \frac{d^2\varphi}{dx^2}(x) dx$, comparing the Normal Distribution with the Logistic Distribution and the GPD separately. We did this by finding functions $Y = \varphi(x)$ such that Y is the Logistic or GPD transformation of a Normal random variable X. According to the Fragility theorem, a negative integration value would indicate Y being more fragile than X at given values of K (loss threshold) and Sigma (σ). We then found that within a reasonable range of volatility (σ) values, the integrations are all positive. (Please refer to Appendix D on page 17 for more detailed calculations.) That is, the two distributions are less fragile than the Normal distribution. In Table 1 we see that the lower the loss threshold (K), the larger is the fragility difference, implying the more fragile is the Normal distribution. Indeed, the Normal distribution bears a lot more left tail

risks, and thus is not sustainable.

In this way, we believe we have shown that the Normal distribution is a fragile proxy of asset return distributions. Usage of such a fragile distribution within the MV framework tends to produce portfolio solutions that lack robustness.

PNC Solutions

A naïve usage of the MV Model or, for that matter, any model would likely lead to asset allocation solutions that are not robust. So at PNC, we employ a

Table 1
Fragility Integration Comparison

		Logistic Compared with Normal				GPD Compared with Normal			
		σ (Volatility)				σ (Volatility)			
		0.15	0.2	0.25	0.3	0.15	0.2	0.25	0.3
K (Loss Threshold)	-0.5	0.0052	0.00092	0.00029	5E-05	0.00354	0.00052	0.00015	2.56E-05
	-0.6	0.04343	0.00331	0.00093	0.00036	0.02953	0.00185	0.00049	0.00018
	-0.7	0.57658	0.01347	0.00258	0.00096	0.392	0.00753	0.00136	0.00049
	-0.8	12.3797	0.06887	0.00743	0.00225	8.41653	0.03849	0.00391	0.00116
	-0.9	427.649	0.45788	0.02398	0.00529	290.744	0.25588	0.01264	0.00272
	-1	23619.2	3.99039	0.09024	0.0132	16057.9	2.23001	0.04757	0.00679
	-1.1	2075828	45.5297	0.40196	0.03611	1411284	25.4441	0.21189	0.01859
	-1.2	2.9E+08	678.143	2.12894	0.11018	1.97E+08	378.977	1.12226	0.05671

Source: PNC

different portfolio construction process, relying on our four pillars of investment, striving to provide unique asset allocation solutions that best accommodate clients' objectives at their given risk budgets.

The four pillars of PNC investment are:

- investment advisors and relationship managers;
- the Investment Policy Committee (IPC);
- Investment Advisor Research (IAR); and
- the Portfolio Construction Committee (PCC).

The portfolio construction process begins with the investment advisors and relationship managers who work with clients to determine strategic asset allocations that best meet clients' goals and risk budgets. This results in a written Investment Policy Statement. The IPC provides beta exposure and risk control recommendations, determining appropriate market or asset class exposure in our asset allocation profiles. IAR identifies manager alpha and selects investment managers to be used in our portfolio construction. Then, the PCC selects specific managers from the manager pool that IAR identifies to fulfill the asset allocation recommendations from the IPC.

Although maximizing returns and controlling risks are essential to our portfolio constructions, historical returns and volatility are not the only factors that determine our allocation recommendations. For instance, in addition to reviewing past performance, IAR also pays attention to aspects including background and dynamics of investment professionals, consistency of investment process, strategy performance and manager investment style, business and operational structure, and legal and compliance structure. Also, within the PCC, besides evaluating managers' historical returns using a series of rigorous approaches that include a proprietary method to analyze the interaction risk of managers, we also keep our eyes fixed on the changing environment. We believe that although analyzing historical data and detailed information is helpful, forward-looking thinking is crucial, as relationships and environments change over time.

