Vectors

1. Concept of a vector. Representation of vectors using directed line segments. Unit vectors; base vectors i, j, k. Components of a vector. Algebraic and geometric approaches to the following: the sum and difference of two vectors; the zero vector 0, the vector −v; multiplication by a scalar, kv; magnitude of a vector, v; position vectors OA = a. AB = b-a.

- Be able to understand what is meant by vector and scalar:

Vector = magnitude with direction, with i, j, k representing direction in plane x, y, z respectively.

Scalar = magnitude only

$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = xi + yj + zk$ in 3D

$v = \begin{pmatrix} x \\ y \end{pmatrix} = xi + yj$ in 2D

Be able to calculate sum and difference of vectors:

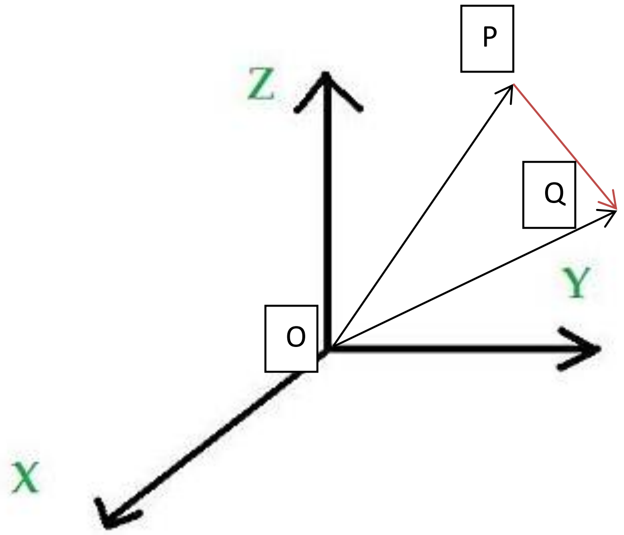
You subtract numbers in the same plane.

Consider $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\omega = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$v + \omega = \begin{pmatrix} x + a \\ y + b \\ z + c \end{pmatrix}$$

$$v - \omega = \begin{pmatrix} x - a \\ y - b \\ z - c \end{pmatrix}$$

Be able to calculate a vector using vectors from origin (aka position vectors):



To find $\overrightarrow{PQ}$, just need to draw the vectors, like in the left. We can see that:

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

Be able to find the magnitude:

Magnitude is just Pythagoras theorem!

$\|v\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, and also z if the vector is in 3D plane.

Be able to multiply by a scalar:

Scalar is just a number, constant, or a length.

$$v \times k = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times k = \begin{pmatrix} xk \\ yk \\ zk \end{pmatrix}$$

Be able to use unit vectors:

$$\hat{u} = \frac{v}{|v|}$$

We divide the vector by its own length to make it into a "unit". It is as simple as that! We usually use unit vectors to check if two vectors are collinear.

If $\hat{u} = \hat{v} \times k$, then it means that the vectors have same direction but with different magnitude, hence they are collinear.

2. The definition of the scalar product of two vectors. Properties of the scalar product. The angle between two vectors. Perpendicular vectors; parallel vectors.

- Be able to understand dot product:

Dot product = Type of multiplication that gives a scalar value and it is noted with a dot.

It can be expressed in two ways:

$$\vec{v} \cdot \vec{u} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = v_x u_x + v_y u_y + v_z u_z$$

$$\vec{v} \cdot \vec{u} = |v||u|cos\theta$$

The second expression implies that if the vectors are perpendicular to each other, they will have a dot product of 0, since $cos\left(\frac{\pi}{2}\right) = 0$.

Be able to use important properties of dot product:

1. $\vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$

2. $\vec{k} \cdot (\vec{v} + \vec{u}) = \vec{k} \cdot \vec{v} + \vec{k} \cdot \vec{u}$

3. $(k\vec{v}) \cdot \vec{u} = k(\vec{v} \cdot \vec{u})$

They essentially operate like normal algebra.

