Functions & equations

1. Concept of function f : x→f (x) : domain, range; image (value). Odd and even functions. Composite functions $f∘g$. Identity function. One-to-one and many-to-one functions. Inverse function f^{-1}, including domain restriction. Self-inverse functions.

- <u>Be able to understand domain, range and different types of functions.</u>

Domain = possible x values.

Range = possible y values.

One-to-one = function with one x for one y.

Many-to-one = function with many x for the same y, such as $y=x^2$.

One-to-many = function with one x to many y. This is not a proper function.

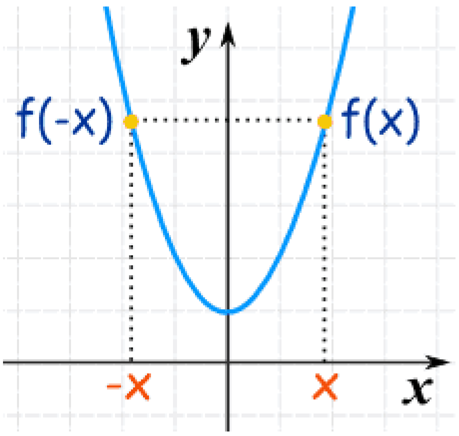
Many-to-many = function with many x to many y. This is not a proper function.

Vertex = point

Axis of symmetry = line of mirror reflection.

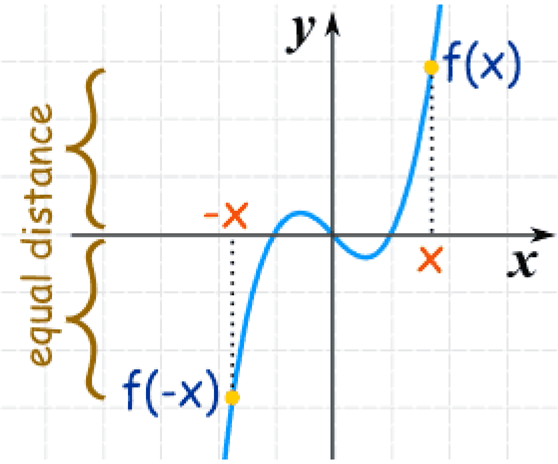
<u>Be able to understand even and odd functions:</u>

An even function is a function that is symmetrical along the y axis.



We can see that it naturally leads to $f(x) = f(-x)$.

An odd function is a function that is symmetrical upside down.



We can see that in this case, $f(x) = -f(-x)$

<u>Be able to understand an identity function:</u>

Identity function has the same output as its input.

$$f(x) = x$$

<u>Be able to understand an inverse function:</u>

This is a function where you get back x from y, in reference to your original function. It is denoted as $f^{-1}(x)$.

Consider $f(x) = y$, where $y = \frac{2x+7}{3}$.

To find the inverse function $f^{-1}(x)$, we just need to swap x with y and rearrange to make y the subject. Thus:

$$x = \frac{2y+7}{3}$$

And by rearranging, $f^{-1}(x) = y = \frac{3x-7}{2}$.

Be careful with inverse functions when it comes to domain. Your domain and range changes from your original function (they actually swap).

<u>Be able to understand a composite function:</u>

It is a function within a function. You are probably already familiar with this.

$$(f^{\circ}g)(x) = f(g(x))$$

2. The graph of a function; its equation y = f(x). Investigation of key features of graphs, such as maximum and minimum values, intercepts, horizontal and vertical asymptotes and symmetry, and consideration of domain and range. The graphs of the functions $y = |f(x)|$ and $y = f(|x|)$. The graph of $y = \frac{1}{f(x)}$ given the graph of $y = f(x)$.

- Be able to work with maximum, minimum, intercepts, horizontal asymptote, vertical asymptotes, line of symmetry, domain and range:

Things you have to be very comfortable manipulating are radical functions, absolute functions, rational functions, curve asymptotes, split functions, reciprocal functions (and perhaps oblique asymptotes).

Be able to find different asymptotes:

Vertical asymptote = any undefined x would be vertical asymptote.

