# Algebra

**1. Arithmetic sequences and series, sum of finite arithmetic series, geometric sequences and series, sum of finite and infinite geometric series, sigma notation.**
- Be able to distinguish between arithmetic sequence and geometric sequence.

Arithmetic sequence = addition of constant, $u_n = u_1 + (n-1)d$
Geometric sequence = multiplication of constant, $u_n = u_1 r^{n-1}$
Sum of arithmetic sequence = see formula booklet and watch proof (very helpful)
Sum of geometric sequence = see formula booklet and watch proof (very helpful)

Be able to use sigma notation.

Finite sum with variable = $\sum_{i=0}^{n} i = 0 + 1 + 2 + 3 + \cdots + n$
Infinite sum with variable = $\sum_{i=0}^{\infty} i = 0 + 1 + 2 + 3 + \cdots + \infty$
Finite sum with constant = $\sum_{i=0}^{n} k = k + k + k + k + k \ldots = n \times k$

Be able to split sigma notations.

$$\sum_{i=0}^{n}\big(f(i) + g(i)\big) = \sum_{i=0}^{n} f(i) + \sum_{i=0}^{n} g(i)$$

Be able to change the upper and lower bounds.

$$\sum_{i=k}^{n} f(i) = \sum_{i=k+r}^{n+r} f(i-r)$$

$$\sum_{i=k}^{n} f(i) = \sum_{i=k-r}^{n-r} f(i+r)$$

$$\sum_{i=5}^{100} x^2 + 3x = \sum_{i=1}^{96} (x+4)^2 + 3(x+4)$$

The idea here is that $5^2 + 3 \cdot 5 = (1+4)^2 + 3 \cdot (1+4)$. So you are not changing anything. You are only breaking the numbers apart.

Be able to use simple interest and compound interest.

Simple interest = an addition of fixed money per year. This means that we can express it in an arithmetic sequence.

Compound interest = multiplied money per year, such as you add 5% more money of the money you had in your previous year. This means that we can express in geometric sequence.

Questions might however not give you the yearly compound interest. For instance, if the interest rate is 5% and you compound it 4 times a year and not once per year, you have to adjust the interest rate.

What you do is you simply divide.

$$Adjusted\ interest\ rate = \frac{interest\ rate}{number\ of\ times\ you\ compound\ in\ a\ year}$$

Ex, a compound interest of 6% is compounded every 4 months for 3 years. Every 4 months = compounding 3 times a year. So in total, you compound 3x3 = 9 times, at 0.06/3

$$Money\ after\ 3\ years = Money\ in\ the\ beginning \times (1 + \frac{0.06}{3})^9$$

<u>Be able to use geometric sequence in form of annuity.</u>

Annuity is when you add money every year with a certain interest rate. In this case it is 5%.

Year 1 = 1000

Year 2 = 1000 + 1000·1.05

Year 3 = 1000 + 1000·1.05 + $1000 \cdot 1.05^2$

Year n = 1000 + 1000·1.05 + $1000 \cdot 1.05^2$ … + $1000 \cdot 1.05^{n-1}$

This forms a geometric sequence where r = interest rate, $u_1$ = money added and n = number of years.

**2. Exponents and logarithms, laws of exponents, laws of logarithms, change of base.**

- <u>Be able to know the basic concepts of logarithms.</u>

Logarithm is just another form of expressing a number. Consider $a^x = b$

We have expressed *b* in terms of *a* and *x*, and we can express *a* in terms of *b* and *x* as such.
$a = \sqrt[x]{b}$

But how do express the little *x* in terms of *a* and *b*? This is where we use logarithms.

$$x = log_a b$$

Here are the four underlying properties in logarithms.

1. $log_a xy = log_a x + log_a y$

2. $log_a \frac{x}{y} = log_a x - log_a y$

3. $log_a x^n = n \times log_a x$

4. $log_a x = \frac{log_b x}{log_b a}$, where *b* is the new base.

Be able to use natural logarithms.

