# Estimating Signaling Games in International Relations: Problems and Solutions

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**Abstract**

Signaling games are central to the study of politics but often have multiple equilibria, leading to no definitive empirical prediction. We demonstrate that these indeterminacies create substantial problems for currently used maximum likelihood techniques: they fail to uncover the parameters of the canonical crisis-signaling game, regardless of sample size, *even if* the equilibrium in the game generating the data is unique. To overcome this problem, we propose three estimators that outperform current best practices and are well suited to problems in international relations. We fit a signaling model to data on economic sanctions and find that standard maximum likelihood produces unintuitive estimates that are highly sensitive to modeling or software choices. Our solutions remedy the problems and uncover a novel $U$-shaped relationship between audience costs and the propensity for leaders to threaten sanctions.

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# 1 Introduction

Political scientists use signaling games to analyze a variety of phenomena across practically all subfields. Due to their importance in the crisis-bargaining literature, scholars of international relations structurally estimate increasingly more complicated signaling models to answer questions about sanction threats, crisis-escalation, audience costs, and more (Bas, Signorino and Whang 2014; Kurizaki and Whang 2015; Lewis and Schultz 2003; Wand 2006; Whang 2010; Whang, McLean and Kuberski 2013). Advocated by the movement for empirical implications of theoretical models, the structural approach allows researchers to account for strategic interdependence in the data generating process, estimate theoretical parameters, e.g., audience costs, and conduct counterfactual policy analysis in the absence of experimental conditions.

Despite this ubiquity and usefulness, political scientists have yet to overcome two substantial hurdles that arise when estimating signaling games. First, previous endeavors adopt a variant of the ML routines in Signorino (1999). These routines maximize a likelihood function which computes an equilibrium for every observation at every guess of the parameters. While straightforward, the procedure sidesteps a substantial problem in practice: An equilibrium is computed as if it is unique. When multiple equilibria exist—a possibility in these games—they create an indeterminacy in the likelihood, leading to inconsistent estimates (Jo 2011*a*). Second, even if a selection rule guaranteeing uniqueness were imposed, MLE can be computationally inefficient because multiple equilibria create discontinuities in the likelihood function and because current routines repeatedly compute equilibria in a resource-intensive manner. Due to these feasibility concerns, several scholars abandon the structural enterprise for reduced-form alternatives (Gleditsch et al. 2016; Trager and Vavreck 2011).

In this paper, we study the canonical signaling model of discrete crisis bargaining in international relations and illustrate the pitfalls that multiple equilibria create when estimating these games. In particular, we demonstrate that current best practices return incorrect estimates not only when there are multiple equilibria in the signaling game generating the data

but also when the equilibrium is unique. In the latter case, MLE fails because it evaluates likelihood functions at parameter guesses that produce multiple equilibria. To overcome these pitfalls, we adapt three estimators from the dynamic games and industrial organization literature, where equilibrium multiplicity is well known (e.g., De Paula 2013; Ellickson and Misra 2011). The estimators are *data-driven* in the sense that they use observables to estimate key parameters describing equilibrium play, allowing us to back out the exogenous parameters of interest without specifying an equilibrium selection rule. Because the estimators lead to well-defined empirical models, they vastly outperform current best practices. Through a series of experiments and applications, we argue that these solutions are quite well suited for the simpler, but far more influential, models in political science.

In doing so, this paper makes several contributions to the methodology and international relations literatures. Our analysis should give researchers pause when following current best practices for estimating crisis-signaling games (Whang 2010; Whang, McLean and Kuberski 2013). Instead, we offer three alternatives that are computationally more feasible, better performing in finite samples (reducing mean-squared error by orders of magnitude), and implemented using the open-sourced R programing language. Additionally, our approaches do not require equilibrium computation or selection, making them less sensitive to pre-estimation modeling or software choices. Furthermore, while we focus on signaling games, the estimators can also be used for other games with multiple equilibria, e.g., those with simultaneous moves.[1]

Although others have noted the problems that can arise with multiple equilibria and traditional MLE (Jo 2011*a*; Signorino 1999), solutions for researchers are scarce.[2] There is a misconception that ML techniques perform well if there is a unique equilibrium in the signaling

[1]MLE is also standard for estimating the extensive-form QRE models in McKelvey and Palfrey (1998) and Signorino (1999). Because these games always have a unique equilibrium, the problems we discuss do not arise in these contexts.

[2]Signorino (1999) concludes, "An important question then is: Given multiple equilibria, how are we to assign probabilities over outcomes? This is a surprisingly little-studied area in political science or in econometrics, but it must be addressed if we are to conduct similar statistical analyses of more realistic models of international interaction."

game generating the data (Bas, Signorino and Whang 2014; Jo 2011*a*). As we show below, this conclusion relies on finding starting values for optimization routines that are sufficiently close to the true values in the data generating process. Without privileged starting values, likelihood functions may be evaluated at parameters in which there are multiple equilibria, leading to the problems discussed above. In fact, MLE shows consistently poor performance even with a global optimization routine and a unique equilibrium in the game generating the data. To avoid these problems, Jo (2011*a*) suggests that refinements may reduce the number of equilibria. We show formally that almost all equilibria of the crisis-signaling game satisfy a regularity condition, one of the strongest refinements in the literature, however. If researchers want to reduce the number of equilibria *a priori*, then *ad hoc* selection is required. We show how this type of selection can lead to discontinuous likelihood functions and additional computational complexities, dramatically reducing the feasibility of traditional methods.

Our empirical application relates to recent debates over the effects of audience costs on the initiation of international crises (Crisman-Cox and Gibilisco 2017; Kurizaki and Whang 2015; Schultz 1999; Weeks 2012). We fit a crisis-signaling model to data on economic sanctions and uncover a novel $U$-shaped relationship between audience costs, i.e., the costs leaders pay for backing down in a crisis that they initiated, and the likelihood of leaders to threaten sanctions. Leaders with large or small audience costs more freely threaten sanctions, as the former can credibly commit to such threats and the latter need not worry about the consequences of backing down. In contrast, leaders with moderate audience costs almost never threaten sanctions because their threats are less credible and backing down entails nontrivial costs. Furthermore, we demonstrate that the vast majority of observations are located on the left-hand-side of the $U$-shaped curve. That is, larger (more negative) audience costs increase the likelihood of leaders to initiate threats, a result which holds in 97% of observations and is significant at conventional levels.

## 2 Model

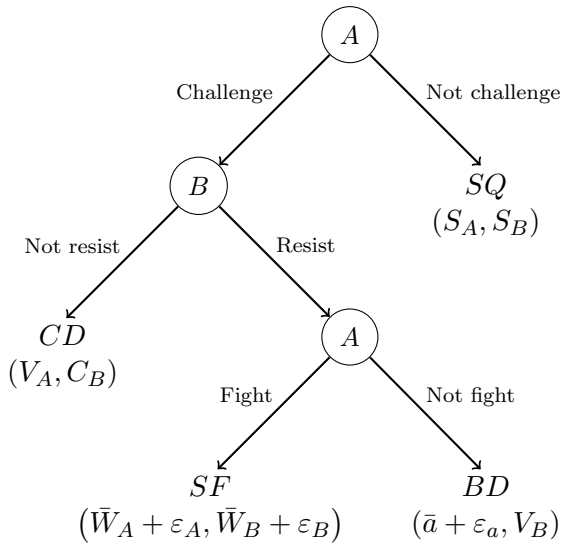
We consider the canonical signaling game in discrete choice crisis bargaining. Throughout, we adopt the specification from Lewis and Schultz (2003) and follow their notation. Several scholars have built upon their initial effort by modifying the assumptions governing private information (Bas, Signorino and Whang 2014; Wand 2006; Whang 2010). Nonetheless, the structure of the model appears in a variety of settings (Gurantz and Hirsch 2017; Kurizaki 2007; Schultz 1999).

There are two states $A$ and $B$ competing over a good or a policy that is currently owned or controlled by $B$. At the beginning of the game, the states observe private information. State $A$ observes $(\varepsilon_A, \varepsilon_a)$, where $\varepsilon_A$ and $\varepsilon_a$ are additively separable payoff shocks to $A$'s utility for war and backing down, respectively. Likewise, $B$ observes $\varepsilon_B$ which is an additively separable payoff shock to its war utility. All shocks are independently and identically distributed according to the standard normal distribution.

Interaction proceeds according to Figure 1. First, $A$ decides whether or not to challenge $B$ for control over the good or policy, and if $A$ does not challenge, then the game ends at node $SQ$ with payoffs $S_i$ for each state $i$. Second, after a challenge, $B$ decides whether or not to resist $A$. If $B$ does not resist, i.e., $B$ concedes to $A$'s demands, then the game ends at node $CD$, and payoffs are $V_A$ and $C_B$ for states $A$ and $B$, respectively. Finally, if $B$ does resist, then $A$ must decide whether to fight or not. When $A$ fights or stands firm, the states receive $\bar{W}_i + \varepsilon_i$ at node $SF$. Similarly, when $A$ backs down and does not fight, the games ends at node $BD$ with $A$ receiving $\bar{a} + \varepsilon_a$ and $B$ receiving $V_B$.

Perfect Bayesian equilibria (equilibria, hereafter) for the game can be represented as choice probabilities. Let $p_C$ and $p_F$ denote the probability that $A$ challenges and fights $B$, respectively, and let $p_R$ denote the probability $B$ resists. Let $p = (p_C, p_R, p_F)$ denote a profile of choice probabilities. Furthermore, let $\theta$ denote the vector of payoffs, i.e., $\theta = \left(\bar{a}, C_B, \left(S_i, V_i, \bar{W}_i\right)_{i=A,B}\right)$. The following result is due to Jo (2011*a*) and characterizes the equilibria of the game in terms of a system of nonlinear equations.

