Self-attraction into spinning eigenstates of a mobile wave source by its emission back-reaction

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The back-reaction of a radiated wave on the emitting source is a general problem. In the most general case, back-reaction on moving wave sources depends on their whole history. Here we study a model system in which a pointlike source is piloted by its own memory-endowed wave field. Such a situation is implemented experimentally using a self-propelled droplet bouncing on a vertically vibrated liquid bath and driven by the waves it generates along its trajectory. The droplet and its associated wave field form an entity having an intrinsic dual particle-wave character. The wave field encodes in its interference structure the past trajectory of the droplet. In the present article we show that this object can self-organize into a spinning state in which the droplet possesses an orbiting motion without any external interaction. The rotation is driven by the wave-mediated attractive interaction of the droplet with its own past. The resulting “memory force” is investigated and characterized experimentally, numerically, and theoretically. Orbiting with a radius of curvature close to half a wavelength is shown to be a memory-induced dynamical attractor for the droplet’s motion.

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I. INTRODUCTION

Moving wave sources are submitted to a back-reaction resulting from the energy-momentum conservation law. This process is complex when the emission itself depends on the source motion, a general case for isolated sources. The interplay between waves and sources is pivotal in understanding the very nature of elementary particles and their interactions [1,2]. The back-reaction can be considered as a signature of a spatiotemporal nonlocality. This has driven prequantum models of the electron [3–5] and is still present a signature of a spatiotemporal nonlocality. This has driven interactions [1,2]. The back-reaction can be considered as standing the very nature of elementary particles and their interplay between waves and sources is pivotal in understanding the source motion, a general case for isolated sources. The process is complex when the emission itself depends on the energy-momentum conservation law. This nonlocality is in current field equation theories [2,6,7]. This nonlocality is central in the theory of the recently observed gravitational attractors had yet been observed.

II. PATH-MEMORY-DYNAMICS MODEL

A walker is composed of a droplet bouncing on a vertically vibrated liquid bath [12–21] have been shown to possess a wave-driven path-memory dynamics [22–24]. The drop can be considered as a pointlike source which is dressed and piloted by the wave field emitted by its previous impacts on the surface. These waves are sustained for a tunable characteristic memory time. For long memory times, the dynamics of these “walkers” becomes complex and bears similarities with quantum particles [25–33]. In particular, in confined geometries, quantized eigenstates emerge resulting from the interplay between the drop trajectory and its associated wave field mode [27–35]. These states were observed in several geometries and interpreted as attractors of the dynamics. In the absence of external force, however, none of these nontrivial attractors had yet been observed.

In the present article, we explore the possible existence of an intrinsic set of such nontrivial attractors. Using this model system, we investigate if the memory-based back-reaction of the wave can produce stable dynamical eigenstates in the absence of any external interactions. In particular, we study the ability to produce self-orbiting states with quantized angular momentum sustained only by a memory force built by the drop past motion. The possibility of such modes was theoretically demonstrated by Oza et al. [29]. It was obtained in their investigation of the circular orbits induced by a Coriolis force transverse to the drop motion. They found that in the very long memory limit, orbital states could result only from the wave field in the case of a vanishing Coriolis force. However, these authors concluded that these modes would be intrinsically linearly unstable for all realistic walkers [13,29]. Here we examine experimentally and theoretically another configuration in which such modes can be generated using magnetic walkers gently released from a harmonic potential confinement.
memory parameter $M_e = \tau / T_F$ with $T_F = 25\, \text{ms}$ the Faraday period. The wave field $h(\mathbf{r}, t)$ in $\mathbf{r}$ and at time $t$ is thus given by [22,23,27]

$$h(\mathbf{r}, t) = h_0 \sum_{j=-\infty}^{0} e^{-i\omega_j} J_0(k_F |\mathbf{r} - \mathbf{r}_j|),$$

where $\mathbf{r}_j$ is the position of the $j$th past bounce occurring at time $t_j$ and $h_0$ is the amplitude of a single bounce disturbance. The past trajectory of the droplet is thus encoded within its associated wave field. For this reason, the droplet is said to have a “path-memory dynamics” [22]. The force driving the droplet into motion originates from the wave field asymmetry. At each bounce, the droplet lands on a local slope that gives it a kick.

In the memory range hitherto explored and in the absence of external perturbations walkers are observed to move in a straight line [12,20–24]. However, if set into an initial curved trajectory, one can imagine other possible dynamical solutions due to the persistency of previously generated waves. In the following, we study the characteristics for sustaining, in a circular self-orbiting mode, the walker confined by its own centripetal memory force exerted by the wave field it has emitted during its previous bounces.

III. EXPERIMENTS

The experimental difficulty of testing this idea is that these memory-induced modes require preparing an initial wave field corresponding to a given past trajectory. A “preparation” process is thus needed to force the walker into a trajectory close to that expected for the self-organized mode. The corresponding wave field can therefore be built over time. For this purpose, we first set the drop in rotation on a small straight line [12,20–24]. However, if set into an initial curved trajectory, one can imagine other possible dynamical solutions due to the persistency of previously generated waves. In the following, we study the characteristics for sustaining, in a circular self-orbiting mode, the walker confined by its own centripetal memory force exerted by the wave field it has emitted during its previous bounces.