Natural log is just a normal logarithm with a specific base. The base here is $e$.

$$lnx = log_e x$$

You will see later, especially in calculus, how handy natural logs are when deriving and integrating.

**3. Counting principles, including permutations and combinations. The binomial theorem: expansion of $(a + b)^n$ , $n \in N$.**

Be able to use permutations and combinations.

Permutation is arrangement with order. So it matters if you choose red before blue from a pocket of coloured beads where you are told to choose one red and one blue.

$$P_k^n = \frac{n!}{(n-k)!}$$

Combination is arrangement without order. It does not matter whether you choose red before blue or blue before red, as long as you just pick one red and one blue. In other words, combination is permutation with overlaps of same combination excluded.

$$C_k^n = \frac{n!}{(n-k)! \times k!}$$

Where *k!* is the possible arrangements with the same combination. Since we want to have only one from each combination, we divide by *k!*.

Also, you will notice as you solve, and in the formula above that in combinations, selecting = discarding. The number of combinations you can select is the same number of combinations you can discard.

Be able to use binomial theorem.

Binomial system = two-named theorem just like in biology. In this case, the "name" is the variable and constant.

Coefficient of binomial theorem = combination of choosing the term from.

For coefficients in $(x+1)^n$, they follow Pascal's triangle.

| Exponent | Pascal's Triangle |
|---|---|
| 0 | 1 |
| 1 | 1 1 |
| 2 | 1 2 1 |
| 3 | 1 3 3 1 |
| 4 | 1 4 6 4 1 |
| 5 | 1 5 10 10 5 1 |
| 6 | 1 6 15 20 15 6 1 |

We can thus write expansion in terms of combinations.

$$(a+x)^n = \binom{n}{0} a^{n-0} + \binom{n}{1} a^{n-1}x + \binom{n}{2} a^{n-2}x^2 + \cdots \binom{n}{n} a^0 x^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} x^r$$

Now you can expand higher orders without multiplying out every single term.

<u>Be able to the coefficient of the term with a certain power, including the constant.</u>

Ex, find the coefficient containing $x^7$ in $(3x\text{-}5y)^{10}$. To solve this, simply find *r* by writing out the expression of binomial expansion.

$$(3\mathrm{x}-5\mathrm{y})^{10} = \sum_{r=0}^{10} \binom{10}{r} (3x)^{10-r} (5y)^r$$

We can see that when *r=3*, we will have the term $x^7$. So then you plug in 3 as r and calculate!

**4. Proof by mathematical induction.**
- <u>Be able to know the general stages of mathematical induction.</u>

Step 1: Try n=1 and state that it is true. Obviously to prove anything, the most basic, first numbers must be true.

Step 2: Replace n with k and assume all values of k is true.

Step 3: Show that k+1 is true. Always work with the left side.

Step 4: Summarize your finding very briefly in words.

<u>Be able to prove general formula for sum.</u>

Ex, prove that $S_n = 1 \times 3 + 2 \times 4 + 3 \times 5 \ldots n \times (n+2) = \frac{1}{6} n(n+1)(2n+7)$

Step 1: When n=1, $1 \times 3 = \frac{1}{6}(1)(1+1)(2+7)$

Step 2: Assume true for all *k*. $S_k = 1 \times 3 + 2 \times 4 + 3 \times 5 \ldots k \times (k+2) = \frac{1}{6} k(k+1)(2k+7)$

Step 3: We want to show that $S_{k+1} = \frac{1}{6}(k+1)\big((k+1)+1\big)(2(k+1)+7)$

$$S_{k+1} = \frac{1}{6}(k+1)(k+2)(2k+9)$$

We show this by working with left hand side.

$$S_{k+1} = S_k + (k+1) \times ((k+1)+2)$$

Step 4: Summarize that following mathematic induction, it is true for all $k$.

Be able to prove divisibility/multiplicity and algebraic formulae.

Do this with your teacher because it is ultimately practice.

