Calculus

1. Informal ideas of limit, continuity and convergence. Definition of derivative from first principles. The derivative interpreted as a gradient function and as a rate of change. Finding equations of tangents and normals. Identifying increasing and decreasing functions. The second derivative. Higher derivatives.

- Be able to understand the basic concepts:

Limit = a concept to explain something approaching but not completely being at the point. The notations are very simple. $\lim_{x\to L^+} f(x)$ is when you approach the certain value L from the positive side. $\lim_{x\to L^-} f(x)$ is when you approach from the negative side.

Continuity = if $\lim_{x\to L^+} f(x) = \lim_{x\to L^-} f(x)$. Thus limits from both sides must approach the same value in order for there to be a continuous line at a particular point.

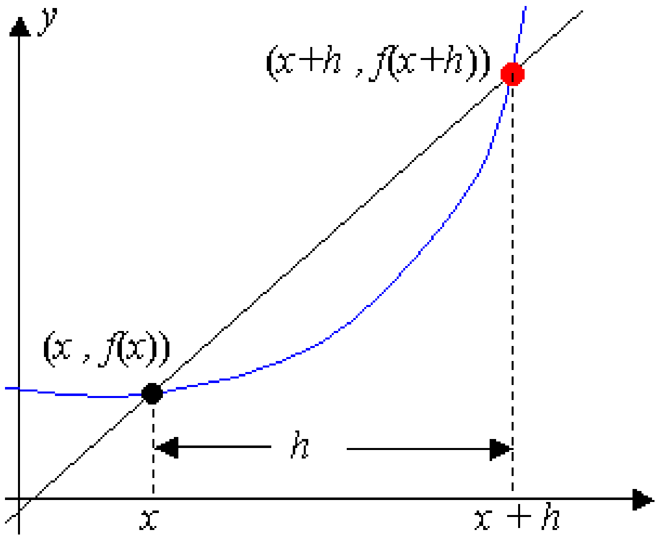
Convergence = as $\lim_{x\to\infty} f(x)$. The larger the x gets, you are starting to get a straight horizontal line.

Divergence = as $\lim_{x\to\infty} f(x)$. The y value approaches infinity as well.

Be able to understand the basic ideas of derivatives:

Derivative = slope at a point = $\frac{\Delta y}{\Delta x}$

$as\ \lim_{\Delta\to 0} \Delta = d$. In other words, *d* is simply an infinitesimal small change in x or a "smaller Δ". It is not really 0, but approaches 0. So to find out the slope at exactly one point, we use the idea of this limit.



These are just two arbitrary coordinates on the blue curve. To find the slope at exactly one point, we simply approach *h* to 0 so that the delta becomes like a point.

$$\lim_{h\to 0}\frac{y_2 - y_1}{x_2 - x_1}\ or\ \lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x}$$
$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

You may ask how this is valid if denominator approaches 0. There is a separate proof for this, but essentially, *h* will get cancelled out so it becomes a valid expression.

Thus the expression above is the fundamental start of derivatives, and derivatives are simply a way to calculate slope. All the different "rules" you will learn is based on the expression above!

Some basic notations are:

$$first\ derivative = \frac{dy}{dx} = f'(x) = f^1(x)$$

$$second\ derivative = \frac{d^2y}{dx^2} = f''(x) = f^2(x)$$

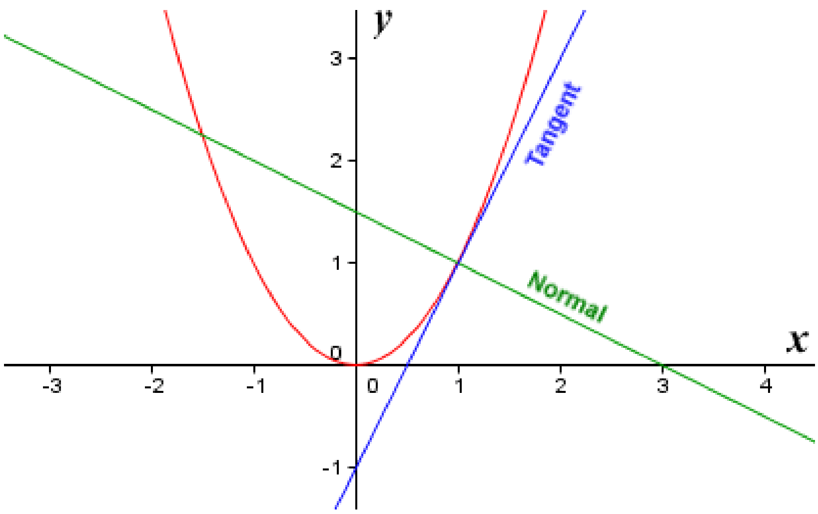
$$arbitrary\ derivative = \frac{d^ny}{dx^n} = f^n(x)$$

The first rule you will learn is the power rule. This is again, all originated from the expression $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.
Essentially, if we put $f(x) = x^n$ and to some hokus pokus, $f'(x) = nx^{n-1}$.

<u>Be able to find equation of tangents and normals:</u>

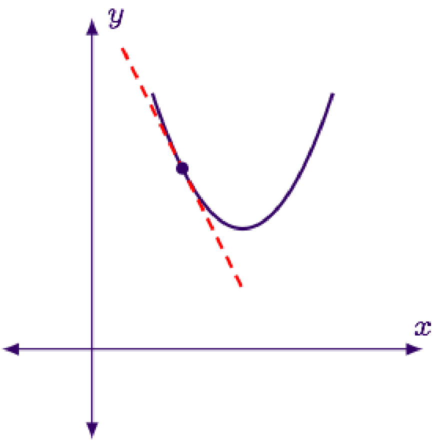
Slope of tangent = this is simply $f'(x)$ because we are finding out the slope of the tangent at a particular point.

Slope of normal = this is the negative reciprocal of the tangent, $-\frac{1}{f'(x)}$. This is just the way to sort of turn the tangent 90 degrees and get its slope.