Be able to find the angle between two vectors:

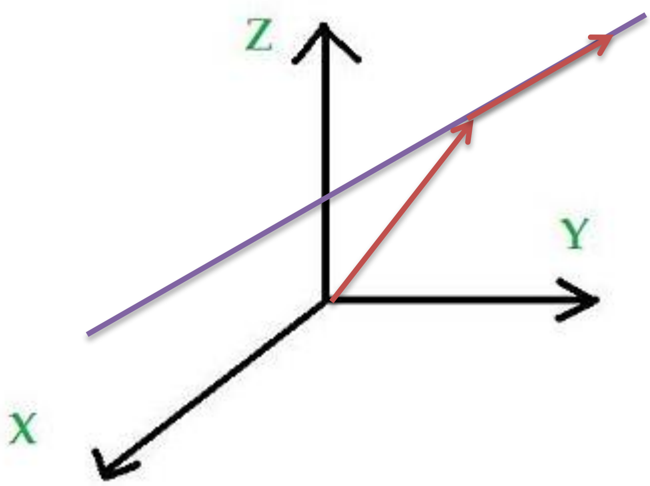
You only need to rearrange the formula above with cosine to find the angle!

3. Vector equation of a line in two and three dimensions: r = a +λb. Simple applications to kinematics. The angle between two lines.

- Be able to write equation of a line with vectors:

Consider this arbitrary line. This line is basically a collection of many points, and we need two things to express all the possible points.

First, we need a vector to get to the line (anywhere on the line) from the origin. Second, we need a vector with same direction to be able to move along the line (forwards and backwards).



It is rather intuitive and to put in words:

$$L: r = the\ vector\ to\ the\ line + scaling\ factor \cdot the\ vector\ along\ the\ line$$

We can express this in various ways and you need to be able to use all of them. These one are vector forms (the second one is the expanded form):

$$L: r = \vec{r_o} + \lambda \vec{d}$$

$$L: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_{0x} \\ r_{0y} \\ r_{0z} \end{pmatrix} + \lambda \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix}$$

This one is parameter form, which is just expressing x, y, z separately:

$$\begin{cases} x = r_{0x} + \lambda d_x \\ y = r_{0y} + \lambda d_y \\ z = r_{0z} + \lambda d_z \end{cases}$$

This one is Cartesian form and it is reached by simply making our scaling factor the subject:

$$\lambda = \frac{x - r_{0x}}{d_x} = \frac{y - r_{0y}}{d_y} = \frac{z - r_{0z}}{d_z}$$

Notice that direction vector can be easily identified by just looking at the denominator!

<u>Be able to find equation of a line from two points on the line:</u>

To find the vector that gets us from the origin to the line, you can use either one from the two provided (since both are on the line). To find the direction vector, you just need to take the different of the two given vectors (in either direction since you will have a scaling factor anyways).

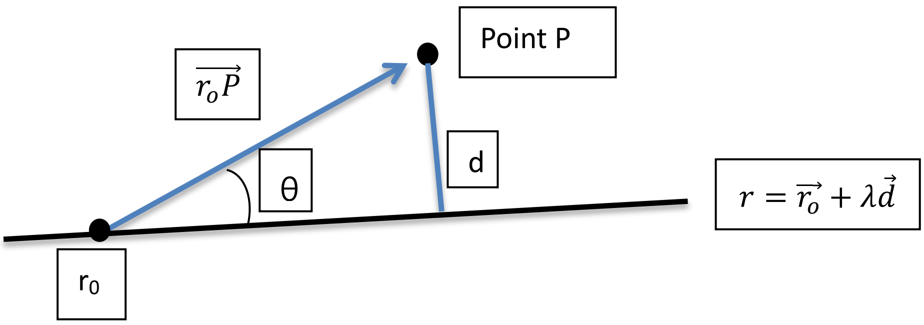
<u>Be able to make a vector line into a line in Cartesian diagram in 2D plane:</u>

$$Slope = \vec{d} = \binom{x}{y}$$

We know that slope is rise over run, so slope can be calculated using the dimension from the vector. $Slope = \frac{y}{x}$.

<u>Be able to find the shortest distance between a point and a line:</u>

The best way is to draw the problem.



From the point "P" and the line "r", we are trying to find the distance "d". We can use trigonometry to express the distance.

$$sin\theta = \frac{d}{|\overrightarrow{r_o P}|}$$

If we are given the angle, then it is very simple to calculate. But most of the time, we are not.