**5. Complex numbers: the number i = $\sqrt{-1}$ ; the terms real part, imaginary part, conjugate, modulus and argument. Cartesian form z = a + ib . Sums, products and quotients of complex numbers.**

- Be able to understand what complex numbers are.

$$Complex\ number\ =\ real\ number\ +\ imaginary\ number$$

$$z = a + ib$$

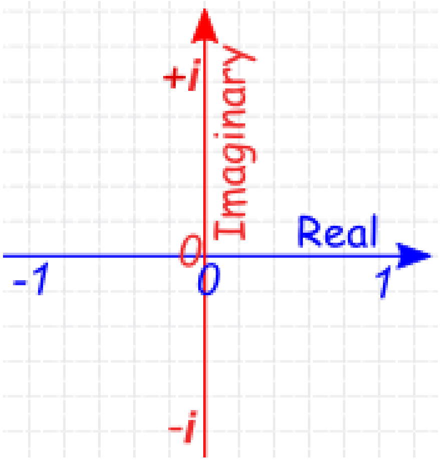
Be able to find real and imaginary parts.

Ex, $z = 7 - 10i$. Then $Re(z) = 7,\ Im(z) = -10.$

Be able to find conjugate of z.

Conjugate of a complex number is the same number with opposite sign on imaginary part. Conjugate of $z = a + ib$ is $z^* = a - ib$

In an Argand diagram (a coordination plane for complex numbers), a conjugate would mean reflection on the $Re(z)$ axis.




Be able to find the modulus of z.

A modulus is just the length/magnitude of z. Just like usual, we use Pythagoras theorem to find the length of a segment.

$$\text{Modulus} = |z| = \sqrt{(Re(z))^2 + (Im(z))^2}$$

<u>Be able to add/subtract complex numbers.</u>

Collect like terms like usual.

$$a + ib + c + id = (a + c) + i(b + d)$$

<u>Be able to multiply complex numbers.</u>

Expand like you normally do.

$$(a + ib)(c + id) = ac + iad + ibc - db = (ac - db) + i(ad + bc)$$

<u>Be able to divide complex numbers.</u>

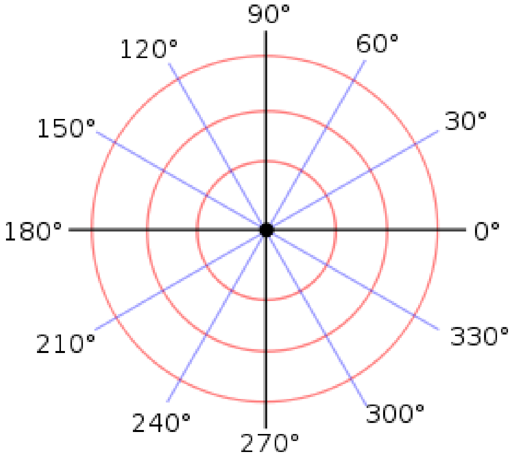
Remove denominator by multiplying by its conjugate. This way we can remove any imaginary number because two values will cancel out just like in (a-b)(a+b), as well as $i^2$ will just become -1.

$$\frac{c + id}{a + ib} = \frac{c + id}{a + ib} \times \frac{a - ib}{a - ib}$$

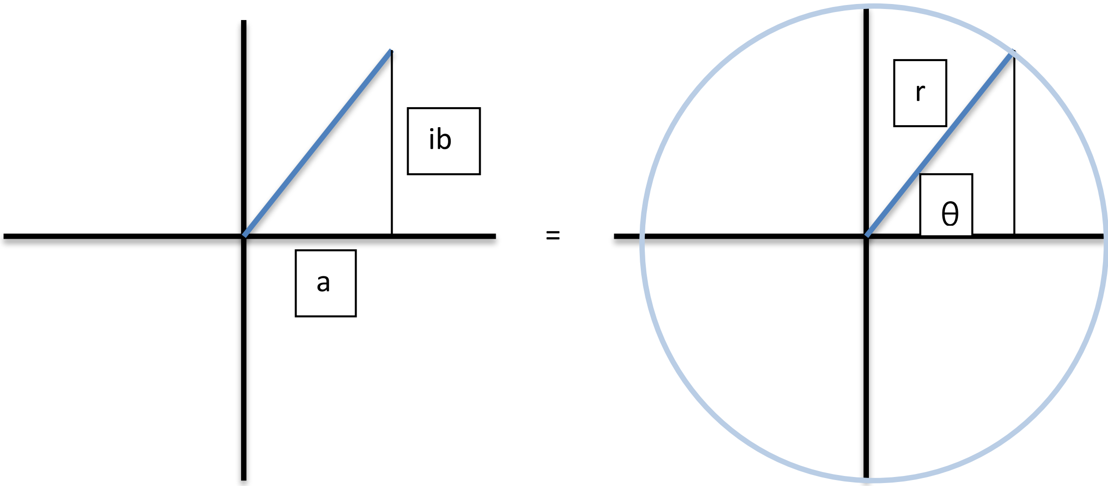
Expand as usual.