In terms of risk, rather than relying solely on volatility, our four pillars framework manages four layers of risks that correspond to components of portfolio construction: investor's risk tolerance, λ ; market exposure, β ; individual product exposure, α ; and overall portfolio risk, Ω :

$$\begin{aligned} \lambda + (\beta + \alpha + \Omega) &= \text{client portfolio} \\ IPS + (IPC + IAR + PCC) &= \text{client portfolio} \\ \text{client} + (PNC) &= \text{client portfolio}. \end{aligned}$$

Please refer to our May 2013 *PNC's Integrated Goals Investing: Pnc's Integrated Investment Approach Evolved* and August 2012 *Four Pillars of PNC Investment Management* white papers for more information.

Table 2 (page 12) is a short summary of our six traditional portfolios' performance statistics. The data are based on a 10-year time series from January 2004 to December 2013. Our portfolios are based on different return and risk combinations. We utilize these portfolios as the building blocks to accommodate investors' return objectives and risk tolerance.

Summary

As statistician George E. P. Box once said, “All models are wrong, but some are useful.” Indeed, we understand that a sophisticated practitioner rarely invests based solely on his mathematical models, though he might refer to those models for some reference. In fact, Warren Buffett has been quoted as

saying that “when we make a decision, there ought to be such a margin of safety... We are very inexact ... How certain we are is the most important part... You’d be amazed at how inexact we are”

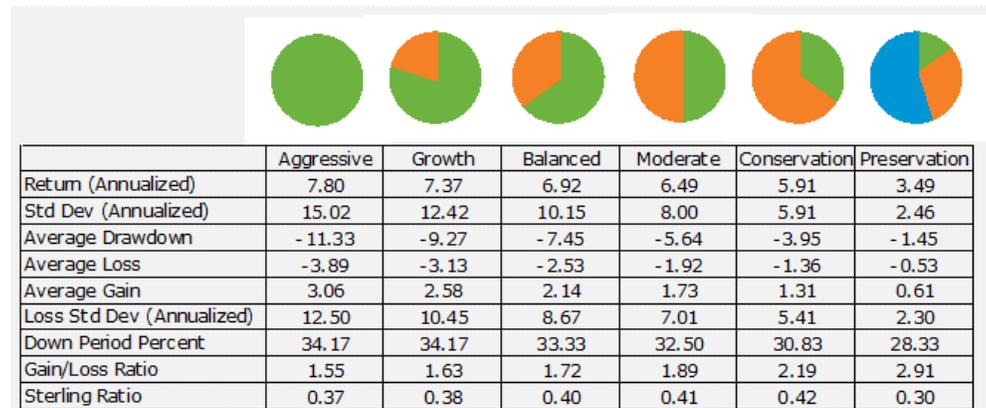
(*Seeking Wisdom: From Darwin to Munger*, by Peter Bevelin). Moreover, even the creator of the MV optimization process does not count too much on the MV procedure when making his own investments. In *Your Money & Your Brain*, author Jason Zweig recounts a story about Harry Markowitz when he

was trying to figure out his retirement plan while working at RAND Corporation. “I should have computed the historical covariances of the asset classes and drawn an efficient frontier,” Markowitz said, “but I visualized my grief if the stock market went way up and I wasn’t in it—or if it went way down and I was completely in it. So I split my contributions fifty-fifty between stocks and bonds.”

All told, a naïve usage of the MV or likely any other optimization would almost certainly lead to biased and fragile asset allocation solutions with unanticipated risks, in our opinion. We would agree with John Kenneth Galbraith when he said that “there are two kinds of forecasters: those who don’t know, and those who don’t know they don’t know.” We firmly believe that portfolios optimized to certain input forecasts are by definition more fragile since any divergence from that forecast would propagate through the optimization process and corrupt the output. So at PNC, we employ a well-defined portfolio construction process, relying on our four pillars of investment management, striving to provide intelligently designed asset allocation solutions that best accommodate clients’ objectives at their given risk budgets.