In those cases we use cross product.

$$cross\ product: \overrightarrow{r_oP} \times \vec{d} = |\overrightarrow{r_oP}||\vec{d}|sin\theta$$

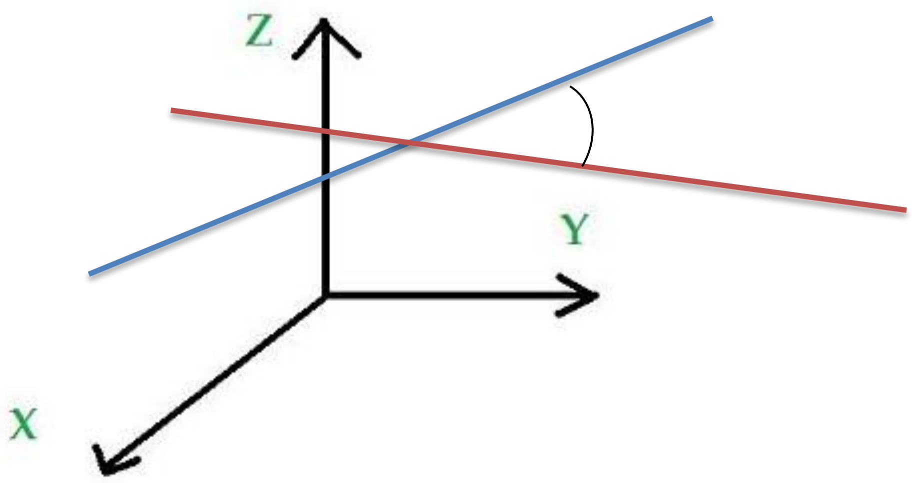
$$sin\theta = \frac{\overrightarrow{r_oP} \times \vec{d}}{|\overrightarrow{r_oP}||\vec{d}|}$$

And we substitute the $sin\theta$ and rearrange.

$$\frac{\overrightarrow{r_oP} \times \vec{d}}{|\overrightarrow{r_oP}||\vec{d}|} = \frac{d}{|\overrightarrow{r_oP}|}$$

$$d = \frac{|\overrightarrow{r_oP} \times \vec{d}|}{|\vec{d}|}$$

<u>Be able to find the angle between two lines:</u>



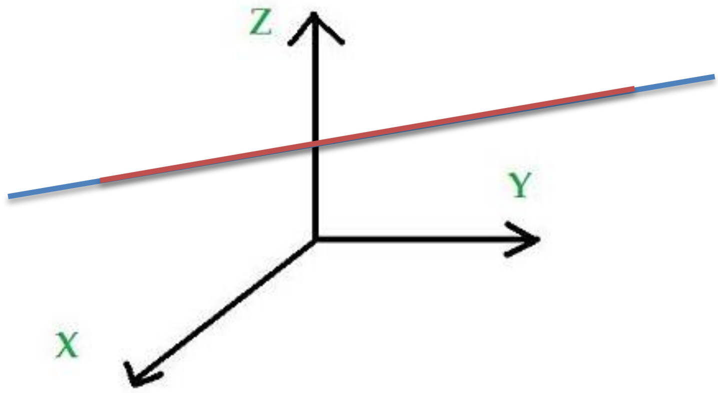
You simply rearrange the dot product. The only vector you use would be of course the direction vector.

$$cos\theta = \frac{\overrightarrow{d_1} \cdot \overrightarrow{d_2}}{|d_1||d_2|}$$

4. Coincident, parallel, intersecting and skew lines; distinguishing between these cases. Points of intersection.

- <u>Be able to identify coincident lines:</u>

Coincident = Lines that lie on top of each other.



How do we know?

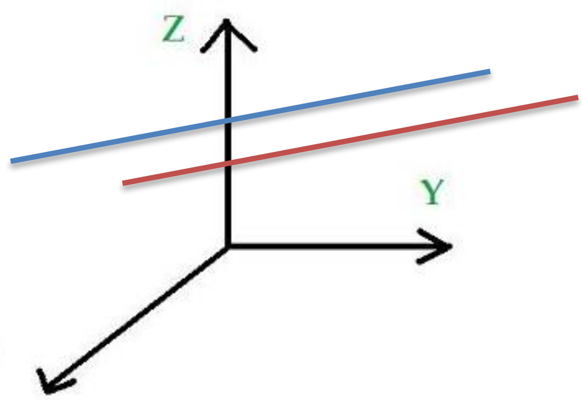
Firstly, the direction must be the same (obviously). So $\overrightarrow{d_1} = k\overrightarrow{d_2}$. If a scalar exists such that the two directions become identical, then both vectors point the same direction. But they can be parallel and not overlap. So we add a second test.