To apply a force on the droplet, the oil droplet is loaded with a ferrofluid [31]. By means of two large Helmholtz coils, the whole bath is submitted to a constant vertical magnetic field that polarizes the droplet. In the center of the cell, an additional radial gradient of magnetic field creates a magnetic trap for the drop. This trap was generated by a sharp cone of pure iron that polarizes the droplet. In the center of the cell, an additional radial gradient of magnetic field creates a magnetic trap for the droplet.

![Scheme of the core of the experimental setup.](a) The experimental difficulty of testing this idea is that these memory-induced modes require preparing an initial wave field corresponding to a given past trajectory. A “preparation” process is thus needed to force the walker into a trajectory close to that expected for the self-organized mode. The corresponding wave field can therefore be built over time. For this purpose, we first set the drop in rotation on a small straight line [12,20–24]. However, if set into an initial curved trajectory, one can imagine other possible dynamical solutions due to the persistency of previously generated waves. In the following, we study the characteristics for sustaining, in a circular self-orbiting mode, the walker confined by its own centripetal memory force exerted by the wave field it has emitted during its previous bounces.

![Top view of the walker in a self-orbiting mode in a long memory regime (Me ≈ 140). The droplet has a diameter D ≈ 0.7 mm and a velocity V ≈ 7 mm s⁻¹.](b) The magnetic field that initially trapped the walker, had been turned off 3 s before this picture was taken. The recent trajectory is superimposed. The nondimensional radius of the free orbit is $R_c/\lambda_F = 0.385 \pm 0.01$.

As shown in Fig. 2(b) the droplet remains trapped in the orbiting motion [Figs. 1(b), 2(b), and 2(d); see Movie S1 in the Supplemental Material [36]]. This effect is optimized when the magnetic field is progressively removed with a characteristic transition time of 1 s.

![View of the walker in a self-orbiting mode in a long memory regime (Me ≈ 140). The droplet has a diameter D ≈ 0.7 mm and a velocity V ≈ 7 mm s⁻¹.](b)

FIG. 1. Experimental realization of self-orbiting modes. (a) Scheme of the core of the experimental setup. The whole bath is submitted to a global magnetic field generated by two coils that are not shown here. A magnetic field maximum is created by the concentration of flux lines due to a sharp rod of pure iron. (b) Top view of the walker in a self-orbiting mode in a long memory regime (Me ≈ 140). The droplet has a diameter D ≈ 0.7 mm and a velocity V ≈ 7 mm s⁻¹. The magnetic field that initially trapped the walker, had been turned off 3 s before this picture was taken. The recent trajectory is superimposed. The nondimensional radius of the free orbit is $R_c/\lambda_F = 0.385 \pm 0.01$. The drop remains trapped in the orbiting motion [Figs. 1(b), 2(b), and 2(d); see Movie S1 in the Supplemental Material [36]]. This effect is optimized when the magnetic field is progressively removed with a characteristic transition time of 1 s.
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FIG. 2. Typical trajectories observed when the confinement is turned off for two values of the memory parameters. (a) Short memory (Me ≈ 10). (b) Long memory (Me ≈ 140) (see Movie S1 [36]). (c,d) Temporal evolution of the normalized trajectory radius Rc/λF along these two trajectories (black: magnetic field on, green: transition time, and red: no central force).

lifetime originates from the presence of experimental noise. In the experiment, possible origins for the noise are the presence of wave reflection from the bath walls or ambient air flows. Though difficult to measure, these various noise sources could easily generate variations of a few percent in the wave field amplitudes.

To clarify this point we performed a numerical analysis of existence of the self-orbiting modes and of their stability in the presence of noise.

IV. THE NUMERICAL INVESTIGATION OF THE SELF-ORBITING MODES

We use a discrete iterative model for the walker dynamics reported by our group [22,25,27,31,37] which describes accurately the walker’s behavior observed in various experimental situations (the details are given in Ref. [37] and in Appendix A). In this model, both the droplet trajectory and the resulting wave field are computed iteratively. The wave field is given by Eq. (1). The drop motion results from several forces. The wave force, or memory force, is related to the local slope of the interface at the point of bouncing. It is opposed by a tangential frictionlike force Fγ due to both the shearing of the air layer between the droplet and the bath and to the formation of the dip in the interface that will be the source of a new wave. The above-described experiment is simulated by adding an external central force Fm to initially trap the drop. As in the experiment, the force is switched off after some time. Figure 3(a) shows the temporal evolution of the orbital radius for short and long memory, respectively, and insets Figs. 3(b,c) associated trajectories for Me = 11, and Me = 27. (d) Temporal evolution of the normalized radius of curvature Rc/λF for a fixed value of the memory (Me = 27) and several values of noise amplitude expressed in radius unit ε = σ/R, with R the instantaneous radius: pink, ε = 2 × 10⁻²; blue ε = 1.6 × 10⁻²; red ε = 1.2 × 10⁻²; black ε = 4 × 10⁻³.