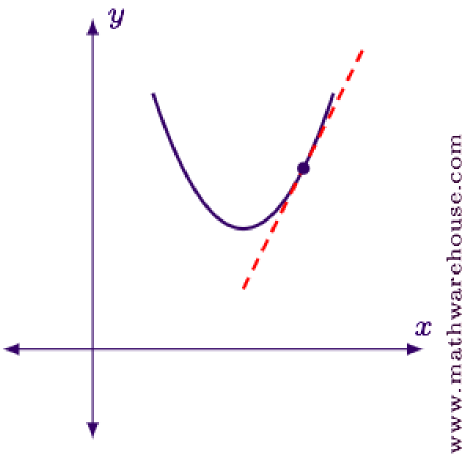


<u>Be able to identify increasing and decreasing graphs:</u>

This is the same as usual. If slope/derivative is positive, it points up. If slope/derivative is negative, it points down.



When the tangent line has a negative slope, the function is decreasing at that point.



When the tangent line has a positive slope, the function is increasing at that point.

Nothing surprising here!

2. Derivatives of x^n, sin x, cos x, tan x, e^x and ln x. Differentiation of sums and multiples of functions. The product and quotient rules. The chain rule for composite functions. Related rates of change. Implicit differentiation. Derivatives of sec x, csc x, cot x, a^x, $\log_a$ x, arcsin x, arccos x and arctan x.

- <u>Be able to use different rules of differentiation:</u>

All of these are in the formula booklet and you just need some practice to work with them! No worries.

Chain rule = used when you can identify a function within a function, $f(g(x))$. For example, $f(x) = (x^3 + x + 3)^7$. Why is this two functions? Well, we have one inner function $u = x^3 + x + 3$, and an outer function that is $v = u^7$.

Product rule = used when you simply have two functions multiplying with each other. For example, $f(x) = (x^5 + 4x)(x^2 + 9)$. You can of course expand, but you do not need to. You can just use the product rule.

Quotient rule = used when you have a fraction!

Most of the time, you have to use a combination of rules because IB does everything to make you fail. So prepare for things like $f(x) = (x^5 + 4x)(x^3 + x + 3)^7$. Here you have to product rule AND chain rule.

Be able to differentiate the functions in formula booklet:

As I said, practice!!! The wider range of experience you have, the easier it will be to identify when to use what.

Be able to differentiate implicitly:

This is a crucial technique that ALWAYS comes up on exams. Basically, you are deriving a function where y isn't the subject, ex $7x^2 + 6xy + y^2 = 8$. You can rearrange to make y the subject, but why bother when you can differentiate just the way it is?

Derivate of the function above will be $14x + 6(xy' + y) + 2y \cdot y' = 0$. You will notice that I used power rule, product rule, and chain rule. When you derive the term y, you must treat it as a composite function because within y there is another function with the terms x.

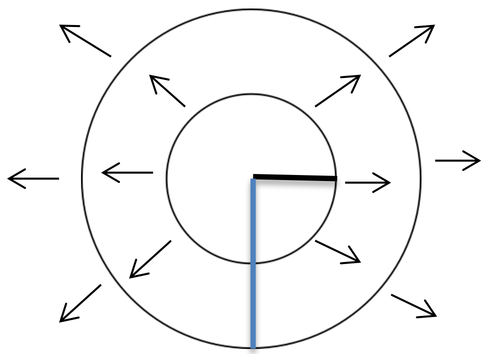
So make sure you know this! It saves time and increases points during exams ;)

Be able to solve related rates of change:

In order to solve using related rates of change, I advise you to start writing derivative denotations the formal way. For instance, $\mathrm{x}' \to \frac{dx}{dy}$ and $\mathrm{y}' \to \frac{dy}{dx}$.

This is very easy to get confused while solving because we are introducing a third variable, which is time (hence the name related *rates* of change). So it is very important to write the information explicitly! For example,

Two circles expand. At a particular time *t*, outer circle's radius = 9m and expands at a rate = 1.2m/s. Inner circle's radius = 1m and expands at a rate = 1.5m/s. What is the rate of change of area between the circles at that time t?



At first it may feel like "wth, you are asking me to calculate the mass of my computer mouse if I have two dogs and one paper". But I will give a "step-by-step template" that I used.

1. Write down the information explicitly. The information we have for outer circle are $\mathrm{R(t)} = 9$ and $\frac{dR}{dt} = 1.2m/s$. Note that we don't just write R' because you must make clear *in what respect* the R is changing. In this case it is in respect to time. The information for inner circle are $\mathrm{r(t)} = 1$ and $\frac{dr}{dt} = 1.5m/s$.

2. Write down the expression of what you want to find out. In this case, we have to find out the rate of change of the area between circles. So first we need to know how to calculate the area between circles.

$A = \pi R^2 - \pi r^2$. Nothing too difficult here.

Now, to find out the rate of change in area, we simply differentiate it in respect to time. Because remember, rate is simply something *per time unit*. And since we don't have any direct variables of time in the expression above, we use chain rule.

$$\frac{dA}{dt} = 2\pi R \cdot \frac{dR}{dt} - 2\pi r \cdot \frac{dr}{dt}$$

3. Plug in the known numbers! We now have a expression for rate of change in area so all we have to do is to plug in the numbers.

$$\frac{dA}{dt} = 2\pi \cdot 9 \cdot 1.2 - 2\pi \cdot 1 \cdot 1.5 = 58.4\frac{m^2}{s}$$

Make sure to get the units correct.

<u>Be able to use trig calculus:</u>

This is also practice by exposing yourself to many situations.

$$(sin\, x)' = cosx$$

$$(cos\, x)' = -sinx$$

$$\int sinxdx = -cosx + c$$

$$\int cosxdx = sinx + c$$

$$\int tanxdx = ln|secx| + c$$

The integral of tan seems very complicated but it is actually very easy to derive. Write out tanx as sinx over cosx, and then use substitution method to solve! You will need formula booklet too.