Secondly, a point in one line can be found by using a certain scaling factor in the other line. So consider two lines $L_1 : r = \overrightarrow{r_1} + \alpha\overrightarrow{d_1}$ and $L_2 : r = \overrightarrow{r_2} + \beta\overrightarrow{d_2}$. If these two lines and coincidental, then point r_1 can be found using a specific β in L_2. This verifies that they are both parallel and on top of each other since they contain the same points.

<u>Be able to identify parallel lines:</u>

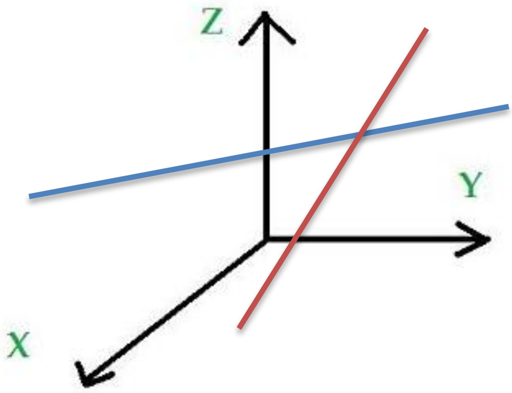
Parallel = Lines in the same direction.

All that is needed for this is $\overrightarrow{d_1} = k\overrightarrow{d_2}$.



Be able to identify the intersection point of two lines:

Intersection = Two lines overlap at some point.



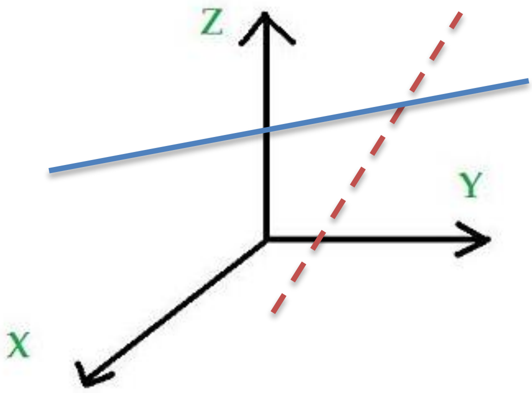
Just like finding intersection in Cartesian diagram, we use simultaneous equations. A simple way to set this up is via parametric equation.

$$\begin{cases} x_1 = x_2 \\ y_1 = y_2 \\ z_1 = z_2 \end{cases}$$

Your objective is to find the either of the scaling factors and eliminate the other. Once you have found it, substitute it into its original line to get coordinates for one point.

Be able to identify skewed lines:

Skewed = Lines neither intersect nor is parallel.

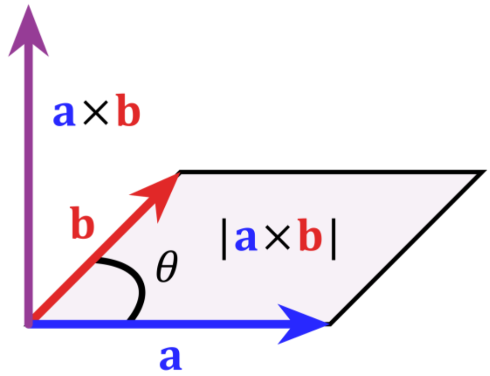


You just need to verify that there is no intersection point.

5. The definition of the vector product of two vectors. Properties of the vector product. Geometric interpretation of |v × w|.

- Be able to understand cross product:

Cross product = Type of multiplication that gives another vector perpendicular to the plane of the two vectors.



Algebraically, you use matrix to calculate it or use the formula provided in formula booklet (which I believe is the simpler way).

$$a \times b = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \begin{bmatrix} a_y & a_z \\ b_y & b_z \end{bmatrix} \hat{\imath} - \begin{bmatrix} a_x & a_z \\ b_x & b_z \end{bmatrix} \hat{\jmath} + \begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix} \hat{k}$$

To calculate the area, we take the absolute value of this new vector.

$$Area\ of\ parallelogram = |a \times b|$$

If it is the area of a triangle, then simply $Area\ of\ triangle = \frac{1}{2}|a \times b|$

Be able to use important properties of cross product:

1. $\hat{a} \times \hat{b} = -\hat{b} \times \hat{a}$

Be able to find volumes of a shape using vectors:

To find the volume, we need base x height in terms of vectors. Since we know how to calculate area, we need to do the following:

$$Volume = \left|(\vec{a} \times \vec{b}) \cdot \vec{c}\right|$$

The cross product is essentially the base and then we take dot product with the remaining vector to calculate the value of volume. Note that absolute value is just to prevent a negative value. If you multiply as such $\left|(\vec{b} \times \vec{a}) \cdot \vec{c}\right|$, you will get the opposite sign but same value (remember property of cross product).