FIG. 3. Numerical investigation of the stability. (a) Temporal evolution of the normalized radius of curvature Rc/λF along three trajectories (pink, Me = 11; blue, Me = 15; black Me = 27). The central force is turned off at t = 0 s. Insets (b,c) associated trajectories for Me = 11, and Me = 27. (d) Temporal evolution of the normalized radius of curvature Rc/λF for a fixed value of the memory (Me = 27) and several values of noise amplitude expressed in radius unit ε = σ/R, with R the instantaneous radius: pink, ε = 2 × 10⁻²; blue ε = 1.6 × 10⁻²; red ε = 1.2 × 10⁻²; black ε = 4 × 10⁻³. (e) Probability of capture p in the self-orbiting mode (color code on the right) when varying the noise amplitude ε and the memory parameter Me. Each probability is calculated over 20 realizations.

These results permit a general description of the stability of the self-orbiting modes. In the simulations they exist above a critical memory threshold Me crítica = 16. In the absence of noise [see Fig. 3(a)] the orbit is briefly disturbed by the external
force switch-off and restabilizes after undergoing a brief wobbling motion of decreasing amplitude. In the presence of a noise of moderate amplitude, the wobbling is observed to be sustained without any disruption of the self-orbiting [see Fig. 3(d)]. There is a noise amplitude threshold [depending on the memory, see Fig. 3(e)] above which the wobbling amplification leads to a disruption of the orbital motion. As in the experiment this disruption occurs after a time delay that has a statistically distributed value. The complex dependence with memory of the noise level required to destabilize the initial orbit is beyond the scope of the present article.

V. DISCUSSION AND CONCLUSION

It is interesting to discuss these results in the light of the theoretical work of Oza et al. [29]. Their analysis has been triggered by Coriolis experiments with walkers [27,28]. They demonstrate that self-orbiting modes could be obtained in the limit of vanishing rotation and very large memory.

This limiting case is very difficult to realize experimentally in the case of a force generated by the rotation of the entire bath. Stopping the bath would indeed generate secondary flows both in air and liquid phases. Even in the current magnetic force configuration which is far less disturbing, we find that self-orbits ultimately destabilize. The question is on whether it results from an intrinsic instability of these modes or from noise-induced effects. While our model shows the latter, a linear stability analysis had led Oza et al. [29] to conclude to an instability for any realistic walkers.

At least two possible origins of this divergence come to mind. While our model maintains the discrete iterative character of the phenomenon, the MIT model [13] takes a continuous limit. This limit lends itself to a convenient analytical framework. However, it may depart from the particular case of walker dynamics as the bouncing is dominated by discontinuities. In various dynamical systems, it has been shown that the stability of discrete dynamics can strongly differ from their continuous counterpart. As for the stability itself, Oza et al. [29] performed a linear stability analysis of these states. However, the walker dynamics is intrinsically nonlinear. This is why we proposed a Monte Carlo approach. Our results show that the self-orbits can be maintained over a long time with or without the presence of wobbling motion. It suggests that even if the orbits were linearly unstable, the nonlinearities play a stabilizing role. This would account for the observed wobbling orbits observed in the simulations. A stability analysis of the circular orbit would consider it unstable while, in a nonlinear perspective, the wobbling does not lead to a bursting of the orbital motion and the angular momentum of the droplet is preserved.

In the orbiting states, the dynamical interplay of the droplet motion and the waves can be best understood by using a decomposition of the wave field reflecting the symmetries of the problem. Using Graf’s theorem, it is possible to reformulate the global wave field [given by Eq. (1)] on a basis of Bessel functions centered on the orbit axis of symmetry [31,34,38] (see Appendix B). The radius of the orbit R determines the efficiency of excitation of each eigenmode. Conversely, when evaluated at the droplet position, the contribution of each central Bessel function to the tangential and radial forces can be obtained (see Appendix B). It is convenient to analyze the drop dynamics by decomposing the memory force \( F_{Me} = -C[\nabla h]|_{R_0} \) into its tangential \( F_{Me}^T \) and radial \( F_{Me}^R \) components [see Fig. 4(a)], with C the wave coupling constant.

For the self-orbiting mode to exist, the self-propulsion of the walker must be maintained for increasing Me. Velocity results from the balance between \( F_{Me}^T \) and a friction force \( F_{\gamma} = -\gamma V; V = 1/\gamma F_{Me}^T \). Both C \( \times h_0 \) and \( \gamma \) are constants that we estimate from the experiments [21,22]. Their values are in accordance with those predicted by the hydrodynamic analysis done by Moláček and Bush [19,24].