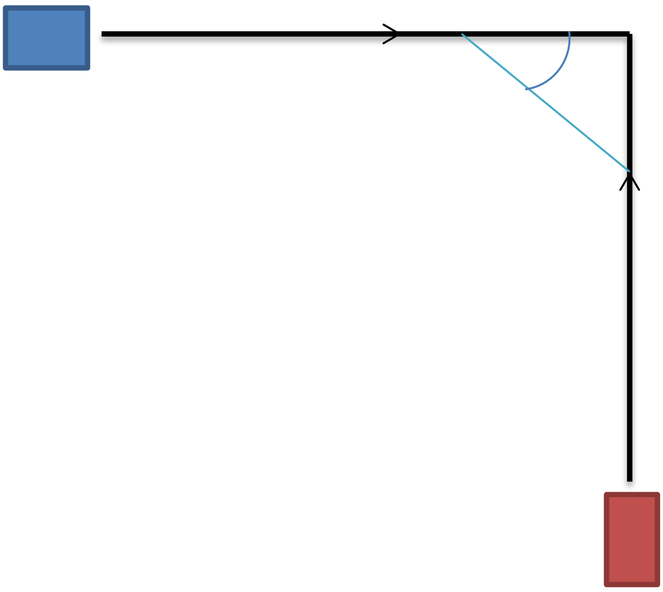
<u>Be able to perform higher derivatives of trig functions:</u>

The only special thing with deriving trig functions is that it repeats after 4 derivatives.

$f(x) = sinx,\ f'(x) = cosx,\ f''(x) = -sinx,\ f'''(x) = -cosx,\ f^4(x) = sinx$

<u>Be able to calculate related rates involving trigs:</u>

This is very similar to how we calculated related rates without trigs. However, we usually have up to 4 variables instead of 3. Time, x, y, and angle. But an example is always good for seemingly daunting things.



Blue boat is approaching at speed $x(t) = 50 - 15t$, and red boat is approaching at speed $y(t) = 40 - 12t$, where t is in hours. How fast is the bearing changing in blue boat's perspective after two hours?

1. Write down the information explicitly. For blue boat, $\frac{dx}{dt} = -15$ and after two hours $x(2) = 20$. For red boat, $\frac{dy}{dt} = -12$ and after two hours $y(2) = 16$.

2. Write down the expression of what you want to find out. We want to know how fast the angle is changing, hence $\frac{d\theta}{dt}$. So we have to somehow connect time, angle, x and y. We first connect angle with x and y because that is very easily done by basic trigonometry.

$$tan\theta = \frac{opposite\ (y)}{adjacent\ (x)}$$

After 2 hours, we know that $\theta = tan^{-1}\left(\frac{16}{20}\right)$. Keep this information because we will need it.

We then find the expression by deriving in respect to time $tan\theta = \frac{opposite\ (y)}{adjacent\ (x)}$. We get

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - y \cdot \frac{dx}{dt}}{x^2}$$

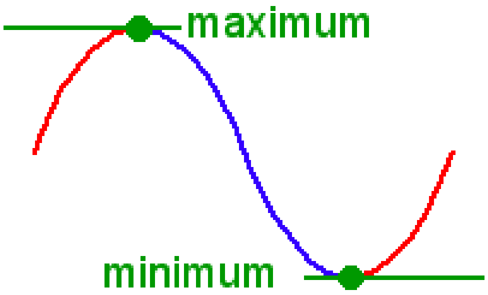
3. Plug in, and you will get that $\frac{d\theta}{dt} = 0$. This means that bearing does not change!

3. Local maximum and minimum values. Optimization problems. Points of inflexion with zero and non-zero gradients. Graphical behaviour of functions, including the relationship between the graphs of f, f ′ and f ″.

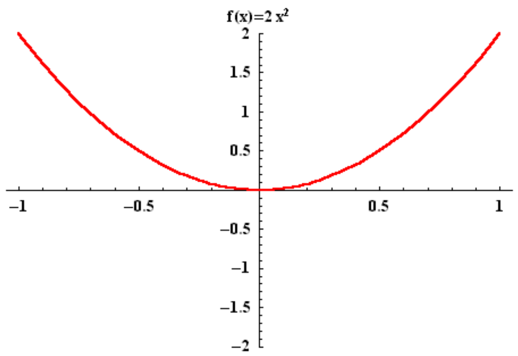
Be able to understand what it means by higher derivatives:

Higher derivate is just finding out the slope at one point of the previous derivative. Certain derivatives tell us certain information and we need to be able to understand them.

$f'(x) = 0$ means that there is a stationary point. Its tangent at that point has slope 0, which is basically just a horizontal line. But, that point can be either a maximum or minimum.



$f''(x) > 0$ means that the slope of function is getting higher and higher. This means that the function has a concavity upwards:



So we see that if second derivative (at the point where $f'(x) = 0$) is larger than 0, that stationary point is a minimum.

$f''(x) < 0$ uses the same idea. We will have a maximum as the shape will be a concavity downwards. This doesn't need to only be used to calculate minimum and maximum. You can simply use this technique to know if the function is increasing or not.

$f''(x) = 0$ means that we have an inflexion point where the function is neither increasing nor decreasing. In other words, it is right at the point between concave up and concave down.

Be able to sketch given f(x) and its f′(x) and f′′(x)…$f^n(x)$:

There are certain coordinates you must specify.

1. x and y interceptions.

2. Vertical and horizontal asymptotes.

3. Max, min and inflexion points.

This is simple, but something that needs practice!

Be able to use derivatives in different applications, kinematics:

So we have learnt all this so we must be able to use it in "real-life situation".

In kinematics, it is about displacement, velocity and acceleration.

$$s(t) = displacement\ at\ time\ t$$

$$v(t) = \frac{ds}{dt}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = instantaneous\ acceleration$$

One of the keys in solving problems related to kinematics is knowing what the signs of $v(t)$ and $a(t)$ imply.

When $v(t) = +$ it means that the object is moving forwards (whatever you define "forward" as). Conversely, $v(t) = -$ means that the object is moving the opposite direction of your defined "forward". When $a(t) = +$ it means the object is getting faster, and $a(t) = -$ means the car is slowing down.