**Figure 1:** The canonical crisis-signaling game.




**Result 1** (Jo, 2011*a*) *An equilibrium $\tilde{p}$ exists, and $\tilde{p}$ is an equilibrium if and only if it satisfies the following system of equations:*

$$\tilde{p}_C = 1 - \Phi\left(\frac{S_A - (1-\tilde{p}_R)V_A}{\tilde{p}_R} - \bar{W}_A\right)\Phi\left(\frac{S_A - (1-\tilde{p}_R)V_A}{\tilde{p}_R} - \bar{a}\right) \equiv g(\tilde{p}_R;\theta), \tag{1}$$

$$\tilde{p}_F = \Phi_2\left(\frac{\bar{W}_A - \bar{a}}{\sqrt{2}}, \bar{W}_A - \frac{S_A - (1-\tilde{p}_R)V_A}{\tilde{p}_R}, \frac{1}{\sqrt{2}}\right)(g(\tilde{p}_R;\theta))^{-1} \equiv h(\tilde{p}_R;\theta), \tag{2}$$

*and*

$$\tilde{p}_R = \Phi\left(\frac{h(\tilde{p}_R;\theta)\bar{W}_B + (1-h(\tilde{p}_R;\theta))V_B - C_B}{h(\tilde{p}_R;\theta)}\right) \equiv f \circ h(\tilde{p}_R;\theta). \tag{3}$$

In words, for a fixed $\theta$, an equilibrium is completely pinned down by $B$'s probability of resisting. In addition, the functions $f$, $g$, and $h$ are best-response functions. Specifically, the functions $g$ and $h$ compute $A$'s best response to $B$'s probability of resisting $p_R$, and function $f$ denotes $B$'s best response to $A$'s probability of fighting. Furthermore, Jo (2011*a*) illustrates that multiple equilibria exist in a nontrivial set of parameters, i.e., there exists several solutions to the equation $f \circ h(p_R;\theta) = p_R$.

Throughout, we refer to *high-stakes* games as those where concessions are sufficiently valuable, i.e., a game is high-stakes if $V_B > C_B$ and $V_A > S_A$. In words, this condition means that the initiating country $A$ prefers $B$'s concession (outcome $CD$) to the status-quo (outcome

$SQ$). Likewise, the target country $B$ prefers $A$ backing down in the game (outcome $BD$) to initially conceding to $A$ (outcome $CD$). Notice that the condition does not limit how costly the war payoffs can be nor $A$'s own cost of backing down. High-stakes games are a natural way to think about this game and match the intuitive restrictions discussed in Schultz and Lewis (2005, p. 123–4). As we discuss below, high-stakes games are overwhelmingly prevalent in our empirical application.

# 3 Estimation: Problems and Solutions

We consider $D$ independent dyads or games. For each dyad $d$, researchers observe explanatory variables $x_d$. These observed variables, along with the true structural parameters, $\beta^*$, determine the system of equations defining equilibria. In the standard manner, the variables $x_d$ and parameters $\beta$ determine the model's payoffs, $\theta$, which we parameterize as follows:

$$\theta(x_d, \beta) = \begin{bmatrix} S_{dA} \\ S_{dB} \\ V_{dA} \\ C_{dB} \\ \bar{W}_{dA} \\ \bar{W}_{dB} \\ \bar{a}_d \\ V_{dB} \end{bmatrix} = \begin{bmatrix} x_{dS_A} \cdot \beta_{S_A} \\ 0 \\ x_{dV_A} \cdot \beta_{V_A} \\ x_{dC_B} \cdot \beta_{C_B} \\ x_{d\bar{W}_A} \cdot \beta_{\bar{W}_A} \\ x_{d\bar{W}_B} \cdot \beta_{\bar{W}_B} \\ x_{d\bar{a}} \cdot \beta_{\bar{a}} \\ x_{dV_B} \cdot \beta_{V_B} \end{bmatrix}, \tag{4}$$

where $\theta(x_d, \beta^*)$ are the true payoffs in the data generating process. Each $x_d$ vector above contains zero or more explanatory variables. As in Lewis and Schultz (2003), identification depends on there being at least one variable (including the constant) for each player that does not appear in all of that player's utilities.

Hereafter, we are interested in the $\beta$ parameters that are common across all games, rather

**Table 1:** Parameters for Monte Carlo experiments.

| Utility | Multiple equilibria | Unique equilibrium |
|---|---|---|
| $S_{dA}$ | 0 | 0 |
| $S_{dB}$ | 0 | 0 |
| $V_{dA}$ | 1 | 1 |
| $C_{dB}$ | 0 | 0 |
| $\bar{W}_{dA}$ | $-1.9$ | $-1.8$ |
| $V_{dB}$ | 1 | 1 |
| $\bar{W}_{dB}$ | $-2.9 + 0.1x_d$ | $-2.45 + 0.1x_d$ |
| $\bar{a_d}$ | $-1.2$ | $-1.2$ |

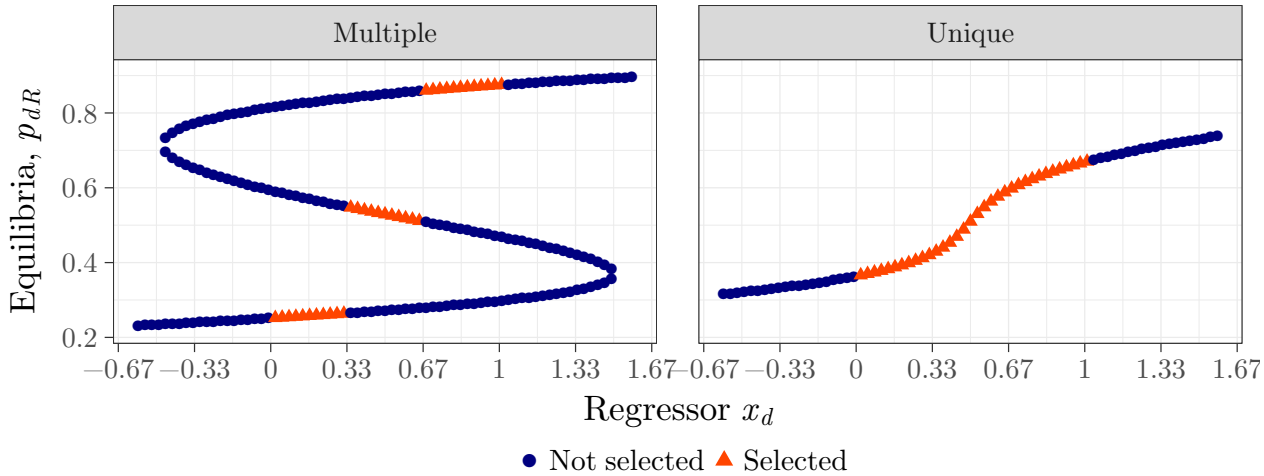
than $\theta_d$. Let $g(p_{dR}; x_d, \beta)$, $h(p_{dR}; x_d, \beta)$, and $f(p_{dF}; x_d, \beta)$ denote Result 1's equations that are parametrized by payoffs $\theta(x_d, \beta)$, and dyad $d$'s data generating equilibrium is labeled $p^*(x_d, \beta^*) = (p^*_{dC}, p^*_{dF}, p^*_{dR})$. Furthermore, researchers observe $T \geq 1$ observations from each equilibrium, where for each dyad $d$ and observation $t$, an outcome $y_{dt}$ is a terminal node, $y_{dt} \in \{SQ, CD, SF, BD\}$.[3]

Throughout it will be helpful to consider two numerical examples. Table 1 contains two sets of parameters that we use to demonstrate cases of unique and multiple equilibria. In both settings we include one regressor, which enters $B$'s war payoff, specifically, $\bar{W}_{dB} = \beta^1_{\bar{W}_B} + \beta^2_{\bar{W}_B} x_d$. The regressor $x_d$ is distributed uniformly over the unit interval. The first column includes values which are identical to those in Jo (2011*a*), while the second column tweaks these values slightly to create a scenario where the underlying game has a unique equilibrium.

There are three things to notice about the parameters in Table 1. First, we normalize the status-quo payoffs $S_i$ and $B$'s concession payoff to zero, following standard identification assumptions (Bas, Signorino and Whang 2014; Jo 2011*a*; Lewis and Schultz 2003). Second,

[3]We assume that two states play from the same equilibrium conditional on $x_d$, rather than allowing the equilibrium to vary over within-game observations. The assumption reflects an anarchic international system in which states find it difficult to repeatably coordinate on different equilibria due strategic reasons. Furthermore, several forces may incentivize states to focus on a single equilibrium over time, including persistent international norms/institutions (Keohane 1984), a focal point specific to these two states (Schelling 1960), or other factors that emerge from their long time spans and repeated interaction. We discuss possibilities for future work that would relax this assumption in the conclusion.

**Figure 2:** The equilibrium correspondences for numerical examples




**Caption:** The $x$-axis denotes the value of the regressor $x_d$, and the $y$-axis denotes the solutions to Equation 3 in Result 1 when the remaining parameters take values in Table 1. Solutions are computed using a line-search method. Colors and shapes denote the equilibrium selection the numerical examples and Monte Carlo experiments.

the differences in the two columns are minor: by making small adjustments to the parameters we can easily move into and out of situations were multiple equilibria exist. Third, these parameters reflect reasonable payoffs. The games in both columns are high-stakes. In addition, both war and backing down from threats are worse than the status quo, and actors only receive positive payoffs when their opponent backs down.

To illustrate the two settings, Figure 2 graphs the game's equilibrium correspondence as we vary the single regressor $x_d$ along the $x$-axis. The figure's $y$-axis denotes $B$'s equilibrium probability of resisting, which completely pins down an equilibrium in the game by Result 1. In the left-hand panel of Figure 2, there are multiple solutions to Equation 3 in Result 1 for values of $x_d$ between 0 and 1. Furthermore, the orange triangles in the plots illustrate how we determine which equilibria generate the data in our Monte Carlo experiments. Specifically, when $x_d \in \left[0, \frac{1}{3}\right)$, we use the smallest equilibrium probability of resisting $p_R$ to generate the data for dyad $d$. When $x_d \in \left(\frac{2}{3}, 1\right]$, we use the largest. Finally, we use the moderate equilibrium in the remaining case. The right-hand side of Figure 2 graphs the equilibrium correspondence under parameters shown in the third column of Table 1, where there is an unique equilibrium for all values of $x_d$.

The goal is to estimate $\beta^*$. We first explicate the commonly used ML approach. Subsequently, we adapt three estimators from the dynamic games literature to the crisis-signaling environment. Our three proposals are *data driven* in the sense that they use observed data to uncover each dyad's data generating equilibrium even though multiple ones may exist in the signaling game governing the data generating process. Broadly speaking, we consider two classes of estimators. One class consists of pseudo-likelihood procedures where we use observed covariates across dyads to identify key characteristics describing equilibria generating the data. The parameters of interest are subsequently pinned down in a second stage fixing these estimated equilibrium characteristics. In contrast, the second class is a full-information approach where we maximize a likelihood subject to constraints that describe equilibrium play. Here, we use repeated observations within each game to simultaneously estimate the equilibrium generating the data and the parameters that describe equilibrium constraints.

## 3.1 Problems with traditional MLE

The current best practices in the literature closely follow the ML techniques discussed in Signorino (1999) and Rust (1987). For every $\beta$, an equilibrium to game $d$ is computed by solving the system of equations in Result 1; call this solution $p(x_d, \beta)$. Note that this solution is not necessarily unique, and following standard practices, we do not search for all solutions. Technically, $p(x_d, \beta)$ is a "function" that maps from data and parameters into equilibrium choice probabilities.