We compute the equilibrium radius from Eq. (1) for the radius measured experimentally. This latter is close to the first zero of the J0 Bessel function \( R_0 \) (as will be discussed below). Figure 4(d) shows the variations of \( F_{Me}^T \) as a function of the memory parameter Me. In addition, the contribution of each central Bessel function to \( F_{Me}^T \) can be evaluated

![FIG. 4. Theoretical characterization of the forces acting in self-orbits. (a) The computed orbit and wave field at long memory (Me = 100). The dotted lines are the zeros of J0. (b) The wave-induced radial forces and the centrifugal pseudoforce \( F_{\text{cent}} \) (dashed red line) as a function of the normalized orbit radius \( R/\lambda_F \). The estimation wave-induced force is shown for several values of the memory parameters (solid line, from light gray to black: Me = 11, 15, 19, 23, 27) with \( C_{\text{hub}} h_0 = 8.05 \times 10^{-2} N/kg \) and \( m_{\text{eff}} = 3.82 \times 10^{-3} kg \). Equilibrium is possible when the two curves cross. Vertical lines indicate the first zeros of the J0 Bessel function \( R_0 \) and \( R_1 \). (c) The possible orbits (black lines: stable solutions; gray: unstable solutions) (d) The evolution with memory of the contributions of the first Bessel modes to the tangential force \( F_{Me}^T \) with an equilibrium speed \( V = 10.9 \) mm/s which enables to estimate \( \gamma = 7.8 \times 10^{-6} kg/s \). (e) The evolution with memory of the contributions of the first Bessel modes to the radial force \( F_{Me}^R \). The dashed lines represent the contribution of the first three centered Bessel modes. The vertical line indicates the critical memory parameter \( Me_c \); the horizontal line shows the asymptotic value of \( F_{Me} = F_{\text{hess}} \) when \( J_0(k_F R) = 0 \).
using Graf’s theorem calculated at the droplet position. \( F_{\text{Me}}^{\perp} \)

is nearly constant above \( \text{Me}_c \). At long memory, it takes the limiting value of 

\[
\frac{1}{F_{\text{Me}}} \sum_{n=0}^{\frac{1}{2}} F_n^0 \left( k_\text{F} R_0 \right) / (R_\text{F} \omega_0) \]

(see the Appendix B). Apart from the axisymmetric \( J_0 \) Bessel mode, all the Bessel modes could contribute to the local tangential slope. However, when evaluated near the first zero of the \( J_0 \) Bessel mode only Bessel functions of order \( n = 1 \) and \( 2 \) have significant contributions to the tangential propulsion (representing, respectively, approximately 52\% and 38\% of the tangential slope). These modes rotate at the angular frequency \( \omega \) of the droplet maintaining a constant local slope at the point of impact, resulting in a sustained propulsion, as observed experimentally. In other terms such a structure is intrinsically endowed with an “arrow of time”. When their two-dimensional (2D) trajectory is circular, the space-time symmetry is helical.

We can now examine the situation in the radial direction. The droplet rotating on an orbit of radius \( R \) is submitted to the memory-induced force \( F_{\text{Me}}^{\perp} \). This latter satisfies \( F_{\text{Me}}^{\perp} = -C \partial h / \partial r \big|_{r=R} \) and can be computed directly from Eq. \( 1 \). Figure 4(b) shows \( F_{\text{Me}}^{\perp} \) for different memory parameters \( \text{Me} \) as a function of the normalized radius of the orbit \( R / \lambda_\text{F} \) (see Appendix B). The inertial pseudoforce \( F_{\text{inert}} = -m_{\text{eff}} V^2 / R \) is also plotted with the constant droplet velocity approximation. The radial balance can only be reached for a discrete set of radii for which \( F_{\text{inert}} = F_{\text{Me}}^{\perp} \) above a critical memory parameter \( \text{Me}_c \approx 16 \). The number of crossings, associated to larger radii, increases with increasing memory. The solutions in Fig. 4(b) for which the slope of the memory force is positive at the crossing are trivially nonstable. For the others, the radii of the self-orbits decrease towards the zeros of the \( J_0 \) Bessel functions (dotted vertical lines) as the memory increases [Fig. 4(c)]. In the following, we will focus on the smallest orbit observed experimentally associated with the first zero of the \( J_0 \) Bessel function, i.e., with \( R / \lambda_\text{F} = R_c / \lambda_\text{F} = 0.38 \pm 0.01 \).

We can understand this evolution using the decomposition in centered Bessel modes. Figure 4(e) shows the memory-induced force \( F_{\text{Me}}^{\perp} \), as a function of \( \text{Me} \) together with the contributions of centered Bessel modes \( n = 0, 1, 2 \). With increasing memory, the role of the central \( J_0 \) Bessel function is the leading contribution of the wave field to the radial force. At long memory \( (\text{Me} \gg \text{Me}_c) \), the radial force takes the simple expression

\[
F_{\text{Me}}^{\perp} \approx C h_0 k_\text{F} \text{Me} J_0 (k_\text{F} R) J_1 (k_\text{F} R),
\]

(2)