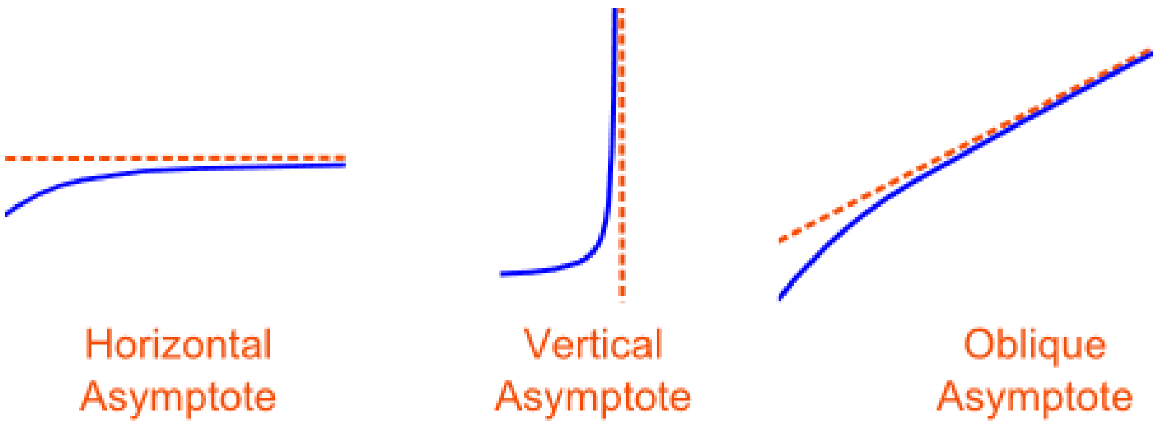
Horizontal asymptote = see where the value goes for large x-values, i.e. substitute x with infinity. Such as:

$$y = \frac{2x+1}{7x+8} = \frac{2+\frac{1}{x}}{7+\frac{8}{x}}$$

As x→∞, $y = \frac{2}{7}$.

If this method does not work, divide using long division (you will learn this technique). You will get a constant and a rational number that will become 0 as x becomes infinite. This is the most secure way to find horizontal asymptote.

Oblique asymptote = use the same principle as horizontal asymptote. But you will get a function and not a constant, and a rational number that will become 0 as x becomes infinite. That function is the oblique asymptote.



Be able to graph $y = \frac{1}{f(x)}$ from a drawn f(x) without function given:

This is simpler than it looks. You have to first consider few important points.

1. Find the x that gives f(x)=1. When f(x)=1, y=1 and you have one very easy point.

2. Find the x that gives f(x)=0. That will be the vertical asymptote for y.

3. Remember that as f(x) gets higher on the graph, your y decreases since $y = \frac{1}{f(x)}$.

4. Then try out some of the points to get the general shape of the graph. You only need a few points in order figure the general shape out once you have the asymptotes.

3. Transformations of graphs: translations; stretches; reflections in the axes. The graph of the inverse function as a reflection in y = x.
- Be able to transform all kind of functions horizontally, vertically, dilate and reflect:

Horizontal shift = f(x-h) is movement to the right and f(x+h) is movement to the left. You might be confused why subtraction leads to a shift of curve to right and vice versa. In very simple terms, think that you are moving the whole coordinate system instead of the curve. So f(x-h), you move whole coordinate system to the left and the opposite for f(x+h). There is a more mathematical reason, but my notes are dedicated for review and tricks to not confuse small concepts **after** you understand mathematically.

Vertical shift = This is just plain addition and subtraction that moves the curve up and down respectively. It is very intuitive.

Dilations = Dilation is a number multiplied with x to change the x's output value. For instance,

$$f(x) = x^2 \text{ can be dilated to } f(x) = 15x^2$$

The multiplying factor can also be within the function, such as:

$$f(x) = x^2 \text{ can be dilated to } f(x) = (15x)^2, \text{which can simply be written as}$$

$$f(x) = 225x^2$$

Reflection = To reflect in y axis, you just have to put the opposite y, hence $-f(x)$. To reflect in x axis, you just have to put the opposite x, hence $f(-x)$.

4. The rational function $x \to \frac{ax+b}{cx+d}$, and its graph. The function $x \to a^x$, a > 0 , and its graph. The function $x \to log_a x, x > 0$.

- Be able to graph inverse functions:

Inverse function is fundamentally just a swap of x and y, and therefore also the domain and range. This means that inverse function will be a reflection on the straight line $y = x$.

You can try this for $f(x) = a^x$ and its inverse $f^{-1}(x) = log_a x$. You will indeed see that they reflect on $y = x$.

5. Polynomial functions and their graphs. The factor and remainder theorems. The fundamental theorem of algebra.

- Be able to understand axis of symmetry:

Consider $f(x) = ax^2 + bx + c$ (which can also be expressed as $f(x) = (x-p)(x-q)$). Axis of symmetry is simply a point where the function turns. To understand the formula, we need to derive it.

$$f(x) = ax^2 + bx + c$$

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

The you just have to complete the square.

$$f(x) = a\left((x + \frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a}\right)$$

$$f(x) = a(x + \frac{b}{2a})^2 + some\ constant$$

Now we can see that the lowest point in $f(x)$ is when $x = -\frac{b}{2a}$. Therefore the point of symmetry is when $x = -\frac{b}{2a}$.

Be able to complete the square:

This is just practice. All you have to remember is that you are not changing anything in the function. You are simply adding and subtracting, but it is the same function as you started with.

Be able to factorize basically everything:

This is also practice. You should be able to factorize at least all second order functions (the functions that are factorize-able).

6. Solving quadratic equations using the quadratic formula. Use of the discriminant $\Delta = b^2 - 4ac$ to determine the nature of the roots. Solving polynomial equations both graphically and algebraically. Sum and product of the roots of polynomial equations. Solution of $a^x = b$ using logarithms. Use of technology to solve a variety of equations, including those where there is no appropriate analytic approach.