So, if you calculated that a car is travelling at $v(t) = +$ but $a(t) = -$, the car is still moving forward but slowing down.

Be careful when solving distance and displacement. Distance is the area under the velocity curve and displacement is the integral of the velocity curve. In other words,

$$Total\ distance = \int_{t1}^{t2} |v| dt$$

$$Total\ diplacement = \int_{t1}^{t2} v dt$$

Be able to use derivatives in different applications, economics:

In economics, it is about p, r and c which is profit/unit, revenue/unit and cost/unit respectively. The relationship between these three is logical and is as following:

$$p(x) = r(x) - c(x)$$

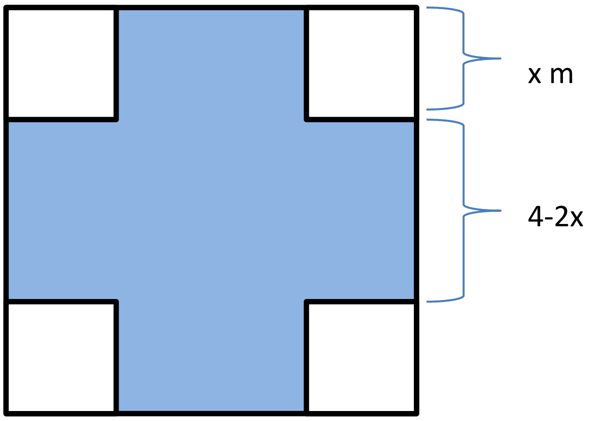
The quantity that gives maximum profit can be found out by simply setting $p'(x) = 0$. This means that $0 = r'(x) - c'(x)$, hence $r'(x) = c'(x)$.

If you want to find out the optimal quantity by trial and error, you can do it using first principle of derivatives $p'(x) \approx \frac{p(x+1)-p(x)}{(x+1)-x}$. So, $p(x+1) - p(x) = 0$.

Be able to use derivatives in different applications, optimization:

In optimization, it is about finding the value where the derivative of certain dimension is 0. Ex,

There is a 4m by 4m box with “x”m squares on the edges. Edges will be cut to make a box. What x will make the box hold the most volume?



$$V(x) = (4 - 2\mathrm{x})^2 \cdot x$$

$$V'(x) = 2(4 - 2\mathrm{x}) \cdot (-2) \cdot x + (4 - 2\mathrm{x})^2 \cdot 1$$

$$V'(x) = 12x^2 - 32x + 16$$

Since we want to find out the maximum, $V'(x) = 0$.

$$12x^2 - 32x + 16 = 0$$

$$3x^2 - 8x + 4 = 0$$

$$(3x - 2)(x - 2) = 0$$

$$x = \frac{2}{3} or 2$$

However, $x \neq 2$ because that would make $4 - 2\mathrm{x} = 0$, hence answer is $x = \frac{2}{3}$. In fact, the values we have found are the minimum and maximum. If we would have taken the second derivative of $3x^2 - 8x + 4 = 0$ and plugged in the x values we got, we will see that one is concave up and the other concave down. But in this question, we are only interested in maximum, which is concave down and the answer is $x = \frac{2}{3}$.

4. Indefinite integration as anti-differentiation. Indefinite integral of x^n, sin x, cos x and e^x. Other indefinite integrals using the results from 6.2. The composites of any of these with a linear function.

- <u>Be able to understand indefinite integration and know the basic rules of indefinite integration:</u>

Indefinite integration = integration of a function that gives out another function, and not a value = anti-derivative.

So basically, you need to think in the opposite direction of derivatives. It makes sense after a while.

$$\int f(x)dx = F(x) + C, thus\ F'(x) = f(x)$$

When $f(x) = x^n$, $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$

When $f(x) = (ax+b)^n$, $\int (ax+b)^n dx = \frac{1}{(n+1)a}(ax+b)^{n+1} + C$

When $f(x) = e^x\ or\ a^x, look\ at\ the\ formula\ booklet.$

When $f(x) = e^{mx+c}$, $\int e^{mx+c} dx = \frac{1}{m} e^{mx+c} + C$

When $f(x) = a^{mx+c}$, $\int a^{mx+c} dx = \frac{1}{m(lna)} a^{mx+c} + C$

When $f(x) = \frac{1}{x}, look\ at\ the\ formula\ booklet.$ You will see in the booklet that x is put in absolute value, and it is because $e^y \neq -x$. So there is error in domain if do not put the absolute value.

<u>Be able to find particular antiderivative:</u>

This is basically finding out the "C" in indefinite integrals. You will therefore be given a certain $F(a) = b$. Ex,

Given $f(x) = 7 - 6cosx$ and $F(1) = 6$, find $F(x)$.

This is very a very simple integration problem. $\int f(x)dx = 7x - 6sinx + C.$

$$F(1) = 7 - 6sin(1) + C = 6$$

$$7 - 6\frac{\pi}{2} + C = 6$$

$$C = 3\pi - 1$$

Therefore, $F(x) = 7x - 6sinx + 3\pi - 1.$

5. Anti-differentiation with a boundary condition to determine the constant of integration. Definite integrals. Area of the region enclosed by a curve and the x-axis or y-axis in a given interval; areas of regions enclosed by curves. Volumes of revolution about the x-axis or y-axis.

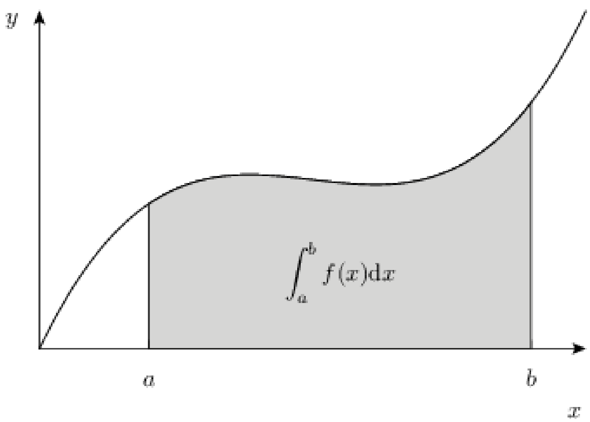
- Be able to understand definite integrals:

Definite integral = integration that gives a value (within a certain boundary). Here are some basic rules that should make sense to you.