\( F_{\text{Me}}^{\perp} \) is the product of three terms: \( J_1 (k_\text{F} R) \) arises from the radial derivative at \( k_\text{F} R \) of the \( J_0 \) mode, \( J_0 (k_\text{F} R) \) is the \( J_0 \) mode excitation amplitude by the droplet bounces at a radius \( R \), and \( \text{Me} \) is the number of bounces that contributes to the global wave field [22]. We can now understand why the radii \( R / \lambda_\text{F} \) of the self-orbiting modes tend to the zeros of the \( J_0 \) Bessel function as \( \text{Me} \) increases [see Figs. 4(b) and 4(c)]. It results from a self-organized mechanical balance. Since \( F_{\text{inert}} \) does not depend on the memory parameter, the product \( \text{Me} J_0 (k_\text{F} R) \) has to remain constant to maintain the force balance \( F_{\text{inert}} = F_{\text{Me}}^{\perp} \). Hence, at critical memory parameter \( \text{Me}_c \), the walker maximizes the contribution of each bounce to reach a sufficient wave force to balance inertia. This becomes possible above \( \text{Me}_c \) and with a radius that is at a maximum of \( J_0 (k_\text{F} R) \). As memory increases, the number of bounces that contributes to the wave force increases; their individual contributions to the global wave field must decrease. In the long memory limit, each individual contribution tends to zero. The self-orbit radius then satisfies

\[
\frac{R}{R_0} \approx 1 + \frac{\beta}{\text{Me}} \quad \text{with} \quad \beta = \frac{m_{\text{eff}}}{C h_0} \left( \frac{V}{k_\text{F} R_0 J_1 (k_\text{F} R_0)} \right)^2,
\]

(3)

where \( k_\text{F} R_0 \) is a zero of the \( J_0 \) Bessel function. Equations (2) and (3) are similar to those found in Oza et al. [29]. This fine-tuning balancing the centrifugal inertial force can be interpreted as a spatio-temporal self-organization of the walker. This self-orbiting mode is an attractor of the dynamical system.

The previous analysis takes the point of view of the particle: The wave field is only considered through its interaction with the droplet only and not \( \text{per se} \). For instance, the added angular momentum brought by the wave field to the droplet is the result of a local coupling and is thus different from the angular momentum of the whole wave field. We now discuss the global features of the wave field, its energy \( E_{\text{wave}} \), and angular momentum \( L_{\text{wave}} \).

As the wave is monochromatic, the energy stored in the wave field \( E_{\text{wave}} \) is proportional to the square of the surface displacement \( h(\mathbf{r}, t) \) integrated over the surface of the bath (see Appendix C). This energy is normalized by the wave field energy induced by a single bounce modeled as a \( J_0 \) Bessel function. Figure 5(a) shows the dependence of the normalized energy \( E_{\text{wave}} \) stored in the wave field as a function of the memory parameter \( \text{Me} \) for a self-orbiting mode and for a droplet moving in a straight line.

While \( E_{\text{wave}} \) increases with memory for the rectilinear motion, it slowly decreases for the self-orbiting mode, reaching a finite value in the high memory limit. However, the wave energy resulting from a single bounce does not depend on the considered trajectory: Each impact generates a wave modeled by a \( J_0 \) Bessel function of equal amplitude. Since all the individual bounces produce coherent waves, each wave source interferes constructively or destructively depending on their relative position. The contribution of each new source to the energy of the global wave field can thus be either positive or negative. Hence, the energy of the global wave field holds a wave information of the droplet’s past trajectory. It is the wavelike signature of the droplet path.

Since centered Bessel functions form an orthogonal basis, it is possible to decompose the dimensionless wave field energy on these Bessel modes:

\[
E_{\text{wave}} = \sum_{n \geq 0} E_{\text{wave}, n},
\]

(4)

where \( E_{\text{wave}, n} \) is the energy of the \( J_n \) centered Bessel function normalized by the energy of a \( J_0 \) Bessel function corresponding to a single bounce. Figure 5(a) shows the contributions of Bessel modes of order \( n = 0, 1, 2 \) to the global wave field energy. Only these first three modes contribute significantly to the wave field energy. The energy of the centered \( J_0 \) is given by \( E_{\text{wave}, 0} = \text{Me}_c^2 J_0^2 (k_\text{F} R) \) (see Appendix C). At first sight, this energy could diverge with memory. However, as mentioned previously, this mode is responsible for the radial force balance of the droplet [see Fig. 4(c)]. Hence, the radius of the self-orbit undergoes a self-tuning to maintain \( E_{\text{wave}, 0} \) constant. As a result, with increasing memory, \( k_\text{F} R \) tends to a zero of the \( J_0 \)
Bessel function [see Eq. (3)]. This self-organization leads to a memory independent finite limit at long memory satisfying $E_{\text{wave},0} \simeq (k_0 R_0 \beta)^2 J_1^2(k_0 R_0)$. The energy of the other modes only depends marginally on the memory parameter. At long memory, $E_{\text{wave},n>0} \simeq 2 J_n^2(k_0 R_0)/(n \omega T)^2$. When evaluated at $k_0 R_0$, the amplitude of the Bessel functions tends rapidly to 0 with increasing $n$. Hence, only modes $n = 1$ and 2 contribute significantly to the wave field energy. The variations of the contribution between the different modes just above $\text{Me}$ originate in the evolution of the self-orbiting mode towards $R_0$. Figures 5(b)–5(e) show the wave field associated to the self-orbiting mode in the long memory limit together with its decomposition in centered Bessel modes. In agreement with the energy analysis, the $J_1$ Bessel mode is dominant. Note that, in the infinite memory limit, the amplitudes of all the nonzero Bessel functions are fixed. In contrast, $J_0$ being to oppose inertia, its amplitude depends on droplet mass.