Table 2

Statistics of PNC Tactical Portfolios – Traditional Asset Classes



Source: Morningstar Direct, PNC. (Statistics based on index level performance of tactical allocations)

Green = Equity; Orange = Fixed Income; Blue = Cash

References

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Appendix A: Calculating Idiosyncratic Risk

In their 2009 paper “High Idiosyncratic Volatility and Low Returns: International and Further U.S. Evidence,” Ang, Hodrick, Xing, and Zhang calculate idiosyncratic volatility based on the Fama French three-factor model:

$$r_i = \alpha_i + \beta_{1i}MKT + \beta_{2i}SMB + \beta_{3i}HML + \varepsilon_i$$

where r_i is the daily excess return of stock i , MKT is the value-weighted excess return of the market portfolio over the one-month U.S. T-bill rate, SMB is the return of the smallest one-third of firms less the return of the largest one-third firms by market capitalization, HML is the return of the top one-third book-to-market ratio firms minus the bottom one-third book-to-market firms, and ε_i is the regression residual. The idiosyncratic volatility is calculated as the standard deviation of the residual ε_i .

Appendix B: Coherent Risk Measure

Let L be the loss associated with a given portfolio, then a risk measure ρ must satisfy the following properties to be identified as coherent:

- Monotonicity:
If $L_1 \leq L_2$ almost surely, meaning, $P(L_1 \leq L_2) \approx 1$, then we have $\rho(L_1) \leq \rho(L_2)$.
- Translation invariance:
If we add a cash position, C , to the portfolio, then $\rho(L + C) = \rho(L) - C$
This means by adding cash, the risk of portfolio reduces by the exact amount.
- Positive homogeneity:
If we increase the portfolio by λ times, then we have $\rho(\lambda \times L) = \lambda \times \rho(L)$
This says that risk increases at the same scale as the portfolio increase.
- Sub-additivity:
With two portfolios having loss values L_1 and L_2 , we have $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$
This axiom claims that diversification reduces, or at least maintains the sum of risk levels.

Appendix C: Fragility Integrant Functions

For X following a normal distribution, let σ (volatility) be the λ parameter defined in the Fragility function, then we have the cumulative distribution function $F_\sigma(x)$ and $\frac{\partial F_\sigma(x)}{\partial \sigma}$ as:

$$F_\sigma(x) = \frac{1}{2} \left[1 + \frac{1}{\sqrt{\pi}} \int_{\frac{x-\mu}{\sqrt{2\sigma^2}}}^{\frac{x-\mu}{\sqrt{2\sigma^2}}} e^{-t^2} dt \right]$$

$$\begin{aligned} \frac{\partial F_\sigma(x)}{\partial \sigma} &= \frac{1}{2} \frac{1}{\sqrt{\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{(x-\mu)}{\sqrt{2}} \left(\frac{-1}{\sigma|\sigma|} - \frac{1}{\sigma|\sigma|} \right) = \frac{x-\mu}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left(\frac{-1}{\sigma|\sigma|} \right) \\ &= \frac{\mu-x}{\sigma|\sigma|\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \end{aligned}$$

For X following a Logistic distribution, let s be the λ defined in Fragility function, then we have the cumulative distribution function $F_\sigma(x)$ and $\frac{\partial F_\sigma(x)}{\partial \sigma}$ as:

$$F_\sigma(x) = \frac{1}{1 + e^{-\frac{(x-\mu)}{\sigma}}}$$

$$\frac{\partial F_\sigma(x)}{\partial \sigma} = \frac{x-\mu}{\sigma^2} \frac{-1}{\left(1 + e^{-\frac{(x-\mu)}{\sigma}}\right)^2} e^{-\frac{(x-\mu)}{\sigma}} = \frac{(\mu-x)e^{-\frac{(x-\mu)}{\sigma}}}{\sigma^2 \left(1 + e^{-\frac{(x-\mu)}{\sigma}}\right)^2}$$

For X following a Generalized Pareto distributed distribution, the cumulative distribution function is:

$$F_\sigma(x) = \begin{cases} 1 - \left(1 + \xi \left(\frac{x-\mu}{\sigma}\right)\right)^{-\frac{1}{\xi}} & \text{for } \xi \neq 0 \\ 1 - e^{-\left(\frac{x-\mu}{\sigma}\right)} & \text{for } \xi = 0 \end{cases}$$

where

- ξ is the shape parameter;
- μ is the location parameter; and
- σ is the scale parameter.