An alternative way is like this using the dot product with angle (perhaps more intuitive because it is literally base x height).

$$Volume = |\vec{a} \times \vec{b}||\vec{c}|cos\theta$$

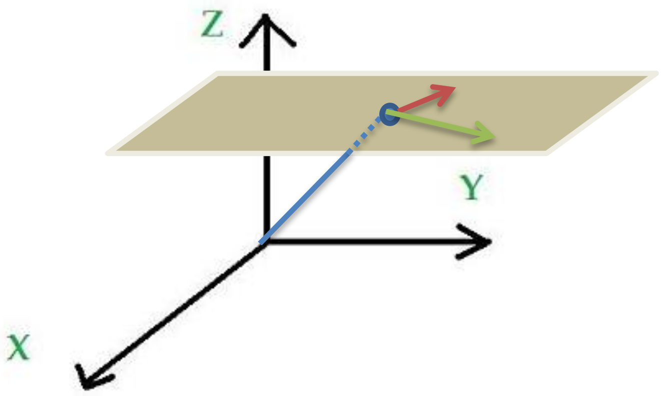
<u>Be able to perform triple cross product:</u>

$\vec{a} \times (\vec{b} \times \vec{c})$
$= the\ vector\ that\ is\ perpendicular\ to: vector\ a\ and\ the\ perpedicular\ vector\ of\ b\ and\ c.$

The actual calculation is exactly the same as before.

6. Vector equation of a plane r = a +λb + μc. Use of normal vector to obtain the form r·n = a·n. Cartesian equation of a plane ax + by + cz = d.
- <u>Be able to write equations of a plane with three non-collinear vectors:</u>

The basic idea is the same as in expressing a line with vectors. First you need a vector to get to the plane. Then you need two vectors this time on the plane (you only needed one in line equation) to be able to cover all points on that plane.



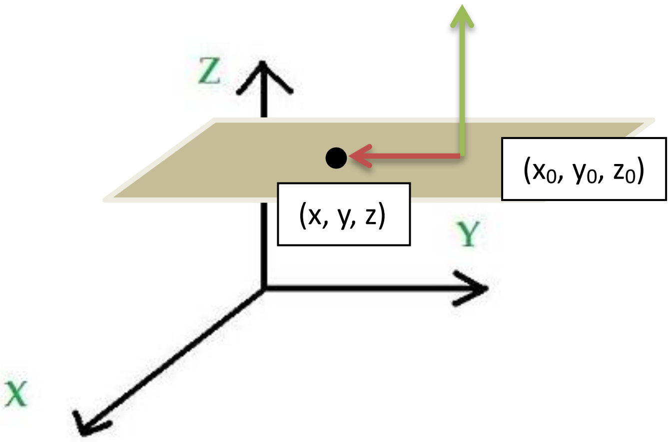
So we express it like this:

$$\pi: \overrightarrow{r_0} + \alpha\vec{v} + \beta\vec{u}$$

We can see that there are two scaling factors this time.

Be able to write equations of a plane with one point on the plane and a normal vector:

A normal vector is any vector perpendicular to the plane.



We are essentially going to express this plane using dot product of the red and green vector. Let the normal vector (green) have coordinates (a, b, c). Since normal vector and red vector is perpendicular, dot product is 0.

$$\vec{r} \cdot \vec{n} = 0$$

Point "P" can be any point on the plane with coordinates (x, y, z) so the vector "r" can be:

$$\vec{r} = (x - x_0, y - y_0, z - z_0)$$

If we multiply the dot product out, we get:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

If we rewind back, this is just the same as:

$$r \cdot n = a \cdot n$$

Note that $ax_0 + by_0 + cz_0$ is constant. This form $ax + by + cz = d$ is called the Cartesian form.