1. $I = \int_a^b f(x) = [F(x)]_a^b = F(b) - F(a)$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
4. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

Be able to calculate areas enclosed by boundaries on x-axis:

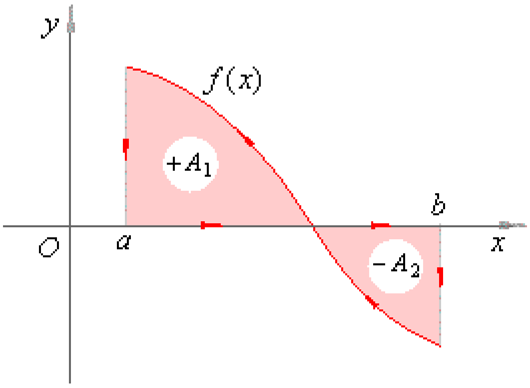
Integral is essentially the area under a curve (of course, there are double integrals, partial derivatives and Green's theorem that make the idea of "area" a bit hairier, but that doesn't bother us).



In this case, the integral is the same as the area because all area is on the positive side of the y-axis. Thus,

$$A = I = \int_a^b f(x) = [F(x)]_a^b$$

However, you may see a curve that cuts the x-axis.

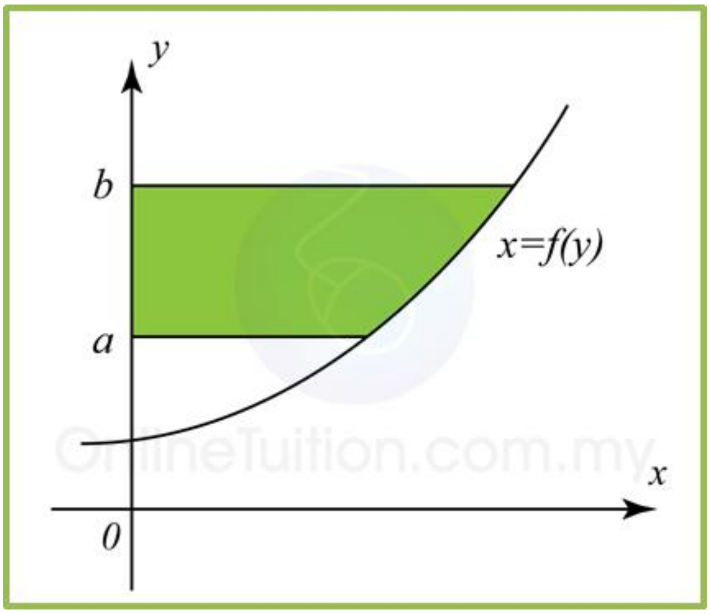


In this case, $A \neq I$. Area would have a higher value than the integral because integral is a method of summing, so if you have one negative area (negative side of y-axis) you will lower your total value. So instead, just divide it into two separate integrals with third point as the x-intercept.

$$A = \left|\int_a^i f(x)dx\right| + \left|\int_i^b f(x)dx\right|$$

Be able to find areas enclosed by boundaries on y-axis:

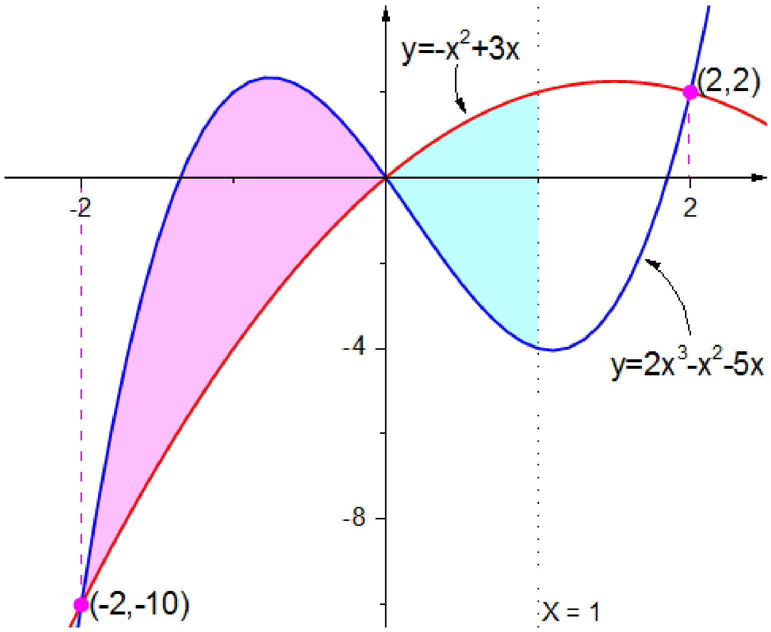
The idea is the same as on x-axis. But you just have to swap the subject of the function since your boundaries are given on y-axis.



So, first rearrange $y = f(x)$ into $x = f(y)$.

Then we write the integral in terms of y! $A = \int_a^b f(y)dy$. That is it!