Using $p(x_d, \beta)$, we define the probability of reaching each of the terminal nodes as

$$\Pr[y_{dt} \mid p(x_d, \beta)] = \begin{cases} (1 - p_{dC}) & \text{if } y_{dt} = SQ \\ p_{dC}(1 - p_{dR}) & \text{if } y_{dt} = CD \\ p_{dC}p_{dR}(1 - p_{dF}) & \text{if } y_{dt} = BD \\ p_{dC}p_{dR}p_{dF} & \text{if } y_{dt} = SF. \end{cases} \tag{5}$$

Under this setup, the log-likelihood takes the form

$$L(\theta \mid Y) = \sum_{d=1}^{D} \sum_{t=1}^{T} \log \Pr[y_{dt} \mid p(x_d, \beta)], \tag{6}$$

and the traditional ML (tML) estimates maximize this log-likelihood.

As described in Jo (2011*a*), the current approach evaluates the likelihood function as if a unique equilibrium exists. That is, for each guess of the parameters, we compute an equilibrium, $p(x_d, \beta)$, using a standard (numeric) equation solver. If there are multiple equilibria, then there is an indeterminacy in how analysts evaluate $p(x_d, \beta)$. If the equation solver of choice selects the wrong equilibrium, i.e., not the one in the data generating process, then the likelihood is computed incorrectly, resulting in mistaken inferences. To better see this problem, suppose there are $D$ dyads, and fixing parameters $\beta$, suppose each dyad admits $n > 1$ equilibria. In this case there are $n^D$ possible values of the log-likelihood for one parameter value, as each dyad may be playing any one of the $n$ equilibria. Standard equation solvers return just one of the $n^D$ combinations. As $D$ increases, it is increasingly implausible that the correct selection is made.[4] An implication of this discussion is that two researchers can reach conflicting conclusions even when analyzing the same data if they implement the tMLE with different equation solvers. We illustrate this problem in our empirical application.

Before proceeding, we consider potential fixes to the traditional MLE approach and discuss why they are not appropriate solutions. To this end, we first ask: Can multiplicity in the crisis-signaling game be solved with traditional refinements? If we can use a principled refinement to produce a unique equilibrium, then traditional ML techniques can be safely used so long as they are adjusted to always select the surviving equilibrium. Standard refinements based on off-the-path-of-play beliefs, such as the Intuitive Criterion or Divinity, are inconsequential here as all histories are reached with positive probability in every equilibrium. Because of

[4]Global optimization does not solve this problem. Even if the optimizer guesses the true value, the underlying equation solver may not find the correct equilibrium for each observation, leading the optimizer away from the truth.

this, an analyst may be tempted to use a refinement called regularity, (e.g., Doraszelski and Escobar 2010; Harsanyi 1973; van Damme 1996), which subsumes several other refinements, including perfection, essentialness, and strong stability.[5] Unfortunately, the refinement does not solve the multiplicity in this game. As we show in Appendix A, for almost all parameter values, all equilibria of the crisis-signaling game satisfy regularity.[6] Most importantly, the result demonstrates that the problem with multiple equilibria cannot be "refined away" using standard criteria, and the predictive indeterminacy that plagues traditional maximum likelihood methods still persists.

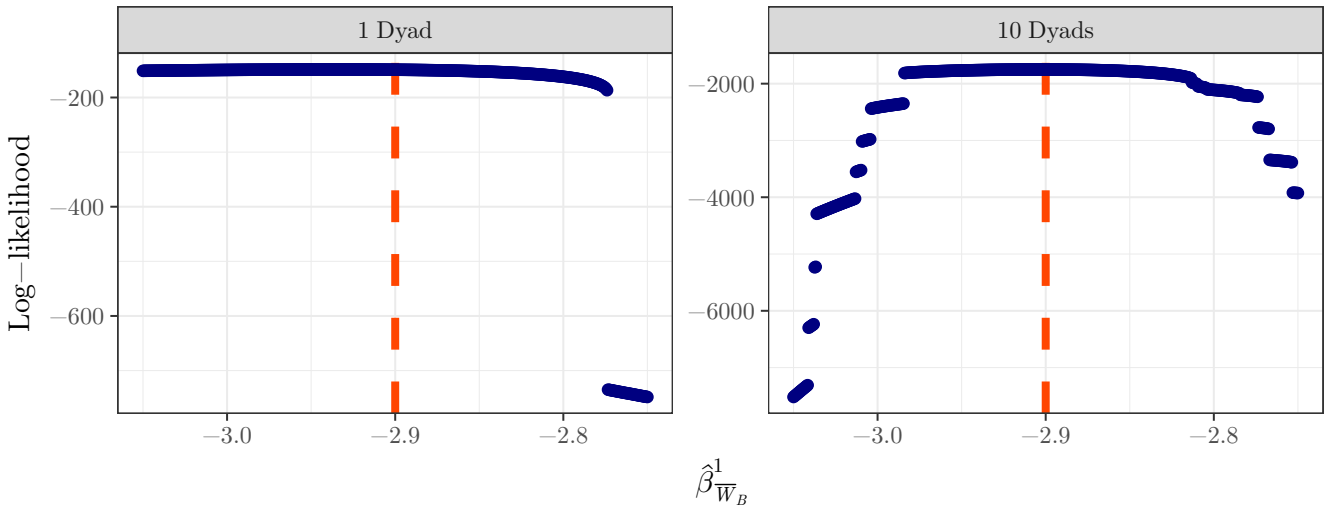
With traditional refinements unavailable, an analyst might consider other possible selection mechanisms. Some options could be selecting the equilibrium that maximizes a convex sum of $A$ and $B$'s payoffs or the equilibrium that solves $\max \tilde{p}_R$. A straightforward approach would be to select the equilibrium that maximizes each dyad $d$'s contribution to the likelifor all $d$hood. Such criteria might remove the indeterminacy in $p(x_d, \beta)$, resulting in a well-defined likelihood. This approach has several drawback, however. First, this selection process forces an additional and consequential modeling choice onto the analyst, one that can heavily influence the resulting estimates. In our empirical application, we show how equilibrium selection can dramatically change substantive conclusions. Second, without further theoretical results, researchers would need to compute all equilibria and then verify which one satisfies their criterion, a computationally demanding task. Third, imposing a selection criterion introduces discontinuities in the likelihood function as the number of equilibria and hence the solution to the criterion varies across different parameter values.

To better illustrate this last point consider Figure 3. Here, we graph the log-likelihood as a function of the parameter $\hat{\beta}^1_{\bar{W}_B}$, where data were generated using the values in Table 1, column 1, the equilibrium selection in Figure 2, and $D \in \{1, 10\}$ with $T = 200$. Furthermore,

[5]For a formal definition, see Appendix A. There are additional reasons to focus on regular equilibria. They can be implicitly expressed as continuous functions of parameters. As such, if we uncover noisy, but sufficiently accurate estimates of $\theta$, then equilibrium choice probabilities will be close to their true values as well. Likewise, comparative statics (predicted probabilities) on regular equilibria will be well behaved.

[6]We say that a property holds for almost all parameters $\theta$, if it does not hold at most in a closed, Lebesgue-measure-zero subset of $\mathbb{R}^8$.

**Figure 3:** Log-likelihood function with an imposed selection rule




**Caption:** The $x$-axis denotes the value of the estimate of $\hat{\beta}^1_{\bar{W}_B}$, where the true value is $-2.9$. The $y$-axis corresponds to the value of the log-likelihood maximized over all equilibria. All other parameters are held at their true values. To generate the data, we fix $T = 200$ and use the parameters in Table 1, column 1, and the selection rule in Figure 2.

for each dyad $d$ and each value of $\hat{\beta}^1_{\bar{W}_B}$ along the $x$-axis, we compute all equilibria and select the one that maximizes the log-likelihood, which is denoted by the $y$-axis. This selection rule represents the best case for traditional methods, but requires additional computational costs as we repeatedly solve the system of equations describing equilibrium play. Although this selection rule removes the indeterminacy that multiple equilibria induce and correctly identifies the true value of $\beta^1_{\bar{W}_B}$, it creates a discontinuous likelihood function. Furthermore, as $D$ increases, so do the number of discontinuities and the number of equilibrium computations. The problems are further exacerbated if we were to estimate more than one parameter. The three estimators we propose below avoid these issues. Not only do they lead to well-defined empirical models and objective functions, but also they do not burden the researcher by requiring a selection rule or the repeated computation of equilibria.

## 3.2 Pseudo-likelihoods

Our first proposal involves the two-step estimator from Hotz and Miller (1993), which is also analyzed as a special case in Aguirregabiria and Mira (2007).[7] In the first step, we produce consistent (in $T$ or $D$) estimates of the equilibrium choice probabilities $p^*_{dR}$ and $p^*_{dF}$, for $d = 1, ..., D$. We label these estimates $\hat{\mathbf{p}}_{\mathbf{R}} = (\hat{p}_{1R}, ..., \hat{p}_{DR})$ and $\hat{\mathbf{p}}_{\mathbf{F}} = (\hat{p}_{1F}, ..., \hat{p}_{DF})$. Ideally, this step is done either non-parametrically or with an extremely flexible estimator. While we are agnostic about how to obtain the first-stage estimates in practice, a non-parametric frequency estimator may be preferred when $T$ is large. When dealing with smaller samples, however, flexible methods that pool information between games, such as kernel regression, basis functions, or generalized additive models, may be needed. We use a random forest to produce first-stage estimates throughout our analysis.

Next, we can consider how actors best respond to these first-stage estimates. These best responses are computed using the equations in Result 1, which take the form:

$$\hat{p}(\hat{p}_{dR}, \hat{p}_{dF}; x_d, \beta) = \begin{bmatrix} g(\hat{p}_{dR}; x_d, \beta) \\ h(\hat{p}_{dR}; x_d, \beta) \\ f(\hat{p}_{dF}; x_d, \beta) \end{bmatrix}. \tag{7}$$

In other words, if actors play the game *as if* they believed their opponents use strategies estimated in the first stage, $\hat{p}_{dR}$ and $\hat{p}_{dF}$, then $\hat{p}$ are their best responses. These best-responses approach their true values, as the first-stage estimates become more precise. Using the first-stage estimates and the associated best-responses, we build the pseudo-log-likelihood function

[7] Another possible two-step estimator is a nonlinear least squares approach based on Pesendorfer and Schmidt-Dengler (2008). In Monte Carlo simulations, the least-squares estimator performed poorly compared to the three likelihood estimators we introduce. These performance issues appear to result from highly nonlinear and numerically unstable first derivatives. Because of these issues, we do not include it here and discourage its use with this game.

as

$$PL(\beta \mid \hat{\mathbf{p}}_{\mathbf{R}}, \hat{\mathbf{p}}_{\mathbf{F}}, Y, X) = \sum_{d=1}^{D} \sum_{t=1}^{T} \log \Pr[y_{dt} \mid \hat{p}(\hat{p}_{dR}, \hat{p}_{dF}; x_d, \beta)]. \tag{8}$$

Notice that when the first-stage estimates equal the equilibrium choice probabilities, the pseudo-log-likelihood is the log-likelihood for Equation 6 with the correct equilibrium selection.