Be able to move between vector form, parametric form and Cartesian form:

Vector form:

$$\pi : \overrightarrow{r_0} + \alpha\vec{v} + \beta\vec{u}$$

Parametric form:

$$\begin{cases} x = x_0 + \alpha v_1 + \beta u_1 \\ y = y_0 + \alpha v_2 + \beta u_2 \\ x = z_0 + \alpha v_3 + \beta u_3 \end{cases}$$

Cartesian form:

To move from parametric to Cartesian form, you need to solve the parametric form through simultaneous equation. Remove $\alpha\ and\ \beta$ through elimination. Then we will get the form $ax + by + cz = d$. Or we could of course find the Cartesian equation if a point and a normal vector is given.

7. Intersections of: a line with a plane; two planes; three planes. Angle between: a line and a plane; two planes.

- Be able to find intersections of two lines:

This is already explained above. You will however notice that there are only two variables (which are the two scaling factors) while you have three equations (one each for x, y, z). You only need two equations for two variables, but you must use the third equation to verify that they agree.

$$\begin{cases} v_x + \lambda d_x = u_x + \mu d_x \\ v_y + \lambda d_y = u_y + \mu d_y \\ v_z + \lambda d_z = u_z + \mu d_z \end{cases}$$

Be able to find intersections of a line and a plane:

This is also just a type of simultaneous equation.

$$\begin{cases} r = \vec{r}_0 + \lambda\vec{d} \\ ax + by + cz = d \end{cases}$$

The key here is to use parametric equation for the line, and substitute in the plane equation. Our parametric equation is:

$$\begin{cases} x = r_{0x} + \lambda d_x \\ y = r_{0y} + \lambda d_y \\ z = r_{0z} + \lambda d_z \end{cases}$$

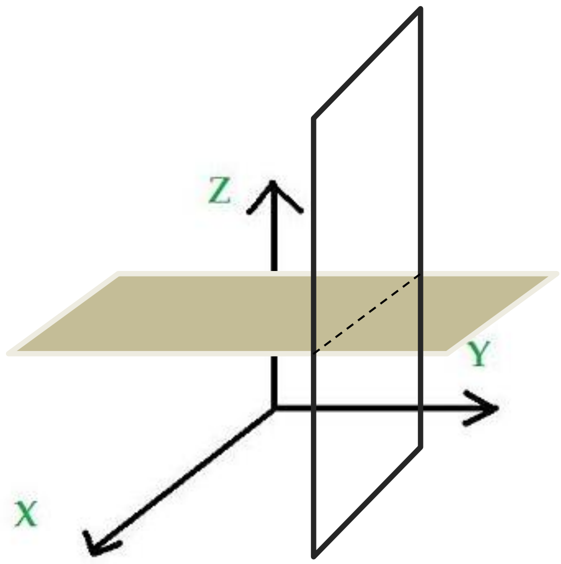
And if we substitute this into the plane equation we get:

$$a(r_{0x} + \lambda d_x) + b\left(r_{0y} + \lambda d_y\right) + c(r_{0z} + \lambda d_z) = d$$

Find the scaling factor λ, and once you have it, just use that on your line equation to find the point of intersection.

Be able to find intersections of two planes:

An intersection of two planes gives a line.



$$\begin{cases} \pi_1: ax_1 + by_1 + cz_1 = d_1 \\ \pi_2: ax_2 + by_2 + cz_2 = d_2 \end{cases}$$

Your main objective here is to make this into a Cartesian form of equation. To do so, you need to solve in respect to one variable (either x, y, z), for instance:

$$z = \frac{x - r_{0x}}{d_x} \text{ and } z = \frac{y - r_{0y}}{d_y}$$

Then you assume z is the scaling factor (assumption here doesn't matter because you will still get the same line). Thus in the end you will get:

$$\lambda = \frac{x - r_{0x}}{d_x} = \frac{y - r_{0y}}{d_y} = z$$

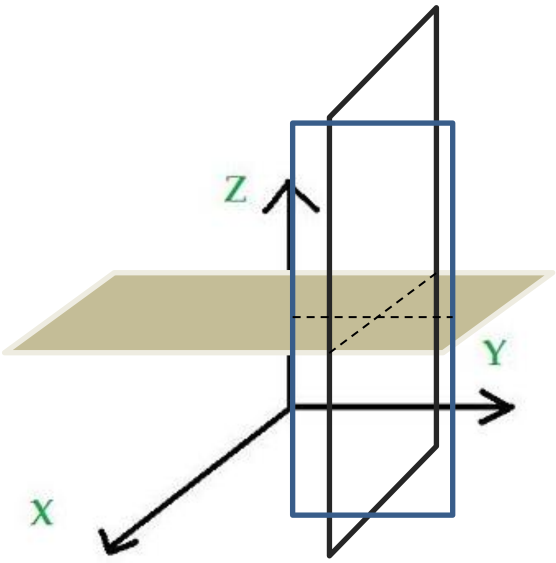
Be able to identify parallel planes:

Parallel means that they have parallel normal vectors so if their cross product equals 0, they are parallel (because sin0 = 0)

$$n_1 \times n_2 = |\overrightarrow{n_1}||\overrightarrow{n_2}| \sin\theta$$

<u>Be able to find intersections of three planes:</u>

This again simultaneous equation in Cartesian form!



When there is a unique solution = It will give a point P(x, y, z).

When there are an infinite number of solutions = The planes are either on top of each other or they form a line. You will reach either 0=0 or $k = \frac{0}{0}$.

To see if they are on top of each other, it means that all three planes are identical. Hence:

$$\pi_1 = k_1\pi_2 = k_2\pi_3$$

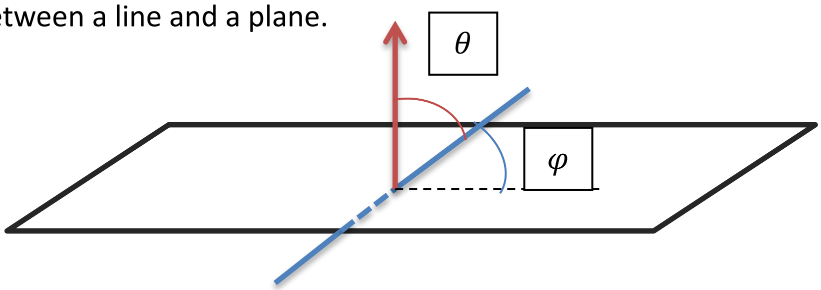
To see if they form a line, you test if the normal vector of one plane is 90° to the cross product of the remaining two planes.

$$n_1 \cdot (n_2 \times n_3) = 0 \; if \; they \; form \; a \; line$$

When there is no solution = There will be a contradictive statement when you have solved for a variable.

<u>Be able to find the angle between a line and a plane:</u>

We know that dot product is very helpful when calculating angles. It is the same for the angle between a line and a plane.

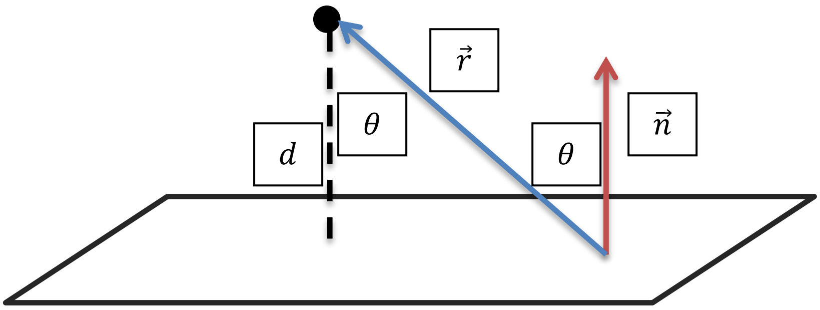


As you can see in the picture, calculate the angle between normal vector and the direction vector of the line first. Then, subtract that value from $\frac{\pi}{2}$ to get the angle between the plane.

$$\varphi = \frac{\pi}{2} - arccos\left(\frac{\vec{n}\cdot\vec{d}}{|\vec{n}||\vec{d}|}\right)$$

<u>Be able to find the distance between a point and a plane:</u>

Again, we use the dot product and trigonometry. You only need to find out the vector from bottom of normal to the point and you are then good to go!



$$cos\theta = \frac{d}{|\vec{r}|}$$

$$d = cos\theta|\vec{r}|$$

How do we find $cos\theta$? Well, through dot product of $\vec{r}$ and $\vec{n}$!

<u>Be able to find the angle between two planes:</u>

Use dot product between the normal vectors. As simple as that.