In our test, we set $\xi = 0$ and let σ (the scale parameter) be the λ parameter, then we have the cumulative distribution function $F_\sigma(x)$ and $\frac{\partial F_\sigma(x)}{\partial \sigma}$ as:

$$\begin{aligned} F_\sigma(x) &= 1 - e^{-\left(\frac{x-\mu}{\sigma}\right)} \\ \frac{\partial F_\sigma(x)}{\partial \sigma} &= -e^{-\left(\frac{x-\mu}{\sigma}\right)} (\mu-x) \left(-\frac{1}{\sigma^2}\right) = \frac{e^{-\left(\frac{x-\mu}{\sigma}\right)}}{\sigma^2} (\mu-x) \end{aligned}$$

With all the partial derivatives defined for the three distributions, we can now take a look at the H function:

Given that:

$$H_{\lambda}^K(x) = \frac{dP_{\lambda}^K}{d\lambda}(x) - \frac{dP_{\lambda}(x)}{d\lambda} = \frac{dP_{\lambda}^K}{d\lambda}(\Omega) - \frac{dP_{\lambda}(\Omega)}{d\lambda}$$

and

$$P_{\lambda}^K(x) = P_{\lambda}(\min(K, x)) \quad \frac{\partial P_{\lambda}}{\partial \lambda}(x) = \int_{-\infty}^x \frac{\partial F_{\lambda}(t)}{\partial \lambda} dt$$

Then for $x \leq K$ and σ as λ , we have:

$$H_{\sigma}^K(x) = \frac{\frac{\partial P_{\sigma}^K}{\partial \sigma}(x)}{\frac{\partial P_{\sigma}^K}{\partial \sigma}(\Omega)} - \frac{\frac{\partial P_{\sigma}}{\partial \sigma}(x)}{\frac{\partial P_{\sigma}}{\partial \sigma}(\Omega)} = \frac{\frac{\partial P_{\sigma}}{\partial \sigma}(x)}{\frac{\partial P_{\sigma}^K}{\partial \sigma}(\Omega) \frac{\partial P_{\sigma}}{\partial \sigma}(\Omega)} \left(\frac{\partial P_{\sigma}}{\partial \sigma}(\Omega) - \frac{\partial P_{\sigma}^K}{\partial \sigma}(\Omega) \right)$$

Therefore, we have:

$$\begin{aligned} \frac{\partial P_{\sigma}}{\partial \sigma}(\Omega) - \frac{\partial P_{\sigma}^K}{\partial \sigma}(\Omega) &= \int_{-\infty}^{\Omega} \frac{\partial F_{\sigma}(x)}{\partial \sigma} dx - \int_{-\infty}^K \frac{\partial F_{\sigma}(x)}{\partial \sigma} dx - (\Omega - K) \frac{\partial F_{\sigma}(K)}{\partial \sigma} \\ &= \int_k^{\Omega} \frac{\partial F_{\sigma}(x)}{\partial \sigma} dx - (\Omega - K) \frac{\partial F_{\sigma}(K)}{\partial \sigma} \\ H_{\sigma}^K(x) &= \frac{\int_{-\infty}^x \frac{\partial F_{\sigma}(t)}{\partial \sigma} dt \left(\int_k^{\Omega} \frac{\partial F_{\sigma}(x)}{\partial \sigma} dx - (\Omega - K) \frac{\partial F_{\sigma}(K)}{\partial \sigma} \right)}{\left(\int_{-\infty}^K \frac{\partial F_{\sigma}(t)}{\partial \sigma} dt + (\Omega - K) \frac{\partial F_{\sigma}(K)}{\partial \sigma} \right) \int_{-\infty}^{\Omega} \frac{\partial F_{\sigma}(t)}{\partial \sigma} dt} \end{aligned}$$