Be able to find areas bounded by two functions:



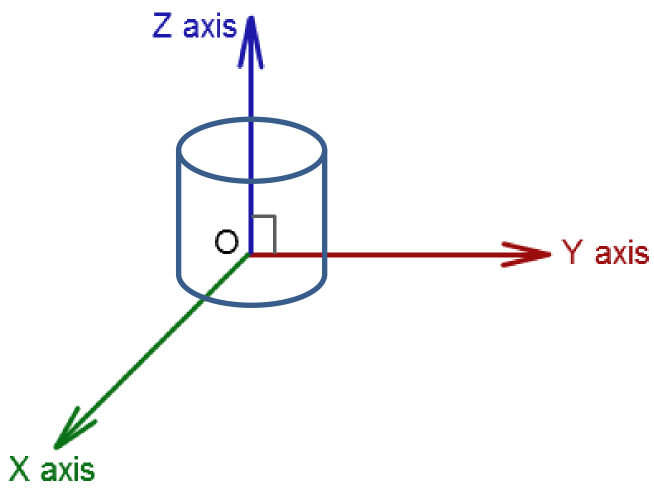
Here, all you need to do is to take integral of the function on top and subtract it with the function below. In this case, blue is above first, and then red is above. So write like this:

$$\int_{-2}^{0}\big((2x^3 - x^2 - 5x) - (x^2 + 3x)\big)dx + \int_{0}^{2}\big((x^2 + 3x) - (2x^3 - x^2 - 5x)\big)dx$$

If it is allowed, I would just use the GDC to graph and identify which one is on top. But you can sketch it as well by identifying the intersection points, and try any number in between to see which function is above and below.

Be able to find volumes and volume of solid of revolution with integrals:

The basic idea is to slice the volume into very thin discs, and add the discs up using integral.



So I will tell you a logical procedure that will help you to calculate any volume! I will take the cylinder as an example

1. Write the expression for the very thin disc.

$$dV = \pi r^2 \cdot dh$$

2. Then to make it into a solid, simply integrate the *dV* to make it into *V*!

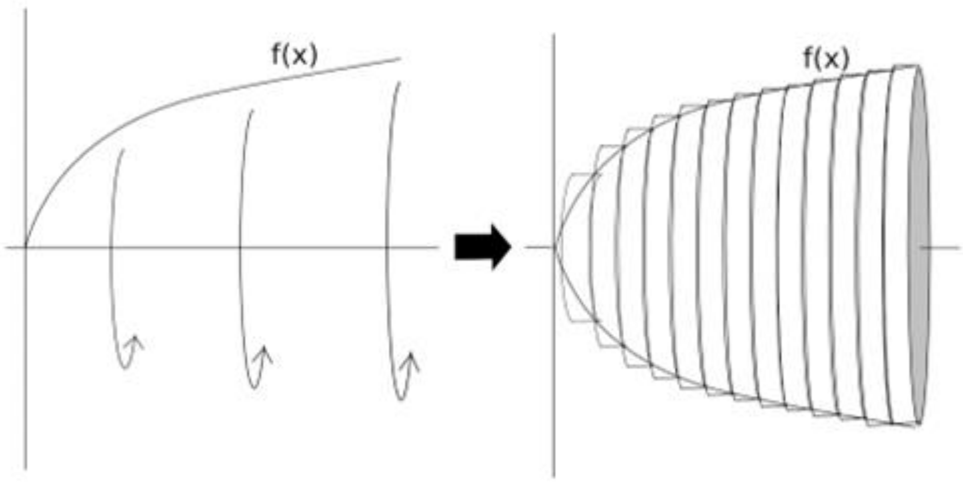
$$\int dV = \int \pi r^2 dh$$

$$V = \pi r^2 \int dh$$

$$V = \pi r^2 h$$

And this is indeed the volume formula for cylinder.

Now, you must also be able to find volumes of revolution. We are going to use the exact same procedure.



1. Write the expression for the very thin disc.

$$dV = \pi y_i^2 dx$$

You will notice that the thin discs have radius as *y* and height as *dx*.

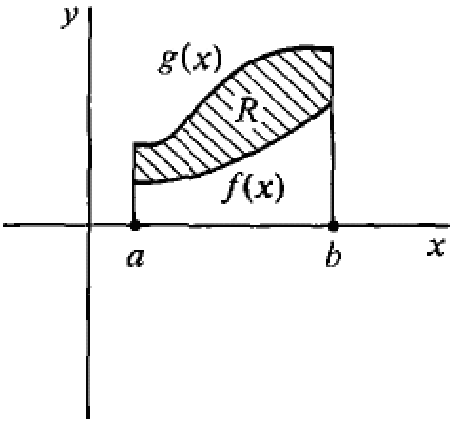
2. We integrate! Of course, you have to write y_i in terms of *x* because we are integrating in respect to *x*.

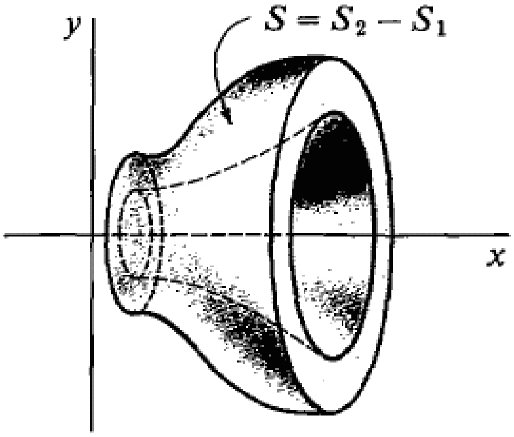
$$\mathrm{V} = \pi \int_a^b y_i^2 dx$$

If the curve is rotated along y-axis, don't get confused! You have all the tools to do it and it is very simple! Just follow the logical procedure of first making thin discs, and then integrating it.

<u>Be able to find volumes of solid of revolution bounded by two curves:</u>

The principle is again the same.





$$Volume = volume\ outer - volume\ inner$$

$$dV = \pi g(x)_i^2 dx - \pi f(x)_i^2 dx$$

$$V = \pi(\int_a^b g(x)_i^2 dx - \int_a^b f(x)_i^2 dx)$$

<u>Be able to find volumes of sphere:</u>

Again, the idea is same. Consider a sphere with equation:

$$x^2 + y^2 = r^2$$

$$dV =$$

$$V = \pi \int_{-r}^{r} (r^2 - y^2) dy$$

Remember that *r* is just a constant!