### 3.2.1 Nested Pseudo-Likelihood

The NPL approach, proposed by Aguirregabiria and Mira (2007), builds on the PL by using best responses to update the first-stage choice probabilities upon knowing the PL estimates. This process is repeated until convergence. More precisely, the NPL algorithm begins with the PL estimates,

$$(\hat{\beta}_0^{NPL}, \hat{\mathbf{p}}_{\mathbf{R},\mathbf{0}}, \hat{\mathbf{p}}_{\mathbf{F},\mathbf{0}}) = (\hat{\beta}^{PL}, \hat{\mathbf{p}}_{\mathbf{R}}, \hat{\mathbf{p}}_{\mathbf{F}}),$$

and for the $k$th iteration, we set

$$\begin{aligned}
\hat{p}_{dF,k} &= h(\hat{p}_{dR,k-1}; x_d, \beta_{k-1}),\ d = 1, \ldots, D \\
\hat{p}_{dR,k} &= f(\hat{p}_{dF,k-1}; x_d, \beta_{k-1}),\ d = 1, \ldots, D \\
\hat{\beta}_k^{NPL} &= \underset{\beta}{\operatorname{argmax}}\ PL(\beta \mid \hat{\mathbf{p}}_{\mathbf{R},\mathbf{k}}, \hat{\mathbf{p}}_{\mathbf{F},\mathbf{k}}, Y, X).
\end{aligned}$$

The algorithm is repeated until the parameters and choice probabilities cease changing (subject to a pre-specified step-tolerance). As discussed in Aguirregabiria and Mira (2007), the NPL approach reduces the analyst's reliance on correct first-stage estimates and is more efficient than the PL estimator when the first-stage estimates are imprecise.

However, the NPL algorithm is not costless: without a particular stability condition on the data generating process, the algorithm may fail to converge (Kasahara and Shimotsu 2012; Pesendorfer and Schmidt-Dengler 2010). Specifically, we need to be concerned with whether

the data generating equilibria are best-response stable. If an equilibrium is best-response stable, then repeatedly evaluating the function $f \circ h$ will coverage to the equilibrium as long as the starting value is not too far away. In contrast, if the data generating equilibrium is unstable, then best-response iteration will not converge to the true equilibrium unless it does so in a finite number of steps. We provide formal definitions of stable and unstable equilibria in the Supplementary Materials.

This concern raises the question as to whether best-response stable or unstable equilibria appear in the underlying game? Restricting our attention to just high-stakes games, we show that if there is a unique equilibrium, it is best-response stable. However, if there are multiple equilibria, then there exists at least one best-response unstable equilibrium and at least two best-response unstable equilibria.[8] As both pseudo-likelihood methods rely on computing at least one best response, we analyze the degree to which these methods are sensitive to unstable equilibria in a Monte Carlo experiment in Appendix B.3, where we find that both the PL and the NPL perform quite well even if best-response unstable equilibria dominate in the data.

## 3.3 Constrained MLE

An alternative approach is to use a full-information CMLE, as proposed by (Su and Judd 2012). Applied to this problem, we maximize the likelihood in Equation 6 subject to the equilibrium constraints in Result 1. To do this we define

$$\bar{p}(p_{dR}; x_d, \beta) = \begin{bmatrix} g(p_{dR}; x_d, \beta) \\ h(p_{dR}; x_d, \beta) \\ p_{dR} \end{bmatrix}. \tag{9}$$

[8] See Appendix A for the formal statements and proofs of these results.

Then the CMLE estimates solve the following problem

$$\max_{\beta,\, \mathbf{p_R}} \sum_{d=1}^{D} \sum_{t=1}^{T} \log \Pr[y_{dt} \mid \bar{p}(p_{dR}; x_d, \beta)], \\ \text{s.t. } f \circ h(p_{dR}; x_d, \beta) = p_{dR},\ d = 1, ..., D. \tag{10}$$

Su and Judd (2012) demonstrate that the CMLE is equivalent to an MLE procedure in which analysts select the equilibrium that maximizes the likelihood of the observed data for every guess of the parameters. Such a modification substantially reduces MLE's feasibility, however, for two of the reasons discussed above: it requires computing all equilibria and it produces a discontinuous likelihood function. The constrained approach avoids both problems.[9] Unlike the pseudo-likelihood procedures there is no need to estimate choice probabilities separately from the main parameters. This improves the estimator's efficiency and eliminates the potential bias resulting from model misspecification or unstable equilibria.

Note that the constrained optimization problem has $D$ auxiliary parameters, which means it may not be consistent in the number of dyads. In fact, if $T = 1$, then the estimator requires more parameters than observations.[10] For larger values of $T$, however, we gain leverage by pooling over dyads. Intuitively, with more draws from the equilibrium within each dyad, we learn the equilibrium choice probabilities with increasing certainty and can then uncover $\beta$. Furthermore, our experiments demonstrate that the CMLE performs well even with a moderate number of within game observations.

Another drawback of the CMLE is that solving the constrained optimization problem is more computationally intense than either of the pseudo-likelihood estimators. In our Monte Carlo simulations, an R implementation of the augmented Lagrangian method (Nocedal and Wright 2006) performs well at solving the problem in Equation 10.[11] More specialized—

[9]Notice that both the objective and constraint functions in Equation 10 are continuous in $(\beta, \mathbf{p_R})$. By not requiring that $\mathbf{p_R}$ satisfy the equilibrium condition at every guess of $\beta$, the CMLE ensures that the objective function is well behaved. If $(\hat{\beta}, \hat{\mathbf{p_R}})$ is a solution to the problem in Equation 10, then equilibrium constraint is satisfied, i.e., $f \circ h(p_{dR}; x_d, \beta) = p_{dR}$ for all dyads $d$.

[10]This is in contrast to the two pseudo-likelihood approaches, which can be used for the $T = 1$ case.

[11]This implementation can be found in R's `alabama` package.

but still open source— software performs even better in the simulations and is required to reliably fit the CMLE to our economic sanctions data; complete implementation details are in Appendix C.

# 4 Performance

We now evaluate the performance of the estimators under different numbers of equilibria in the data generating process. We continue to use the parameter values from Table 1, where $x_d$ is still distributed standard uniform. As a preview, the traditional MLE consistently performs the worst, and the PL, NPL, and CMLE do quite well.

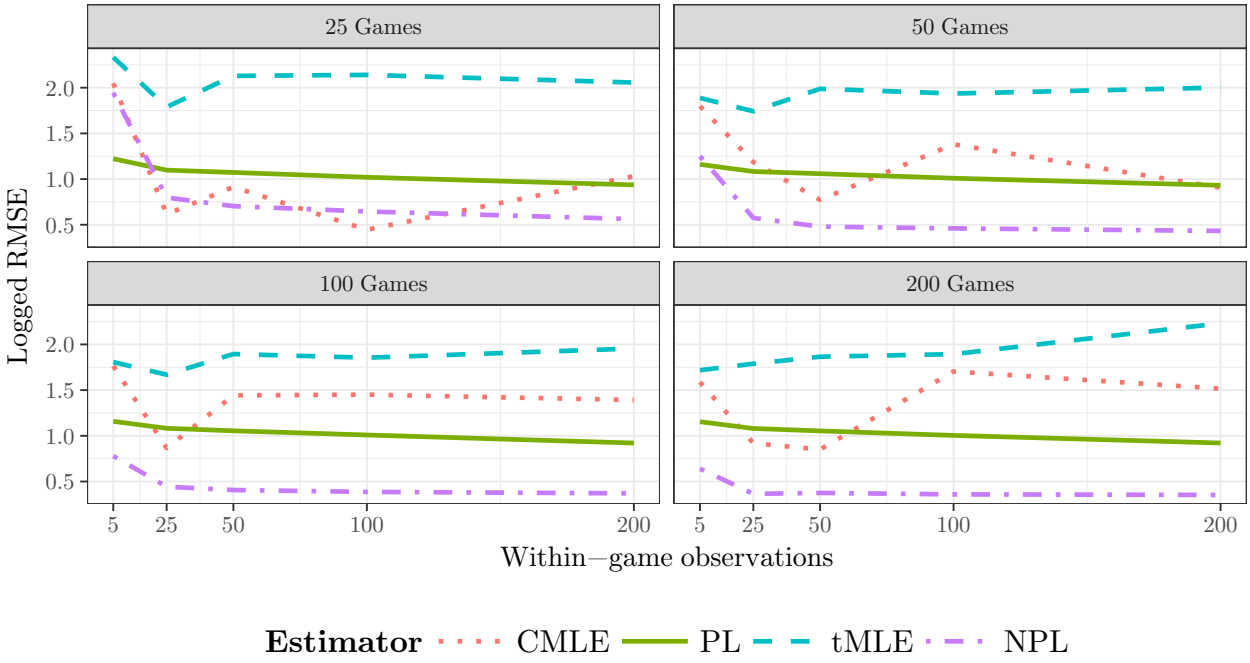
Recall that the pseudo-likelihood estimator requires first-stage estimates of equilibrium choice probabilities. As mentioned above, we use a random forest, a flexible machine learning technique, to produce these estimates. There are two models in the first-stage, where the dependent variables are the nonparametric frequency estimates of the probability that $B$ resists (for $\hat{\mathbf{p}}_{\mathbf{R}}$) and $A$ fights (for $\hat{\mathbf{p}}_{\mathbf{F}}$). We fit the former only with observations in which $A$ challenges, and we fit the latter only with observations in which $B$ resists. For predictors, we include the one regressor, $x_d$.[12]

With our experimental conditions fixed, we now examine how the number of equilibria affects the estimators. We vary the number of dyads, $D$, between 25 and 200 and the number of within-game observation, $T$, between 5 and 200 to create simulated datasets of various sizes. For each combination of $D$ and $T$, we draw $x_d$ from the standard uniform distribution and then select the appropriate equilibrium that generates the data for the corresponding dyad, as shown in Figure 2. Finally, we use the simulated data to estimate the game using all four estimators. We repeat this process for $1,000$ times for each pair of $D$ and $T$ and for each of the parameter settings in Table 1.[13]

[12]Montgomery and Olivella (2017) have a recent discussion of tree-based models.