The projection onto the Bessel mode orthonormal basis enables a calculation of the angular momentum of the wave field since $L_{\text{wave}} = \sum_{n>0} L_{\text{wave},n}$ where $L_{\text{wave},n}$ is the angular momentum of the centered $J_n$ Bessel function. For an angular frequency $\omega$, $L_{\text{wave},n} = n \frac{E_{\text{wave}}}{\omega}$ [39]. As discussed before [Fig. 5(a)], in the case of self-orbiting modes, the energy contributions of all the nonzero Bessel modes rapidly reach a finite limit at long memory. Hence, the normalized angular momentum of the wave field also tends to a finite value. The symmetries of the trajectory are present in the wave field. The trajectory being spatiotemporally helical, it possesses an intrinsic angular momentum. As memory increases, the contributions of all the bounces along the trajectory for one revolution become more alike. Consequently, the axial asymmetry along the droplet path decreases on the scale of one turn. However, when integrated over the memory time, a constant asymmetry is recovered. The constant velocity of the droplet is a direct consequence of the helical symmetry of its trajectory. Note that similar angular momentum exchanges have been observed in optical tweezers using Bessel beams. The orbital angular momentum of the light beam sets trapped microparticles into rotation [40]. In the case of walkers, contrary to standard static external potentials that are limited to spatial symmetries, the dynamical potential associated with the wave field holds the whole spatiotemporal symmetry of the trajectory.

The self-orbiting mode is surprising, in particular because the rotation center does not coincide with the center of mass. As shown above, this is a direct consequence of the role of the wave field. It is interesting to compare this motion with the more standard situation in which two identical droplets orbit symmetrically around their center of mass. This binary motion has already been studied [12,21,37] and it was shown that the orbits could only have discrete sizes. For in-phase bouncing droplets, the discrete set of possible radii is also related to the successive zeros of the $J_0$ Bessel function. However, in this latter case the orbiting results from a mutual attraction: It exists even at short memory, being due to the effect on each drop of the wave emitted by the other.

For self-orbiting modes, the effect of the memory secondary sources in the radial direction is similar to that of a virtual drop that would be bouncing with the same phase and diametrically opposed to the real one [27]. A self-orbit is thus in a sense a binary spinning system composed of a massive pointlike object (the droplet) set in rotation by its interaction with a virtual counterpart. This mirror droplet is the echo of the object (the droplet) set in rotation by its interaction with a wave field. This spatiotemporal nonlocal dynamics of the walkers makes possible the rotation around a center different from the center of mass. It is tantalizing to attribute an effective mass to the droplet of the wave emitted by the other.
behaves as if it was slightly heavier [13,27,29,31,41,42], but in the case of self-rotating states, the wavelike counterpart of the droplet should then be endowed with an identical mass, similar to a twin droplet rotating system. The effective angular momentum of the self-orbiting droplet should then be doubled when compared with that of an undressed droplet. It is interesting to note that a charged self-orbiting droplet would in this case possess a half-integer gyromagnetic ratio that cannot be found when mass and charge have a similar spatial distribution.

We have shown that self-orbits are attractors for free walkers. Thus it is natural to wonder if more generally these attractors play a role in nonisolated dynamics at long memory. Figure 6(a) shows the trajectory of a droplet in a wide potential (defined by $\Lambda \approx 2$; see [35]) at very high values of the memory (e.g., $M_e \geq 500$). The motion is then highly chaotic (intermittence) and in frequent occurrence the droplet is observed to be trapped in orbiting motion. It performs one or several successive orbits before escaping and getting trapped in another self-orbit motion elsewhere. The observation of the waves [Fig. 6(b)] reveals that the local structure of the wave field is then very similar to that observed in self-orbits [see Fig. 1(b)]. These small scale orbits are statistically relevant: if the radius of curvature $R_S$ is measured along the whole trajectory a peak is observed centered at a value $R_S = 0.37\lambda_F$. The presence of these elementary orbits, each endowed with a fixed elementary unit of angular momentum, appears to be general. It is also observed at very high memory for other types of confinement, when the walker is enclosed in corrals, for instance [Fig. 6(c)]. These small orbits are the first building blocks of complex trajectories observed at high values of the memory parameter ($M_e > 100$) in confined geometries. In previous articles investigating the walker dynamics in an attractive potential, it was shown that the dynamics of walkers could be analyzed in terms of time scales [34]. While the short time scale effect is simply responsible for the walker’s propulsion, the intermediate scale can generate spontaneously pivotal structures around which the droplet performs U turns. At a larger time scale, these pivots can become the building blocks of a self-organization into a global coherent orbital motion [34]. However, this is only possible in very narrow ranges of values of the width of the confining potential well. In all other cases the pivots cannot become coherently spatially organized in a global mode and the resulting frustration results in chaotic motion [35]. At long memory, the dynamics of a confined walker is essentially determined by a self-organization between the droplet motion and its wave field. This is why the signature of self-orbiting attractors emerges systematically.