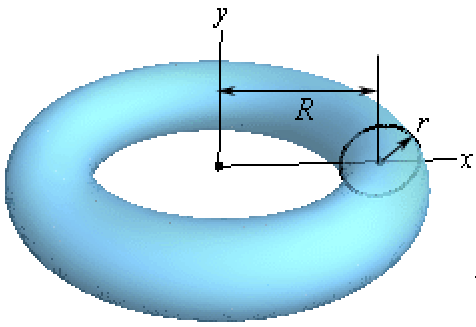
$$V = \pi[r^2 y - \frac{1}{3} y^3]_{-r}^{r}$$

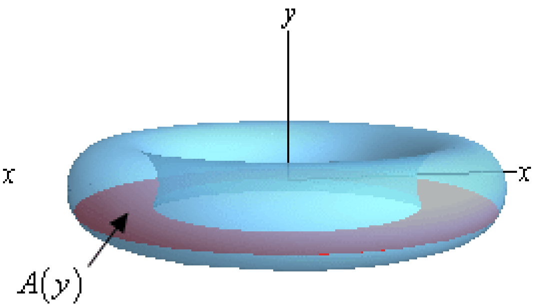
Be able to find volume of torus(?):

This is quite rare, but finding volume of torus is actually quite fun. First, you need to understand the basics of torus equation.

$$(x-a)^2 + (y-b)^2 = r^2$$

The value of *a* determines where the centre of circle lies on the x-axis. If $(x-2)^2$, then the centre of circle lies on x=2. The value of *b* determines where the centre lies on y-axis. If $(y+2)^2$ the centre lies on y=-2.





The idea is to integrate the outer disc from bottom to top, and subtract with the inner disc from bottom to top. Let's say we have a torus $(x-2)^2 + (y+2)^2 = 1$.

$$x = 2 \pm \sqrt{1-(y+2)^2}$$

$$dV = V_{outer} - V_{inner}$$

$$dV = \pi x_{outer}^2 dy - \pi x_{inner}^2 dy$$

We integrate from -3 to -1 because since midpoint of circle on y-axis is in -2 and the radius is 1.

$$\text{V} = \pi \int_{-3}^{-1} [(2 + \sqrt{1-(y+2)^2})^2 - (2 - \sqrt{1-(y+2)^2})^2] dy$$

Then you are most likely to be able to use the GDC.

Be able to calculate the length of a line:

This is derived from Pythagoras theorem.

$$\Delta s^2 = \Delta x^2 + \Delta y^2$$

$$\Delta \mathrm{s} = \sqrt{\Delta x^2 + \Delta y^2}$$

As $\Delta \rightarrow d$,

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{dx^2 + dy^2} \times \frac{dx}{dx}$$

$$ds = \sqrt{(dx^2 + dy^2)(\frac{1}{dx})^2} \times dx$$

$$ds = \sqrt{(1 + (\frac{dy}{dx})^2} \times dx$$

$$s = \int_a^b \sqrt{(1 + (\frac{dy}{dx})^2} \times dx$$

Thus the length of line can be found by deriving the function and plug it in the integral function.

6. Kinematic problems involving displacement s, velocity v and acceleration a. Total distance travelled.

- This was explained above.

7. Integration by substitution. Integration by parts.

- Be able to use different techniques in suitable situations:

This is probably one of most difficult parts when integrating: What to use when. I have therefore made an outline of all the techniques you must be able to use and tips on how to identify which ones you should use.

1. Integration by substitution.

Use this method of substitution when you see a derivative of a function within the equation. This is very helpful with rational functions.

Ex,

$$\int tanxdx = \int \frac{sinx}{cosx} dx$$

Since derivative of sinx is cosx, you can use it to integrate this. Let $cosx = u$, then $-sinxdx = du$. Then replace these two information to the original function!

$$\int tanxdx = -\int \frac{1}{u} du$$

$$\int tanxdx = -ln|u| + C$$

$$\int tanxdx = -ln|cosx| + C = ln|cosx^{-1}| + C = ln|secx| + C$$

Use this method of substitution when you fail to solve by integration by parts. Then try to substitution as a last option. You are basically trying to create the derivative of an expression.

Ex,

$$\int x^3\sqrt{x^2-1}\,dx = \int \frac{1}{2}x^2 2x\sqrt{x^2-1}\,dx$$

Let $u = x^2 - 1$ and $du = 2xdx$. If we now use this information,

$$= \int \frac{1}{2}x^2\sqrt{u}\,du$$

$$= \frac{1}{2}\int (u+1)\sqrt{u}\,du$$

If we expand it out,

$$= \frac{1}{2}\int u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du$$

Then basic integration and substitute back into *x*!

2. Integration by parts.

Use this whenever you see two functions multiplied together. The method is basically another form of product rule.

$$(u \cdot v)' = u'v + uv'$$

It we integrate this back,

$$u \cdot v = \int u'v + \int uv'$$

$$\int uv' = u \cdot v - \int u'v$$

Thus to integrate a multiplied function, one needs to be treated as a derivative (*v'*). So when choosing which ones you are going to put as *u* and *v'*, choose the functions that are most convenient to derive and integrate respectively!

Ex,

$$\int (4x + 5) lnx\, dx$$

Let $u = lnx,\ \ v' = 4x + 5$

Then $du = \frac{1}{x} dx\ \ v = 2x^2 + 5x$

$$= lnx(2x^2 + 5x) - \int \frac{2x^2 + 5x}{x} dx$$

$$= lnx(2x^2 + 5x) - (x^2 + 5x) + C$$

Also, you can use integration by parts by treating *dx* as the derivative of *x*.

Ex,

$$\int lnx\, dx$$

Let $u = lnx,\ \ v' = dx$

Then $du = \frac{1}{x} dx\ \ v = x$

$$= xlnx - x + C$$

3. Integration by special substitution.

Use this when you see a multiplied function with radicals. This may be more helpful than integration by parts. With regular substitution, you only substituted the inner part. This time, we substitute the whole radical.

Ex,

$$\int x\sqrt{2x+1}\,dx$$

Let $u = \sqrt{2x+1}$, then if we rearrange this $x = \frac{1}{2}(u^2 - 1)$.