**6. Modulus–argument (polar) form z = r(cosθ + i sinθ ) = rcisθ = $re^{i\theta}$. The complex plane.**
- <u>Be able to convert from argand diagram to polar diagram.</u>

Complex numbers can be plotted in a polar diagram too (and not only in Argand diagram). We are just changing the ways of expressing a certain value, just like we swap between radians and degrees to express an angle.




What is needed is an angle to show direction, and an *r* to show how long. In argand diagram, we needed how much horizontally and how much vertically. But the idea is exactly the same




Using trigonometry, $b = rsin\theta$ and $a = rcos\theta$

Thus, $z = a + ib$ can be rewritten as $z = rcos\theta + irsin\theta = r(cos\theta + isin\theta)$

For convenience, we shorten $\boldsymbol{r}(\boldsymbol{cos}\theta + \boldsymbol{isin\theta}) = rcis\theta$.

Note that $\{\theta: -\pi < \theta \leq \pi\}$.

Be able to use rcisθ = re$^{i\theta}$.

This is just another way of expressing a complex number. The exponent form is very convenient when dividing and multiplying.

The rule is $re^{i\theta} = r(cos\theta + isin\theta)$

Which means that $e^{i\theta} = cos\theta + isin\theta$.

To express its conjugate in exponent form,

$e^{i(-\theta)} = cos(-\theta) + isin(-\theta)$. However, $cos(-\theta) = cos(\theta)$ due to cosine being an even function and $\sin(-\theta) = -\sin(\theta)$ due to odd function (you can verify by drawing the unit circle if you want to).

So we can rewrite it as $e^{i(-\theta)} = cos(\theta) - isin(\theta)$, which is the conjugate.

Be able to solve $z_1$=$z_2$.

For the two complex number to be identical, its magnitude must be the same (which is the *r*) and also their angle.

So, $|z_1| = |z_2| \leftrightarrow r_1 = r_2$ and, $\theta_1 = \theta_2 + 2\pi k$. It is extremely crucial with $2\pi k$ because it can make one more revolution and still be the same complex number. So there are multiple solutions!!!

Be able to multiply in polar form.

$$z_1 \times z_2 = r_1(cos\theta_1 + isin\theta_1) \times r_2(cos\theta_2 + isin\theta_2)$$

$$r_1(cos\theta_1 + isin\theta_1) \times r_2(cos\theta_2 + isin\theta_2) = r_1 r_2\big(\cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)\big)$$

Why suddenly add the angles? You will see that if you expand as usual, you will get:

$$r_1 r_2(cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2 + i[sin\theta_1 cos\theta_2 + sin\theta_2 cos\theta_1])$$

Now, we can use the double angle theorem to simplify! It is in your formula booklet.

$$cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2 = cos(\theta_1 + \theta_2)$$

$$sin\theta_1 cos\theta_2 + sin\theta_2 cos\theta_1 = sin(\theta_1 + \theta_2)$$

Be able to divide in polar form.

Similarly, using the same expansion and same principle of double angle theorem, we get the same thing, but subtracting the angles instead of adding:

$$r_1(cos\theta_1 + isin\theta_1) \div r_2(cos\theta_2 + isin\theta_2) = \frac{r_1}{r_2}\big(\cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)\big)$$

Be able to multiply in exponential form:

This is an alternative and a much easier way to multiply complex numbers.

$$z_1 \times z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}$$

$$z_1 \times z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

We can verify that the angles are added in exponential form too.