[13]Starting values were drawn from a standard uniform distribution. For the CMLE, starting values for $\mathbf{p}_{\mathbf{R}}$ are the random forest estimates from the first-stage analysis, associated with the two-step estimators.

**Figure 4:** RMSE in signaling estimators with multiple equilibria.

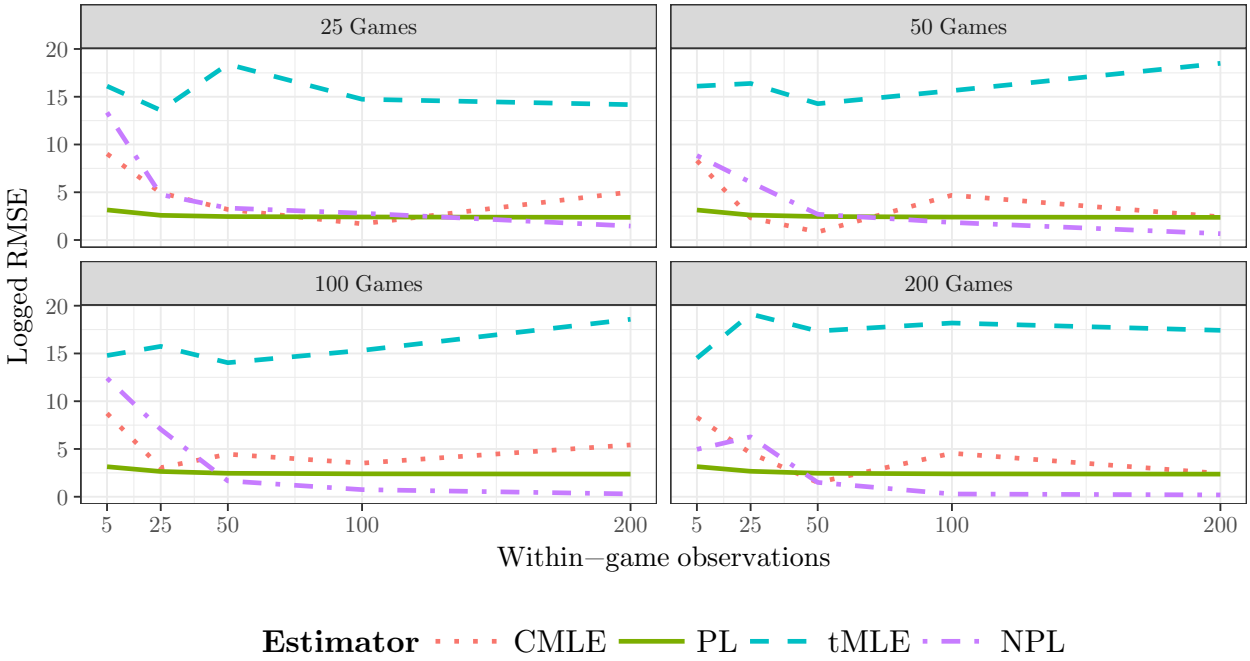



The main results of the experiment are summarized in Figures 4 and 5, which compare the logged root-mean-square error (RMSE) of the estimators.[14] The first thing we note is that the tMLE (blue, dashed line) performs consistently bad and shows no improvement as the amount of data increases in either $D$ or $T$. In many cases, its performance worsens as $T$ increases. In Appendices B.1 and B.2, we graph the bias and variance of the estimators as well as their convergence rates and computational time. The graphs demonstrate that the bias associated with the tMLE is an order of magnitude larger than its worst performing competitor. Furthermore, the large variance of the tMLE demonstrates that the optimization algorithm converges on estimates scattered throughout the parameter space and is not routinely attracted to a local minimum.[15] Overall, we find that equilibrium selection is the root cause of the tMLE's

[14] Multivariate mean-square error is calculated as $\text{tr}(\text{Var}(\hat{\beta})) + [\hat{\beta} - \beta^*]'[\hat{\beta} - \beta^*]$.

[15] As a robustness check, we also use a global optimizer for the tMLE routine. In a simulation with $T = D = 200$ and using the unique equilibrium parameter values from Table 1, we optimize the tMLE objective function with DEoptim (Mullen et al. 2011). While the global optimizer improves the tMLE's performance (RMSE of 10.04, compared to 17.42 in Figure 5), our three estimators continue to outperform it with RMSE values of 2.44, 2.37, and 0.19 for the CMLE, PL, and NPL, respectively, even though the latter methods use the less robust, albeit more computationally feasible, gradient-based optimizers.

**Figure 5:** RMSE in signaling estimators with a unique equilibrium




problems.

Contrast these results to those from the other estimators, which generally all improve with more data. Of these three estimators, the PL (green, solid line) and NPL (pink, dot-dashed) estimators perform remarkably well. The PL method is generally the best performing estimator with small numbers of within-game observations, $T$. Additional analysis in Appendix B shows that the estimator tends to have more bias than the others and that its strong performance is driven by low variance. The NPL greatly improves the bias associated with the PL method, without adding too much variance, and as a result, we see that it is the best performing of our candidates, especially with large $T$. Finally, while the CMLE (red, dotted line) typically outperforms the tMLE, it is also characterized by a low-bias-high-variance trade-off, illustrated in Appendix B.1, which frequently makes it the worst performing of our proposed solutions. When we use specialized software that provides first and second derivative information, the CMLE's performance can greatly improve; see Appendix C for more information.

Notice that the results are quite consistent in both the multiple equilibria setting, i.e., Figure 4, and the unique equilibrium setting, i.e., Figure 5. The tMLE's poor performance in the unique setting is particularly surprising, because uniqueness is standard justification for using the tMLE, as in Bas, Signorino and Whang (2014) and Jo (2011*a*). Furthermore, both papers report Monte Carlo results that illustrate the consistent-like properties of the tMLE routine with a unique equilibrium. To the best of our knowledge, this occurs when the tMLE routine begins with privileged starting values that are sufficiently close to those in the data generating process. We investigate these differences in Appendix D. Without privileged starting values, the tMLE routine will evaluate a likelihood function with parameters guesses under which multiple equilibria exist, leading to difficulties uncovering the true values. Most concerning, these problems persist regardless of the number of observations.

Before turning our attention to economic sanctions, we report the following conclusions.

1. The tMLE routine performs the worst in both multiple and unique settings, regardless of the number of observations.

2. The NPL and PL methods consistently perform the best, but the PL out performs the NPL when the number of within game observations is small, and *vice versa* when the number of within game observations is large.

3. The CMLE has a small bias, large variance trade-off due to the highly nonlinear nature of the constraint function. Its performance can be improved by using more advanced tools.

# 5 Application to Economic Sanctions

Our empirical application is motivated by Whang, McLean and Kuberski (2013, WMK, hereafter) who structurally estimate the process of implementing sanctions as a signaling interaction. The game is reproduced in Figure 6, where $A$'s actions at the first node are now

whether or not to threaten sanctions, $B$ then decides whether or not to resist, and finally $A$ must decide whether to follow through on its threat of sanctions. The outcomes are status quo, concede to the threat, impose sanctions, and back down, which are denoted $SQ$, $CD$, $SF$, and $BD$, respectively.

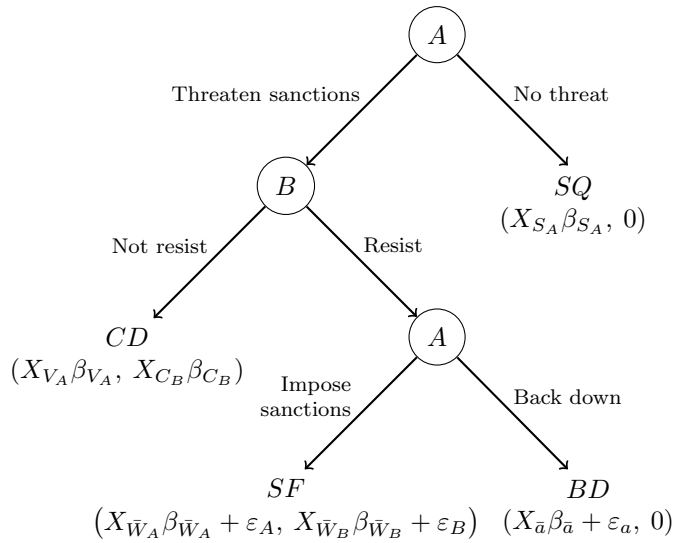
An observation in WMK is a politically relevant directed-dyad decade. A directed-dyad is politically relevant if there exists at least one sanction threat issued from State $A$ to State $B$ in the Threat and Imposition of Sanctions (TIES) dataset during the 1971–2000 period. Within each directed dyad, the authors aggregate the temporal variation based on the most extreme action taken in the decade, dividing the time frame into three groups 1971–80, 1981–90, and 1991–2000.

Like WMK, our analysis considers only politically relevant directed dyads. Unlike their paper, we admit observations at the monthly level, *but* we treat observations within a directed-dyad decade as if they were repeated draws from the same equilibrium.[16] In terms of our setup, a game is a politically relevant, directed dyad decade, and we observe $T = 120$ observations from each game. For our purposes, this approach has two important advantages. First, the CMLE procedure requires multiple observations within game for identification. Without this setup, we could not illustrate this estimator even though it performed quite well in the Monte Carlo experiments. Second, we do not ignore variation within each decade: a directed dyad with only one threat issued in a decade may be substantially different than one with several threats in the same period.

Such a setup is possible because the TIES data records the date of each action taken within the dyad. Following WMK, we use the *Final Outcome* variable to record the outcomes of each dyad-month. When there is no action in a month, we record the status quo, when *Final Outcome* is 1, 2, or 5 we record the outcome as $CD$, a *Final Outcome* of 3 or 4 is $BD$, and $SF$ is recorded for outcomes 6 and above. After dropping irrelevant dyad-decades, we are

[16] A minor difference arising in our application is that WMK add normally distributed shocks to payoffs at every terminal node, and they further estimate the covariance of these shocks. WMK not only report that these covariance estimates are below 0.07 in magnitude, but also fail to reject the null hypothesis that the covariances are equal to zero at the $p < 0.08$ level.

**Figure 6:** Economic sanctions as a signaling interaction.




left with 418 games, each with 120 within game observations that span one of the three time frames, 1971–80, 1981–90, and 1991–2000. The independent variables, their sources, and how they enter the actors' payoffs are listed in Table 2, following the specifications in WMK.