Then if we derive the rearranged expression, $dx = u \cdot du$. Now substitute!

$$= \int \frac{1}{2}(u^2 - 1)u \cdot u \cdot du$$

$$= \frac{1}{2}\int (u^4 - u^2)\,du$$

$$= \frac{1}{2}(\frac{1}{5}u^5 - \frac{1}{3}u^3)$$

4. Integration of higher powers of sin and cos.

Use this when $sin^{2n+1}x$ or $cos^{2n+1}x$ (odd powers). Split the power, use trig identity $sin^2x + cos^2x = 1$, then use substitution method!

Ex,

$$\int cos^3x\,dx$$

$$= \int cosx(1 - sin^2x)\,dx$$

Let $u = sinx,\ du = cosxdx$

$$= \int (1 - u^2)\,du$$

$$= u - \frac{1}{3}u^3 = sinx - \frac{1}{3}sin^3x$$

Use this when $sin^{2n}x$ or $cos^{2n}x$ (even powers). Split power then use double angle formula twice.

Ex,

$$\int cos^4x\,dx$$
$$= \int (cos^2x)^2\,dx$$

$$cos^2x = \frac{cos2x + 1}{2}$$

$$= \int (\frac{cos2x + 1}{2})^2\,dx$$

$$= \frac{1}{4}\int (cos^2 2x + 2cos2x + 1)\,dx$$

$$= \frac{1}{4}\int \left(\frac{cos4x + 1}{2} + 2cos2x + 1\right)dx$$

$$= \frac{1}{4}\left(\frac{sin4x}{8} + \frac{1}{2}x + sin2x + x\right)$$

Use this when in $sin^a cos^b$ either one is odd. Use the same method as with odd power. Break it down and use the trig identity $sin^2x + cos^2x = 1$.

Ex,

$$\int sin^5xcos^4x\,dx$$

$$= \int sin^2xsin^2xcos^2xcos^2xsinx\,dx$$

$$= \int (1 - cos^2x)(1 - cos^2x)cos^2xcos^2xsinx\,dx$$

Let $u = cosx,\ du = -sinxdx.$

$$= -\int (1 - u^2)(1 - u^2)u^2u^2\,du$$

$$= -\int u^8 - 2u^6 + u^4\,du$$

$$= -\left(\frac{1}{9}u^9 - \frac{2}{7}u^7 + \frac{1}{5}u^5\right)$$

$$= -\left(\frac{1}{9}cos^9x - \frac{2}{7}cos^7x + \frac{1}{5}cos^5x\right)$$

5. Integration with trig substitution.

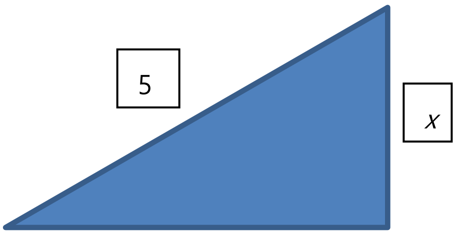
Use this when there is an expression in the form $a^2 \pm b^2$ where either *a* or *b* is *x*. This expression can be under a radical or even as a denominator.

Okay, the basic drill is like this. You substitute x in terms of θ → make dx in terms of $d\theta$ because that variable is what you are integrating for. We do it by taking the derivative, just like normal substitution → integrate → substitute back.

Ex,

$$\int \sqrt{25 - x^2}\, dx$$

How do we know what trig function to substitute with? Of course, you can memorize for different situations, but if you just draw a triangle you don't need to rely on memory! It all starts from Pythagoras theorem.



Note that x can be either as the opposite or adjacent. It doesn't matter as long as you substitute it right! So in this case, I will use sin and put *x* as the opposite side. But this information below that connects x with θ will be your main information source.

$$sin\theta = \frac{x}{5}$$

$$5sin\theta = x$$

$$25sin^2\theta = x^2$$

If we now substitute this we get:

$$\int \sqrt{25 - 25sin^2\theta}\, dx$$

What was the next step? It is to make $dx \to d\theta$. We simply do this by using the same information above.

$$5sin\theta = x$$

$$5cos\theta d\theta = dx$$

If we use this information,

$$\int \sqrt{25 - 25sin^2\theta} \cdot 5cos\theta d\theta$$

$$= \int 5\sqrt{1 - sin^2\theta} \cdot 5cos\theta d\theta$$

$$= 25 \int cos\theta \cdot cos\theta d\theta$$

$$= 25 \int cos^2\theta \, d\theta$$

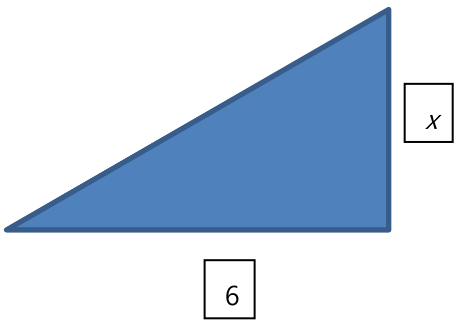
And we know how to integrate trig functions with even powers (use the formula). After you are done, do not forget to substitute back!

$$\theta = arcsin\left(\frac{x}{5}\right)$$

I will take another example. This time we do not have a radicle and we have an addition of squares. But don't get frightened! It works the same way.

Ex,

$$\int \frac{1}{x^2 + 36} dx$$



$$tan\theta = \frac{x}{6}$$

$$6tan\theta = x$$

$$36tan^2\theta = x^2$$

We substitute into the original equation again.

$$= \int \frac{1}{36tan^2\theta + 36} dx$$

Now we substitute the dx.

$$6tan\theta = x$$

$$6sec^2\theta d\theta = dx$$

$$= \int \frac{1}{36tan^2\theta + 36} 6sec^2\theta d\theta$$

$$= \frac{6}{36} \int \frac{1}{tan^2\theta + 1} sec^2\theta d\theta$$

$$= \frac{1}{6} \int \frac{1}{sec^2\theta} sec^2\theta d\theta$$

$$= \frac{1}{6} \int d\theta$$

$$= \frac{1}{6}\theta$$

$$= \frac{1}{6} \arctan\left(\frac{x}{6}\right)$$

That's it! Again, practice is key!