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FIG. 6. Spontaneous transient self-orbiting loops. At very long memory, the trajectory of a confined walker exhibits generically the spontaneous formation of multiple transient self-orbiting loops. (a) Trajectory in a wide potential well at $M_e = 500$. (b) Photograph of the wave field structure in this regime showing, in the vicinity of the drop, the self-organized structure of a small self-orbit. (c) The same transient self-orbits observed when a long memory walker is confined in a circular corral.

APPENDIX A: PATH-MEMORY MODEL

We consider a vibrated bath of silicon oil with an acceleration amplitude $\gamma_m$ typically between 4.2$g$ and 4.5$g$ (the Faraday threshold $\gamma_F$ is at 4.5$g$). In the latter range, a submillimetric drop is bouncing in the period-doubling regime.
and is self-propelled. The horizontal dynamics is decoupled from the vertical one.

1. Vertical dynamics

We restrict ourselves to the period-doubling regime: two oscillations of the bath per drop bounce. The position of the bath is denoted $z_b = z_{b,0} \sin(\omega t)$ with $\omega_0 = 2 \times 2\pi / T_F$ and $T_F = 25$ ms the Faraday period. The vertical dynamics is sequentially decomposed as follows.

The drop takes off for the first time at $t_0$ when the gravity is balanced by the bath acceleration. Once $t_0$ determined, the following takeoffs occur sequentially at times $\{t_0 + nT_F\}_{n\in\mathbb{N}}$. The drop takes off with an initial vertical speed $v_{r,0} = \omega_0 z_{b,0} \cos(\omega_0 t_0)$ and an altitude $z_0 = z_{b,0} \sin(\omega_0 t_0)$. Once in the air, the vertical dynamics of the drops is dictated by the equation $\ddot{z}_b = -g$ with the initial conditions given above.

The landing time $t_l$ corresponds to the time when the drop altitude matches the bath surface again. The time spent in the air is denoted $\Delta t_{air} = t_l - t_0$ while the contact time is given by $\Delta t_{surf} = T_F - \Delta t_{air}$.

Experimentally, the bath oscillates at a frequency of 80 Hz and an amplitude $z_{b,0} \approx 1.8 \times 10^{-4}$ m. Thus, the bath acceleration is about 4.3 g. With these parameters, the initial take-off phase is at $\omega_0 t_0 = 0.21$, $\Delta t_{air} = 19.5$ ms $= 0.78 T_F$, and $\Delta t_{surf} = 5.5$ ms $= 0.22 T_F$. These numerical values are in accordance with those of Moláček and Bush [24].

2. Horizontal dynamics

The drop is modeled as a point. When the drop is in the air, we numerically solve the horizontal equation of motion with a Runge-Kutta method (ODE45, with MATLAB).

The speed just before the $n$th landing is noted $v$ and can be decomposed into a term $v_{||}$ tangential to the surface and one $v_{\perp}$ perpendicular. The transfer of horizontal momentum depends on the angle between the incoming speed and the normal of the surface $n$. If the surface field $h$ is of small amplitude compared to the Faraday wavelength, the relation between the horizontal speed just before $v_{||}(t_n^-)$ and the speed just after $v_{||}(t_n^+)$, at the $n$th landing, is given by $v_{||}(t_n^+) - v_{||}(t_n^-) = ||v_{\perp}|| V_{\perp} h$.

During the “contact” time, the drop loses a part of its energy due to its interaction with the bath which leads to $v_{||}(t_n^+ + \Delta t_{surf}) = v_{||}(t_n^-) e^{-\Delta t_{surf} / T_F}$. The characteristic time is $T_c = 5$ ms which means that $v_{||}(t_n^+ + \Delta t_{surf}) / v_{||}(t_n^-) \approx 0.35$. Note that the total horizontal restitution coefficient $C_\parallel = v_{||}(t_n^+ + \Delta t_{surf}) / v_{||}(t_n^-)$ is about twice as large. This agrees with the measurements of Moláček and Bush [24] who give an approximate value of $\approx 0.7 \pm 0.1$ in the Weber regime of interest.

3. Field evolution

The surface field seen by the drop at the phase of impact $\phi$ is updated at each bounce. At the $n$th landing, the surface field is $h(r, t_n^-)$ [see Eq. (1)].

The gradient of the surface field is taken at the drop position. The remaining coefficient $h_0$ is chosen so that the walking speed matches the experimental one (10 mm/s) in the case $F_{ext} = 0$ and at high memory. The field is initialized as $h(r, t = 0) = 0$. Note that this numerical model contains no free parameter [22]. Only the memory $M_e$ and/or the external forces are control parameters.