We then put every partial derivative functions obtained earlier into the H function above individually and then calculate and derive their plots with a random set of μ , K , and Ω values. According to Ang et al. (2009), $Y = \varphi(x)$ is more fragile than X below threshold K if:

$$\int_{-\infty}^{\Omega} H_{\lambda}^K(x) \frac{d^2\varphi}{dx^2}(x) dx < 0$$

If we define Y as a convex function of X such that $\frac{d^2\varphi}{dx^2}(x) > 0$ over the real axis, then the more negative the H function at each X level, the more fragile the Y . As shown in Chart 5 on page 10, using our definitions of μ , K , and Ω , $H_{\lambda}^K(x_{Normal}) < H_{\lambda}^K(x_{Logistic}) < H_{\lambda}^K(x_{GPD})$ at each level of X , then we can say that the Normally distribution model is the most fragile among the three in our test.

Appendix D: Fragility Integration Functions

To compare the fragility of a Normal distribution with a logistic distribution and a GPD, we further studied the Fragility Integration function:

$$\int_{-\infty}^{\Omega} H_{\lambda}^K(x) \frac{d^2 \varphi}{dx^2}(x) dx$$

Here, we defined a Logistic random variable as a function of a Normal random variable, and obtained the Fragility integration value. We then repeated the same process replacing the Logistic random variable by a GPD random variable. Since we have already obtained the $H_{\lambda}^K(x)$ for a Normal distribution in Appendix C, we will focus on finding the transformation functions $\varphi(x)$ and deriving their second derivatives with regard to x in the following paragraphs.

Logistic Compared with Normal

We know a Normal distribution's CDF is:

$$F_{\sigma}(x) = \frac{1}{2} \left[1 + \frac{1}{\sqrt{\pi}} \int_{\frac{x-\mu}{\sqrt{2}\sigma^2}}^{\frac{x-\mu}{\sqrt{2}\sigma^2}} e^{-t^2} dt \right]$$

And a Logistic distribution's CDF is:

$$F_{\sigma}(x) = \frac{1}{1 + e^{-\frac{(x-\mu)}{\sigma}}}}$$

We can find a function $Y = \varphi(x)$ such that

$$\frac{1}{1 + e^{-\frac{(y-\mu)}{\sigma}}} = \frac{1}{2} \left[1 + \frac{1}{\sqrt{\pi}} \int_{\frac{x-\mu}{\sqrt{2}\sigma^2}}^{\frac{x-\mu}{\sqrt{2}\sigma^2}} e^{-t^2} dt \right]$$

Then we get:

$$y = \varphi(x) = \mu - \sigma \times \ln \left(\frac{1}{\frac{1}{2} \left[1 + \frac{1}{\sqrt{\pi}} \int_{\frac{x-\mu}{\sqrt{2}\sigma^2}}^{\frac{x-\mu}{\sqrt{2}\sigma^2}} e^{-t^2} dt \right]} - 1 \right)$$

To simplify our notations, we define:

$$y_1(x) = \frac{1}{2} \left[1 + \frac{1}{\sqrt{\pi}} \int_{\frac{x-\mu}{\sqrt{2}\sigma^2}}^{\frac{x-\mu}{\sqrt{2}\sigma^2}} e^{-t^2} dt \right]$$

$$y_2(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Then we get:

$$\frac{dy}{dx}$$

$$= \frac{-\sigma}{\frac{1}{2} \left[1 + \frac{1}{\sqrt{\pi}} \int \frac{x-\mu}{\sqrt{2\sigma^2}} e^{-t^2} dt \right]^{-1}} \left(\frac{-1}{\frac{1}{4} \left[1 + \frac{1}{\sqrt{\pi}} \int \frac{x-\mu}{\sqrt{2\sigma^2}} e^{-t^2} dt \right]^2} \right) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\frac{1}{y_1(x)} - 1} \left(\frac{1}{y_1(x)^2} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left[\left(\frac{-1}{\left(\frac{1}{y_1(x)} - 1 \right)^2} \left(\frac{-1}{y_1(x)^2} \right) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) \left(\frac{1}{y_1(x)^2} \right) \right.$$

$$\left. + \frac{1}{\frac{1}{y_1(x)} - 1} \left(\frac{-2}{y_1(x)^3} \right) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right.$$

$$\left. + \frac{1}{\frac{1}{y_1(x)} - 1} \left(\frac{1}{y_1(x)^2} \right) \frac{\mu - x}{\sigma^2} \right]$$

$$= \frac{1}{2\pi\sigma} e^{-\frac{(x-\mu)^2}{\sigma^2}} \left[\frac{1}{\left(\frac{1}{y_1(x)} - 1 \right)^2} \frac{1}{y_1(x)^4} + \frac{1}{\frac{1}{y_1(x)} - 1} \left(\frac{-2}{y_1(x)^3} \right) \right]$$

$$+ \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{\frac{\mu - x}{\sigma^2}}{\frac{1}{y_1(x)} - 1} \left(\frac{1}{y_1(x)^2} \right)$$

$$= \frac{1}{2\pi\sigma} e^{-\frac{(x-\mu)^2}{\sigma^2}} \frac{1}{y_1(x)^3} \left[\frac{1}{y_1(x) \left(\frac{1}{y_1(x)} - 1 \right)^2} - \frac{2}{\frac{1}{y_1(x)} - 1} \right]$$

$$+ \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{\frac{\mu - x}{\sigma^2}}{\frac{1}{y_1(x)} - 1} \left(\frac{1}{y_1(x)^2} \right)$$

$$= y_2(x)^2 \frac{\sigma}{y_1(x)^3} \left[\frac{1}{y_1(x) \left(\frac{1}{y_1(x)} - 1 \right)^2} - \frac{y_1(x)}{1 - y_1(x)} \right]$$

$$+ y_2(x) \frac{\frac{\mu - x}{\sigma^2}}{[1 - y_1(x)]y_1(x)}$$

GPD Compared with Normal

The GPD's CDF is: $F_\sigma(x) = 1 - e^{-\left(\frac{x-\mu}{\sigma}\right)}$

We can find a function $Y = \varphi(x)$ such that

$$1 - e^{-\left(\frac{y-\mu}{\sigma}\right)} = \frac{1}{2} \left[1 + \frac{1}{\sqrt{\pi}} \int_{\frac{x-\mu}{\sqrt{2\sigma^2}}}^{\frac{y-\mu}{\sqrt{2\sigma^2}}} e^{-t^2} dt \right]$$

Then we get:

$$y = \varphi(x) = \mu - \sigma \times \ln \left(1 - \frac{1}{2} \left[1 + \frac{1}{\sqrt{\pi}} \int_{\frac{x-\mu}{\sqrt{2\sigma^2}}}^{\frac{y-\mu}{\sqrt{2\sigma^2}}} e^{-t^2} dt \right] \right)$$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= \frac{-\sigma}{1 - y_1(x)} \frac{-1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{\sigma y_2(x)}{1 - y_1(x)} \\ \frac{d^2y}{dx^2} &= \sigma y_2(x) \frac{-1}{(1 - y_1(x))^2} (-y_2(x)) + \frac{1}{1 - y_1(x)} \sigma y_2(x) \frac{\mu - x}{\sigma^2} \\ &= \frac{\sigma y_2(x)^2}{(1 - y_1(x))^2} + \frac{y_2(x)}{1 - y_1(x)} \frac{\mu - x}{\sigma} \end{aligned}$$

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