We estimate the model using all of the techniques discussed above with identical starting values. In the application $T > 100$ and $D > 25$, so the Monte Carlo experiments suggest that our proposed estimators, i.e., those besides the tMLE, should preform quite well. Following the experimental procedures, we use a random forest and the variables in Table 2 to form first-stage estimates for the pseudo-likelihood estimator; we also initialize the NPL method with these estimates. For standard errors, we use a non-parametric bootstrap to estimate the variance and covariance of the first-stage estimates. The first-stage variance-covariance matrix is then used to compute the analytical standard errors described in Aguirregabiria and Mira (2007) for the PL estimator. Aguirregabiria and Mira (2007) also provide analytic standard errors for the NPL estimator, which do not rely on the variance of the first-stage estimates. Analytical standard errors for the CMLE are computed using (Silvey 1959, Lemma 6) and the tMLE standard errors are from the outer-product of gradients estimator.

**Table 2:** Variables in the economic sanctions model

| Variable | Utilities | Description | Source |
|---|---|---|---|
| Fixed to 0 | $S_B$, $V_B$ | Identification restriction | – |
| Constant | $V_A$, $C_B$, $\bar{W}_i$, $\bar{a}$ | Omitted from $S_A$ for identification | – |
| Econ. Dep$_A$ | $S_A$, $\bar{W}_A$ | $A$'s economic dependence on $B$ | TIES |
| Dem$_A$ | $S_A$, $\bar{a}$, $\bar{W}_A$ | $A$'s Polity2 score | Polity IV |
| Contiguity | $S_A$, $C_B$ | Contiguity between the states | COW |
| Alliance$_B$ | $V_A$, $C_B$ | Alliance between the states (0/1) | COW |
| Costs$_A$ | $V_A$ | Anticipated costs to $A$ | TIES |
| Econ. Dep$_B$ | $C_B$ | $B$'s economic dependence on $A$ | TIES |
| Costs$_B$ | $C_B$ | Anticipated costs to $B$ | TIES |
| Cap. Ratio | $\bar{W}_i$ | (log) ratio of $A$'s capabilities to $B$ | COW |
| Dem$_B$ | $\bar{W}_B$ | $B$'s Polity2 score | Polity IV |

## 5.1 Results

Table 3 displays our main results, where each column contains parameter estimates using the different estimators. There are several notable patterns. First, the techniques derived from the dynamic games literature produce estimates that agree in direction, magnitude, and significance. Models 2–4 match signs for 16 out of 21 coefficients, and when we reject a null hypothesis using one estimator, we generally do the same for one of the others. Second, the tMLE returns estimates that diverge wildly from the other three. The problem appears particularly bad for coefficients that enter the target state's concession payoffs, $C_B$. For example, the tML estimates suggest that contiguity should decrease $B$'s payoff from conceding, a result which is significant at conventional levels. In contrast, the models that accommodate multiple equilibria all agree that continuity should increase $B$'s concessions payoffs, results which are also significant at conventional levels.

Third, the tMLE routine produces substantive implications that are inconsistent with previous findings and theoretical intuitions. Such inconsistencies do not arise with the three estimators that explicitly accommodate multiple equilibria, however. In the MLE results, for example, economic dependence is negative in $B$'s concession payoff, $C_B$. This suggests that greater dependence decreases the target state's preference for peaceful resolutions over

**Table 3:** Economic sanctions application

| | tML Model 1 | Pseudo-Likelihood Model 2 | Nested Pseudo Likelihood Model 3 | CMLE Model 4 |
|---|---|---|---|---|
| $S_A$: Econ. Dep$_A$ | 0.05 | −0.28 | −0.22 | −0.52 |
| | (0.29) | (0.69) | (1.11) | (0.52) |
| $S_A$: Dem$_A$ | 0.00 | −0.01 | 0.03 | 0.01 |
| | (0.00) | (0.06) | (0.07) | (0.03) |
| $S_A$: Contiguity | 0.27* | 0.01 | 0.05 | 0.04* |
| | (0.10) | (0.01) | (0.04) | (0.02) |
| $S_A$: Alliance | −0.06 | −0.18* | −0.17 | −0.16* |
| | (0.08) | (0.03) | (0.10) | (0.07) |
| $V_A$: Const. | −0.06 | −0.29 | 1.60 | 1.31* |
| | (0.08) | (0.75) | (1.65) | (0.47) |
| $V_A$: Costs$_A$ | −0.04 | 0.37 | −0.05 | −0.19 |
| | (0.03) | (0.27) | (0.27) | (0.17) |
| $C_B$: Const. | 0.81 | −0.89* | −2.14* | −4.53* |
| | (0.91) | (0.31) | (0.80) | (2.31) |
| $C_B$: Econ. Dep$_B$ | −0.21 | 1.34* | 2.34* | 2.83* |
| | (0.16) | (0.59) | (1.11) | (0.59) |
| $C_B$: Costs$_B$ | −0.08* | 0.11 | 0.12 | 0.19* |
| | (0.03) | (0.06) | (0.06) | (0.05) |
| $C_B$: Contiguity | −0.25* | 0.09* | 0.12* | 0.10* |
| | (0.02) | (0.03) | (0.04) | (0.03) |
| $C_B$: Alliance | 0.10 | 0.03 | −0.03 | −0.02 |
| | (0.09) | (0.13) | (0.12) | (0.11) |
| $\bar{W}_A$: Const. | −0.15 | −2.43* | −2.42* | −2.46* |
| | (0.78) | (0.10) | (0.13) | (0.08) |
| $\bar{W}_A$: Econ. Dep$_A$ | 0.07 | 0.28 | 0.01 | −0.05 |
| | (0.75) | (0.92) | (1.14) | (0.18) |
| $\bar{W}_A$: Dem$_A$ | 0.01 | 0.00 | 0.04 | 0.00 |
| | (0.01) | (0.07) | (0.07) | (0.03) |
| $\bar{W}_A$: Cap. Ratio | −0.01 | 0.02 | 0.03 | 0.04* |
| | (0.01) | (0.01) | (0.03) | (0.01) |
| $\bar{W}_B$: Const. | −0.38 | 0.51* | −0.91 | −4.42 |
| | (1.13) | (0.25) | (1.11) | (3.15) |
| $\bar{W}_B$: Dem$_B$ | 0.01* | 0.00 | 0.00 | −0.01* |
| | (0.00) | (0.01) | (0.01) | (0.01) |
| $\bar{W}_B$: Cap. Ratio | 0.01 | 0.07 | 0.12* | 0.29* |
| | (0.01) | (0.04) | (0.05) | (0.12) |
| $\bar{a}$: Const. | −0.56 | −2.63* | −2.64* | −2.71* |
| | (0.77) | (0.09) | (0.13) | (0.08) |
| $\bar{a}$: Dem$_A$ | 0.00 | −0.02 | 0.02 | 0.00 |
| | (0.01) | (0.07) | (0.07) | (0.03) |
| Log $L$ | -4102.76 | -3964.53 | -3932.45 | -3927.91 |
| $N$ | $418 \times 120$ | $418 \times 120$ | $418 \times 120$ | $418 \times 120$ |

*Notes:* $^*p < 0.05$

Standard Errors in Parenthesis

escalation. Yet, the remaining estimators suggest the opposite. Here, more dependence implies $B$ should be more likely to prefer peace than escalation, a finding which closely matches previous work (Hafner-Burton and Montgomery 2008; Lektzian and Souva 2003; McLean and Whang 2010). In a similar vein, using the estimates from Models 2–4, all observations are high-stakes games, i.e., the point estimates of the payoffs satisfy the restrictions that $V_B > C_B$ and $V_A > S_A$. Using the estimates the MLE model, no observation is high-stakes.

For another example, consider audience costs, i.e., the initiating state's payoff from backing down, $\bar{a}$. Notice that the relevant constant term is negative, significant, and large in magnitude in all three models that accommodate multiple equilibria.[17] This suggests that states or leaders are indeed punished for backing down after issuing threats. In fact, in Models 2–4, we reject the null hypothesis that audience costs are greater than zero at the $p < 0.05$ level in every observation. In contrast, we cannot reject the null hypothesis that $\bar{a} \geq 0$ in the MLE model for any observation. Our analysis suggests that researchers may underestimate audience costs if estimation techniques do not accommodate the multiplicity of equilibria.

Finally, to better illustrate the tMLE's shortcomings, we also estimate the signaling model with an identical tMLE routine *using a different equation solver*. In Model 1 from Table 3, we use the nonlinear equation solver from our Monte Carlo experiments, which uses Broyden's method. In the second tMLE implementation, we compute all equilibria and select the one that maximizes $p_{dR}$. Appendix E details these results in Table 4.[18] Most surprisingly, the two tMLE results diverge in both sign (for 9/20 estimates) and significance (for 13/20 estimates). Thus, even when analyzing identical data sets, two researchers can reach substantively diverging conclusions if they implement the tMLE using different equation solvers. This suggests that estimates obtained through tMLE routines may be difficult to replicate or compare across studies even with publicly available data.

Overall, the implication is clear: games with multiple equilibria create substantial hurdles

[17] The coefficient relating $A$'s democracy levels to its audiences costs is essentially zero even in the three models that accommodate multiple equilibria. This matches recent work demonstrating that democracy is neither necessary nor sufficient for audience costs (Crisman-Cox and Gibilisco 2017; Weeks 2008, 2012).

[18] Across the two tMLE routines, the optimization method and starting values were fixed.

that tMLE approaches cannot overcome. Traditional methods for estimating signaling games produce severely misleading point estimates and substantive implications. Researchers wishing to estimate these types of models should choose methods that accommodate the possibility of multiple equilibria.