Be able to divide in exponential form:

$$z_1 \div z_2 = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Likewise, we can see that angles are subtracted like in polar form.

**7. Powers of complex numbers: de Moivre's theorem, n^th roots of a complex number.**

- <u>Be able to raise powers of complex numbers:</u>

De Moivre's theorem is a formula for raising powers of complex numbers.

Consider $(4 + 6\mathrm{i})^7$. Expanding this would take quite some time even though you use the binomial expansion. De Moivre solved this problem for us, and it is essentially the same principle as our multiplication in of exponential and polar form.

$$\mathrm{z} \times \mathrm{z} \times \mathrm{z} \ldots \times \mathrm{z} = z^n$$

$$[r(cos\theta + isin\theta)]^n = r^n(cosn\theta + isinn\theta)$$

Or alternatively, $[r(cos\theta + isin\theta)]^n = r^n cisn\theta = r^n e^{in\theta}$

<u>Be able to take roots of complex numbers:</u>

Taking root builds upon De Moivre's theorem (and not the same as division of complex numbers). Simply, root is the 1/n.

$$[r(cos\theta + isin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta + 2\pi k}{n}}$$

Now, where did $2\pi k$ come from? We add it because there are multiple solutions in a root, so we need to take into account that the initial complex number's multiple solutions.

If the root is *n* = 3, then *k* = 0, ±1. If *n = 7*, then *k* = 0, ±2, ±3, ±4. If it is an even number, there will be overlap somewhere so you must look the two *k* that give the same angle.

<u>Be able to understand roots of unity:</u>

It is the same thing as above, but we take root from 1.

$$z^n = 1$$

$1 = e^{i(0+2\pi k)}$ because when θ=0, we have just a line along the real number axis with r=1 as magnitude, hence in complex number terms, it is 1+0i. Then after we replace 1, we solve it as we did above.

$$z^n = e^{i(0+2\pi k)}$$

$$z^{n \times \frac{1}{n}} = e^{i(0+2\pi k) \times \frac{1}{n}}$$

$$z = e^{\frac{i2\pi k}{n}}$$

As above, *k* depends on *n*.

Be able to know another important property:

This may be useful when you want to simplify/expand/solve/etc.

Consider $z = cos\theta + isin\theta,\ z = e^{i\theta},\ \frac{1}{z} = e^{-i\theta}$

$$z + \frac{1}{z} = cos\theta + isin\theta + (cos\theta - isin\theta)$$

$$z + \frac{1}{z} = 2cos\theta$$

Using the same logic, $z - \frac{1}{z} = 2isin\theta$.

If we combine this with De Moivre's theorem, we can get the following.

$$z^n + \frac{1}{z^n} = 2\cos(n\theta)$$

$$z^n - \frac{1}{z^n} = 2isin(n\theta)$$

**8. Conjugate roots of polynomial equations with real coefficients.**
- When calculating *x* in polynomial equations, the discriminant may be less than zero $\Delta < 0$. Where $\Delta = b^2 - 4ac$. Thus this means that if discriminant is less than 0 in this equation, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we have imaginary solutions.

Always remember that due to the $\pm$ sign in the equation, one imaginary solution always has its conjugate as solution too. More of this will be covered in topic 2.

**9. Solutions of systems of linear equations (a maximum of three equations in three unknowns), including cases where there is a unique solution, an infinity of solutions or no solution.**
- Be able to solve system of three equations:

There are different ways to solve this. I personally used simultaneous equations, but some may prefer Gaussian method.

When there is a unique solution, you will reach a definite answer for each of the three variables.

When there are infinite solutions, you will reach $0 = 0$, or $x = \frac{0}{0}$.

When there are no solutions, there will be a contradiction in the equation such as $0 = 2$.

You will get used to this in topic 4 when solving vectors in 3 dimensional planes.