APPENDIX B: COMPUTATION OF THE FORCE EXERTED BY THE WAVE FIELD

This force is due to the bouncing of the drop on a locally slanted surface and is proportional to the local slope. In order to obtain this force we first compute the global wave field $h(r,t)$ and expand it on the wave basis of centered Bessel modes $\{J_n(k_F r)e^{i n \theta}\}_{n \in \mathbb{Z}}$. It gives

$$h(r,t) = h_0 \sum_{n=0}^{+\infty} e^{-i n \phi} \sum_{m=-\infty}^{+\infty} J_n(k_F r) J_m(k_F r) e^{-i n \phi}.$$  

(B1)

In the case of a circular motion, with a radius $R$ and an angular velocity $\omega$, without loss of generality we choose at the time of impact $t$, the angular position of the drop $\theta t$.

We have the following for all $j$: $r_j = R, t - t_j = j T_F$, and $\theta_j = \omega t - \omega j T_F$, where $\theta_j$ is the angular position of the drop at $t = t_j$, an arbitrary constant which depends on the choice of the frame axis. It gives

$$h(r,t) = h_0 \frac{1}{T_F} \sum_{n=0}^{+\infty} (2 - \delta_{n,0}) J_n(k_F R) J_n(k_F r) \tau$$

$$\times \left\{ \frac{\cos [n(\theta t - \omega t)]}{1 + (n \omega t)^2} - \frac{n \omega t \sin [n(\theta t - \omega t)]}{1 + (n \omega t)^2} \right\},$$

(B2)

with $\delta_{n,0}$ the Kronecker symbol.

For the radial force we get

$$F_{M_e}^R = \frac{\delta_0}{T_F} J_0(k_F R) J_1(k_F r)$$

$$- \frac{1}{\tau} \left\{ \frac{\delta_0}{T_F} \sum_{n=0}^{+\infty} J_n(k_F R) [J_{n-1}(k_F R) - J_{n+1}(k_F R)] \right\}$$

$$\times \left[ \frac{\tau^2}{1 + (n \omega t)^2} \right].$$

(B3)

In the long memory limit $\omega t \gg 1$, we have the simplification

$$F_{M_e}^R = \delta_0 \frac{\delta_0}{T_F} \frac{h_0}{T_F} J_0(k_F R) J_1(k_F r) + O\left( \frac{1}{M_e} \right).$$

(B4)

with $M_e = \frac{\delta_0}{T_F}$ which justifies Eq. (2) in the long memory limit. For the tangential force, we get

$$F_{M_e}^\perp = \frac{2 \delta_0}{V T_F} \sum_{n=1}^{+\infty} J_n^2(k_F r) \left[ \frac{(n \omega t)^2}{1 + (n \omega t)^2} \right].$$

(B5)

with $V = \omega R$. In the high memory limit $\omega t \gg 1$, the force can be expressed as

$$F_{M_e}^\perp = \frac{2 \delta_0}{V T_F} \sum_{n=1}^{+\infty} J_n^2(k_F r) = \frac{C_0}{V T_F} [1 - J_0^2(k_F R)].$$

(B6)
APPENDIX C: COMPUTATION OF THE WAVE FIELD ENERGY

Let us evaluate the energy stored in the wave field $E_{\text{wave}}$ normalized by the wave field energy induced by a single bounce modeled as a $J_0$ Bessel function.

The energy of the wave field for a monochromatic wave over a domain $\Omega$ of radius $R$ is proportional to the square of the surface displacement $h(r,t)$ integrated over the surface of the bath. Using the wave basis of the centered Bessel functions \( \{ J_n(kr)e^{i\omega t} \}_{n \in \mathbb{Z}} \), the energy can thus be written as

\[
E \propto h_0^2 \sum_{n=-\infty}^{+\infty} A_n^2 \int_{\Omega} J_n^2(kr) d^2r. \tag{C1}
\]

In the case of a walker orbiting at a radius $R$ with an angular velocity $\omega$, from Eq. (7) we obtain $A_n^2 = \frac{1}{r^2} J_n^2(krR) \frac{1}{\Omega_1} \mathrm{Re} \langle n \rangle^2$. Normalizing the wave energy in the domain $\Omega$ by that of a single bounce $E_{\text{bounce}}$ gives

\[
E_{\text{wave}} = \lim_{R \to \infty} \frac{E}{E_{\text{bounce}}} \simeq \sum_{n=-\infty}^{+\infty} A_n^2. \tag{C2}
\]

It is possible to write $E_{\text{wave}} = \sum_{n=0}^{\infty} E_{\text{wave},n}$ as the energy of the $J_n$ centered Bessel function normalized by the energy of a $J_0$ centered Bessel mode corresponding to a single bounce. In particular, $E_{\text{wave},0} = M e^2 J_0^2(krR)$.

[36] See Supplemental Material at http://link.aps.org supplemental/10.1103/PhysRevE.94.042224 for a stroboscopic film showing the orbit of a droplet in a circular orbit of radius \( R \approx 0.37a_0 \). The dark part of each image is the pure iron pointed rod on which is attached a small lamp: when it is lit the magnetic field is on. At approximately half the film length the current (and the lamp) are switched off and the droplet is observed to keep orbiting. The memory parameter equals \( Me = 140 \).