- Be able to use quadratic formula and state the nature of the roots:

If $\Delta > 0$, there are two real roots.

If $\Delta = 0$, there is one real roots. More correctly, there are two repeated ones. ±0 is just 0.

If $\Delta < 0$, there are two complex roots.

Be able to use Viete's theorem:

This theorem is to link the solutions of a higher order function with the coefficients of the function. We know that a function can be written in two ways: expanded and factorized.

$$f(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$f(x) = (x - p)(x - q) = x^2 - (p + q)x + pq$$

Hence we can confirm that $-(p + q) = \frac{b}{a}$ and $pq = \frac{c}{a}$.

This is useful because you can solve the solutions directly using coefficients.

Be able to solve polynomials with nth power:

Multiplication of polynomials = Use the box/grid method! This is a safer way to expand.

Division of polynomials = It is also very intuitive. You get a quotient and a remainder (if there are any)

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

$$f(x) = q(x) \times g(x) + r(x)$$

When dividing, you can use long division or Horner's algorithm. I personally use long division because it is more intuitive and it is a method that is impossible to forget, while Horner's algorithm is indeed an algorithm that you have to remember (a black out of this method during exam would be fatal wouldn't it?)

Also, Horner's algorithm only works when you divide with a linear equation. Although it is quite fast, I suggest having long division as a base method.

Learn Horner's algorithm (aka synthetic division) and long division via Khan Academy!

Be familiar with factor theorem and remainder theorem:

Factor theorem = if $(ax - b)$ is a factor of a larger polynomial, obviously $f\left(\frac{b}{a}\right) = 0$.

Remainder theorem = if $(ax - b)$ is not a factor of a larger polynomial, $f\left(\frac{b}{a}\right) \neq 0$.

Be able to understand what it means by multiplicity:

Multiplicity = When a factor appears more than once. So if you see that a certain factor is repeated after you have fully factorized, then the number of times it is repeated is called multiplicity.

Be able to solve polynomials algebraically:

Integer zero = Find $f(x) = 0$ through trial and error. Always try the factors of the constant term. If you reach that $f(1) = 0$, then $(x - 1)$ is a factor.

Rational root theorem = This is also trial and error method, but a narrower selection of trial values. Consider $f(x) = ax^3 + bx^2 + cx + d$, and find $f\left(\frac{p}{q}\right) = 0$. P is factors of d and q is factors of a.

Be able to use the conjugate root theorem:

Remember that if $z = a + ib$ is a factor $(\mathrm{x} - (a + ib))$ then its conjugate is also a factor $(\mathrm{x} - (a - ib))$.

Be able to solve polynomials using sum and product of roots:

This is essentially generalizing Viete's theorem for higher order functions.

$$ax^3 + bx^2 + cx + d = 0$$

$$(x - x_1)(x - x_2)(x - x_3) = 0$$

Using these two (like in Viete's theorem), we get that

$$\begin{cases} x_1 + x_2 + x_3 = -\dfrac{b}{a} \\ x_1x_2 + x_1x_3 + x_2x_3 = \dfrac{c}{a} \\ x_1x_2x_3 = -\dfrac{d}{a} \end{cases}$$

In other words:

$$\begin{cases} x_1 + x_2 + x_3 + \cdots x_n = -\dfrac{a_{n-1}}{a_n} \\ x_1x_2 + x_1x_3 + x_2x_3 + \cdots = \dfrac{a_{n-2}}{a_n} \\ x_1x_2x_3 \dots x_n = (-1)^n \dfrac{a_0}{a_n} \end{cases}$$

To generalize these three, we get the following expression:

$$(-1)^k \frac{a_{n-k}}{a_n}$$

Where *n* is the order of polynomial and *k = 1* for sum, *k = 2* for sum&product and *k = n* for product. Remember this because it may be very helpful.

7. Solutions of $g(x) \geq f(x)$. Graphical or algebraic methods, for simple polynomials up to degree 3. Use of technology for these and other functions.
- Be able to solve inequalities:

Linear inequality = Very simple as you only need to be careful with the signs when rearranging.

Quadratic inequality = Remember to put in range.

$$x^2 - 3x - 18 < 0$$

$$(x-6)(x+3) < 0$$

$$-3 < x < 6$$

Modulus inequality = These are the same as absolute value so there are two values here as well! My suggestion here is to square both sides and solve as a quadratic inequality.

$$|2x+3| < 6$$

$$(2x+3)^2 < 36$$

Then solve as usual.

Rational inequality = For any equality with not a 0, you may just multiply out and solve as usual. However, if you get an inequality with a 0, I suggest you make into one fraction, factorize (if possible). Then using the important, critical values, see in which boundaries it satisfies the inequality.

A good technique is Decarte's rule of sign change (ask your teacher or internet for the exact method).