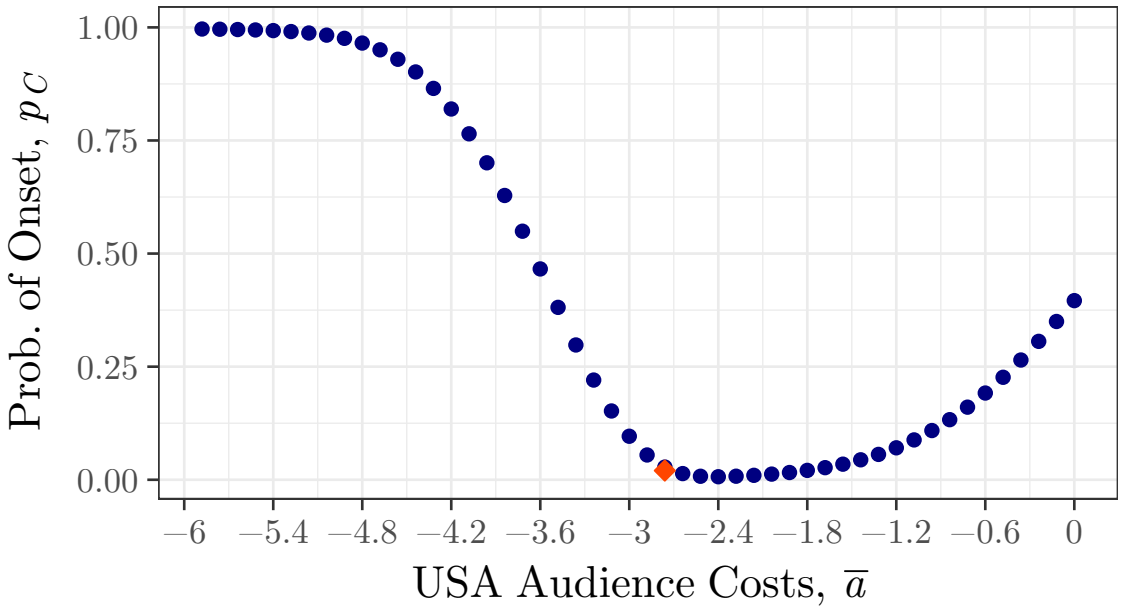
## 5.2 Audience Costs and Comparative Statics

How do audience costs affect the likelihood of leaders threatening sanctions? On the one hand, audience costs should discourage leaders from threatening rivals or initiating crises because leaders anticipate backing down and receiving those larger costs (Kurizaki and Whang 2015; Weeks 2012). On the other hand, audience costs make threats more credible, forcing rivals to concede, and as such, larger costs, and their associated bargaining advantages, should encourage leaders to initiate threats (Crisman-Cox and Gibilisco 2017; Schultz 1999). In the previous section, we analyzed point estimates of audience costs and found that they can be substantial. In this section, we analyze their substantive effects on the equilibrium probability of threatening sanctions, $p_C$. Doing so demonstrates that structural estimation and our proposed methods can help substantive scholars to uncover novel substantive findings in international relations.

For a concrete example, we consider the directed dyad in which the U.S. is the initiating state $A$ and China is the target state $B$ between 1991–2000, which is the most recent decade in the data. We vary the U.S.'s audience cost, $\bar{a}$, from $-6$ to 0 while fixing the remaining payoffs estimated using Model 4 in Table 3.[19] For every value of $\bar{a}$, we compute all equilibria using a line-search method. Then we plot the associated equilibrium probabilities of the U.S. initiating a conflict, $p_C$, in Figure 7. For all values of $\bar{a}$ considered, there is a unique equilibrium, pictured with the blue circles. The orange diamond denotes the value of U.S.

[19] We chose the CMLE in Model 4 for two reasons. First, unlike the PL method, the NPL and the CMLE, upon convergence, produce equilibrium choice probabilities that satisfy the conditions in Result 1. Second, comparing the CMLE and NPL results, the former have the larger log likelihood. Nonetheless, substantively similar results hold if we use the NPL results.

**Figure 7:** Effects of audience costs between U.S. and China dyad, 1991–2000




**Caption:** For each fixed $\bar{a}$, we compute all equilibria in the US-CHN-1990 directed dyad given the results in Table 3, Model 4. We then plot equilibrium probabilities of intiating conflict, $p_C$. The orange diamond denotes the equilibrium estimated using the CMLE; there is a unique equilibrium for all displayed values of $\bar{a}$.

audience costs estimated in the data, around $-2.7$, and the equilibrium estimate from the CMLE.

The figure illustrates three notable results. First, audience costs have a large substantive effect on the probability of threat initiation, covering the entire range between 0 and 1. Second, there is a $U$-shaped relationship between audience costs and threat initiation. Leaders only initiate threats when audience costs are very small or quite large. In the former case, leaders do not pay a cost for backing down, and in the latter case, their threats are quite credible. With intermediate audience costs, however, leaders almost never threaten rivals with sanctions, as their threats are not credible and backing down entails nontrivial costs. Third, if we were to increase the U.S.'s audience costs beginning from the value estimated in the data, then the model predicts an increase in sanction threats toward China. That is, the true value of audience costs tend to fall on the left-hand-side of the $U$-shaped curve, where larger (more negative) audience costs increase the likelihood of interstate threats.

How general is this latter pattern? That is, do larger audience costs encourage leaders to threaten sanctions more generally or is there something unique about the U.S. and China directed dyad? To answer these questions, we compute the marginal effect of making audience costs, $\bar{a}$, more negative on the equilibrium probability of issuing threats. Conclusively, larger (more negative) audience costs increase the likelihood of states threatening their rivals with sanctions. This result holds in 97% of observations, or 406 out of 418 dyads, and demonstrates that the second countervailing effect of audience costs dominates in the economic sanctions data. The median effect in the sample is 0.03, with a non-parametric 95% confidence interval ranging from 0.0004 to 0.17. Linear interpolation suggests that if we double the size of audience costs, i.e., move from $\bar{a} \approx -2.7$ to $\bar{a} \approx -5.4$, then the likelihood of threat initiation should increase by 8.3 percentage points.

# 6 Conclusion

In this paper, we analyze problems that emerge when using standard ML techniques to fit signaling games to data. We demonstrate that these techniques perform poorly not only if there are multiple equilibria in the signaling game generating the data but also if the equilibrium is unique. In the former case, without further information, the likelihood function may select the wrong equilibrium when evaluating different parameter guesses, leading to estimates that do not increase in accuracy with more observations. In the latter case, the likelihood function will often times be evaluated at parameter guesses under which multiple equilibria exist, leading to the similar problems unless starting values are sufficiently close to those in the data generating process. Our analysis should give researchers pause before using these techniques to estimate crisis-signaling games in international relations.

For solutions, we adapt several estimators from the dynamic games literature and show that they are particularly useful for the crisis-signaling game. In a series of experiments and applications, the NPL and PL always perform better than the currently used MLE routines

and tend to outperform both the CMLE estimator. Although the CMLE can dramatically reduce bias when compared to the PL and NPL methods, it tends to have estimates with larger variances. This small bias large variance trade-off diminishes if researchers are able to use specialized constrained optimization software. In general, we propose the following advice when estimating crisis-signaling games.

1. Estimate the game with the PL method, using a flexible first-stage estimator. We recommend random forests, as they tend to outperform splines, basis expansion, regression trees, and other flexible first-stage routines.

2. To verify whether bias in the first-stage estimates has affected the second stage, estimate the game with either the NPL or CMLE techniques. If these converge with smaller variances, then they should be prioritized. If these do not converge or have larger variances, then the PL results should be prioritized.

3. The tMLE routine should not be used; it generally performs worse than the other procedures.

We implement these procedures in our computational appendix. This accessibility should help researchers to uncover theoretically informed parameters rather than engaging in more reduced-form analyses.

Finally, the analysis here raises an important avenue for future research. Throughout, we have assumed that within each dyad or game, states play the same equilibrium for all within period observations $t \in \{1, \ldots, T\}$. However, it could be the case that the dyad switches equilibria over time, that is, $y_{dt}$ and $y_{dt'}$ were generated from two different equilibria for some $t \neq t'$. If this equilibrium selection rule probabilistically depends on equilibrium properties such as Pareto efficiency or the amount of updating, then scholars could attempt to model the equilibrium selection as in Bajari, Hong and Ryan (2010). If this equilibrium selection rule is arbitrary, then scholars could employ techniques that estimate identification bounds around the coefficients of interest as in Ciliberto and Tamer (2009). Either approach would relax

an assumption in our analysis. A major difficulty in this area is that current work considers games of complete information, as these techniques require repeatedly enumerating the entire set of equilibria. With incomplete information and signaling incentives, this task becomes substantially more complicated.

# Appendices

## A Definitions, Results, and Proofs

This Appendix contains the formal arguments for two additional results discussed in the main manuscript. First we define the regularity refinement from Harsanyi (1973) and van Damme (1996).

**Definition 1** *An equilibrium $\tilde{p}_R$ is* regular *if $\delta(\tilde{p}_R;\theta) \neq 1$.*

With this definition we can now state our result concerning the regularity of equilibria.

**Result 2** *For almost all $\theta$, all equilibria of the crisis-signaling game are regular.*

To prove the result and subsequent ones, it is more straightforward to work with the function $F : (0,1) \times \mathbb{R}^8 \to \mathbb{R}$ such that

$$F(p_R;\theta) = f \circ h(p_R;\theta) - p_R,$$

where $\tilde{p}_R$ is an equilibrium if and only if $F(\tilde{p}_R;\theta) = 0$. Likewise, we will also use $\delta(p_R;\theta)$ to be the first derivative of $f \circ h$ with respect to $p_R$ given parameters $\theta$. We state two intermediary results before proving result 2. The first is from Jo (2011*a*) and the second is the parameterized Transversality Theorem.

**Lemma 1** *For all $\theta$, $\lim_{p_R \to 0} f \circ h(pR;\theta) > 0$ and $\lim_{p_R \to 1} f \circ h(p_R;\theta) < 1$.*

Thus, there are no equilibria at the boundaries. In addition, for any fixed $\theta$, there exists $\varepsilon > 0$ and $\nu > 0$ such that $F(\varepsilon;\theta) > 0$ and $F(1-\nu;\theta) < 0$

**Theorem 1** (Transversality Theorem) *Consider an open set $X \subseteq \mathbb{R}^n$. Let $L : X \times \mathbb{R}^s \to \mathbb{R}^n$ be continuously differentiable. Assume that the Jacobian $D_{(x,y)}L$ has rank $n$ for all $(x,y) \in X \times \mathbb{R}^s$ such that $L(x,y) = 0$. Then, for all most all $y' \in \mathbb{R}^s$, the Jacobian $D_x L$ has rank $n$ for all $x \in X$ such that $L(x,y') = 0$.*

We now prove Result 2.

*Proof of Result 2.* Note that $\tilde{p}_R$ is a regular equilibrium if and only if $D_{p_R}F(p_R;\theta) \neq 0$. To prove Result 2, we verify the conditions of the Transversality condition, where in our application, $L = F$ and $(x,y) = (p_R;\theta)$, which means $n = 1$ and $s = 8$. First, note that $F$ is continuously differentiable, because $f \circ h$ is the composition of normal cumulative distribution functions and polynomial functions, and $F$ is defined over the open interval $(0,1)$.

Third and finally, we show that $D_{(p_R;\theta)}F(p_R;\theta)$ has at least one non-zero element (i.e., rank 1) when $F(p_R;\theta) = 0$. To do this, we show a stronger result: for all $(p_R;\theta)$, $D_{(p_R;\theta)}F(p_R;\theta) \neq 0$. To see this, consider $D_{\bar{W}_B}F(p_R;\theta)$. By Result 1, the functions $g$ and $h$ are constant in parameter $\bar{W}_B$, that is, $D_{\bar{W}_B}g(p_R;\theta) = D_{\bar{W}_B}h(p_R;\theta) = 0$. Then we have

$$\begin{aligned}
D_{\bar{W}_B}F(p_R;\theta) &= D_{\bar{W}_B}f \circ h(p_R;\theta) \\
&= D_{\bar{W}_B}\Phi\left(\frac{h(p_R;\theta)\bar{W}_B + (1-h(p_R;\theta))V_B - C_B}{h(p_R;\theta)}\right) \\
&= D_{\bar{W}_B}\Phi\left(\bar{W}_B + \frac{(1-h(p_R;\theta))V_B - C_B}{h(p_R;\theta)}\right) \\
&= \phi\left(\bar{W}_B + \frac{(1-h(p_R;\theta))V_B - C_B}{h(p_R;\theta)}\right) \\
&\neq 0,
\end{aligned}$$

which implies $D_{(p_R;\theta)}F(p_R;\theta) \neq 0$ as required. □

Although the regularity refinement does not generically reduce the number of equilibria, showing that all the equilibria are regular is advantageous for applied empirical research.

Regular equilibria can be implicitly expressed as continuous functions of parameters. This property is particularly important in empirical analyses: if we uncover noisy, but sufficiently accurate estimates of $\theta$, then equilibrium choice probabilities will be close to their true values as well. In addition, comparative statics (predicted probabilities) on regular equilibria will be well behaved, i.e., the equilibrium will not vanish if we vary the data or parameters by some small amount.

Our next result focuses on best response iteration. Before stating the result, we define best-response stable and best-response unstable equilibria.

**Definition 2** *An equilibrium $\tilde{p}_R$ is* best-response stable *if there exists $\varepsilon > 0$ such that for all $p_R^0 \in (\tilde{p}_R - \varepsilon, \tilde{p}_R + \varepsilon)$ the sequence*

$$p_R^k = f \circ h(p_R^{k-1}; \theta),\ k \in \mathbb{N}$$

*converges to $\tilde{p}_R$.*

The next definition introduces best-response unstable equilibria, which is not simply the negation of Definition 2.

**Definition 3** *An equilibrium $\tilde{p}_R$ is* best-response unstable *if there exists $\varepsilon > 0$ such that for all $p_R^0 \in (\tilde{p}_R - \varepsilon, \tilde{p}_R + \varepsilon)$, with $p_R^0 \neq \tilde{p}_R$, the sequence*

$$p_R^k = f \circ h(p_R^{k-1}; \theta),\ k \in \mathbb{N}$$

*leaves the interval $(\tilde{p}_R - \varepsilon, \tilde{p}_R + \varepsilon)$ at least once. That is, there exists $n \in \mathbb{N}$ such that $p_R^n \notin (\tilde{p}_R - \varepsilon, \tilde{p}_R + \varepsilon)$*

With these definitions, we are now ready to state Results 3.

**Result 3** *In high-stakes games where all equilibria are regular, the following hold:*

1. *There is a finite number of equilibria, and every equilibrium is either best-response stable or best-response unstable.*
2. *If there is a unique equilibrium, then it is best-response stable.*
3. *If there are multiple equilibria, then there exists a best-response unstable equilibrium and at least two best-response stable equilibria.*

To prove Result 3, we need two intermediate results.

**Lemma 2** *Assume* $(V_B - C_B)(V_A - S_A) > 0$. *Then* $f \circ h$ *is strictly increasing in* $p_R$, *i.e.,* $\delta(p_R; \theta) > 0$.

*Proof.* The derivative of $f$ with respect to $p_F$ is

$$D_{p_F} f(p_F) = (C_B - V_B) \frac{\phi\left(\bar{W}_B - \frac{(1-p_F)V_B - C_B}{p_F}\right)}{p_F}, \tag{11}$$

where $\phi$ is the probability density function of the standard normal distribution. Notice that if $p_F > 0$, the fraction in Equation 11 divides one positive number by another. Thus, $C_B > V_B$ implies $D_{p_F} f(p_F) > 0$, i.e., $f$ is strictly increasing in $p_F$. Likewise, $C_B < V_B$ implies $D_{p_F} f(p_F) < 0$, i.e., $f$ is strictly decreasing in $p_F$. Similar computations show that $V_A > S_A$ implies $h$ is strictly increasing in $p_R$, and $V_A < S_A$ implies $h$ is strictly decreasing in $p_R$.

There are two cases to consider. First $(V_B - C_B) > 0$ and $(V_A - S_A) > 0$. If $(V_A - S_A) > 0$, then $h$ is strictly increasing. Because the range of $h$ is positive, $(V_B - C_B) > 0$ and the above analysis implies $f \circ h$ is strictly increasing because it is the composition of two strictly increasing functions. A similar argument holds for the second case where $(V_B - C_B) < 0$ and $(V_A - S_A) < 0$ because the composition of two strictly decreasing functions is strictly increasing. □

Notice that a high-stakes game satisfies the assumption in Lemma 2 because high-stakes games have $V_B > C_B$ and $V_A > S_A$. This assumption relaxes Condition **??**, and Result 3 still holds under the more general assumption that $(V_B - C_B)(V_A - S_A) > 0$. The next theorem states a standard result in nonlinear dynamics and fixed point iteration. See Theorem 6.5 in Holmgren (1994).

**Theorem 2** *Consider an equilibrium* $\tilde{p}_R$. *If* $|\delta(\tilde{p}_R \theta)| < 1$, *then* $\tilde{p}_R$ *is best-response stable. If* $|\delta(\tilde{p}_R; \theta)| > 1$, *then* $\tilde{p}_R$ *is best-response unstable.*

To end this Appendix, we prove Result 3.

*Proof of Result 3(1).* We first prove that all regular equilibria are either best-response stable or best-response unstable in high-stakes games. By regularity, $\delta(\tilde{p}_R; \theta) \neq 1$ for some equilib-

rium $\tilde{p}_R$ and fixed parameters $\theta$. By Lemma 2, $\delta(\tilde{p}_R;\theta) > 0$, which means $\delta(\tilde{p}_R;\theta) \neq -1$. Then Theorem 2 implies that equilibrium $\tilde{p}$ is either best-response stable or best-response unstable.

Second, we prove that there is a finite number of equilibria. By assumption all equilibria are regular, which implies $D_{p_R}F(\tilde{p}_R;\theta) \neq 0$ for all $\tilde{p}_R$ such that $F(\tilde{p}_R;\theta) = 0$. Then the Implicit Function Theorem implies that every equilibrium $\tilde{p}_R$ is locally isolated. Because $F$ is continuous, it has closed level sets, so the set of equilibria is closed. Because equilibria fall within the interval $(0,1)$, the set of equilibria is bounded, and therefore compact. As a compact set of locally isolated points, the equilibrium set is finite. □

*Proof of Result 3(2).* Let $\tilde{p}_R^{[1]}$ be the smallest equilibrium, i.e., $\tilde{p}_R^{[1]}$ solves

$$\min\left\{\tilde{p}_R \in [0,1] \mid F(\tilde{p}_R;\theta) = 0\right\}.$$

Such a solution exists because the equilibrium set is non-empty (Result 1) and finite (Result 3(1)). We claim that $\tilde{p}_R^{[1]}$ is best-response stable, i.e., $\left|\delta(\tilde{p}_R^{[1]};\theta)\right| < 1$. To see this suppose not. Then $\left|\delta(\tilde{p}_R^{[1]};\theta)\right| \geq 1$. By Lemma 2, $\delta(\tilde{p}_R^{[1]};\theta) > 0$ because the game is high-stakes, implying $\delta(\tilde{p}_R^{[1]};\theta) \geq 1$. Regularity then implies $\delta(\tilde{p}_R^{[1]};\theta) > 1$.

Because $F$ is continuously differentiable and $D_{p_R}F = \delta(\tilde{p}_R^{[1]};\theta) - 1$, there exists $\varepsilon > 0$ such that $F$ is strictly increasing on the interval $(\tilde{p}_R^{[1]} - \varepsilon, \tilde{p}_R^{[1]})$. Because $F(\tilde{p}_R^{[1]};\theta) = 0$, this implies that there exists a $p'_R \in (\tilde{p}_R - \varepsilon, \tilde{p}_R^{[1]})$ such that $F(p'_R;\theta) < 0$. By Lemma 1, there exists $\nu \in (0, p'_R)$ such that $F(\nu;\theta) > 0$. Then the Intermediate Value Theorem Implies that there exists a $\tilde{p}_R \in (\nu, p'_R)$ such that $F(\tilde{p}_R;\theta) = 0$, but this contradicts the assumption that $\tilde{p}_R^{[1]}$ is the smallest equilibrium. Hence, we conclude that if $\tilde{p}_R^{[1]}$ is the smallest equilibrium, then $\left|\delta(p_R^{[1]};\theta)\right| < 1$, which implies it is stable by Theorem 2. If there is a unique equilibrium, then it must be the smallest equilibrium, and it is stable. □

*Proof of Result 3(3, unstable).* We prove that, if there are multiple equilibria, then there exists a best-response unstable equilibrium. Let $\tilde{p}_R^{[2]}$ be the second smallest equilibrium, i.e., $\tilde{p}_R^{[2]}$

solves

$$\min\left\{\tilde{p}_R \in [0,1] \mid F(\tilde{p}_R;\theta) = 0\right\} \setminus \left\{\tilde{p}_R^{[1]}\right\},$$

where $\tilde{p}_R^{[1]}$ is the smallest equilibrium defined above. Such a solution exists because there are a finite number of equilibria and, by assumption, more than one. We claim that $\left|\delta(\tilde{p}_R^{[2]};\theta)\right| > 1$. To see this suppose not. Then $\left|\delta(\tilde{p}_R^{[2]};\theta)\right| \leq 1$. Because all equilibria are regular, $\delta(\tilde{p}_R^{[2]};\theta) < 1$, implying $D_{p_R}F(\tilde{p}_R^{[2]};\theta) < 0$. This, along with the facts that $F$ is continuously differentiable and $F(\tilde{p}_R^{[2]};\theta) = 0$, implies there exists (arbitrarily small) $\varepsilon > 0$ such that $F(\tilde{p}_R^{[2]} - \varepsilon;\theta) > 0$.

In the proof of Result 3(2), we showed that $\left|\delta(\tilde{p}_R^{[1]};\theta)\right| < 1$ and $F(\tilde{p}_R^{[1]};\theta) = 0$. Then exists (arbitrarily small) $\nu > 0$ such that $F(\tilde{p}_R^{[1]} + \nu;\theta) < 0$ because $F$ is continuously differentiable. So we have $F(\tilde{p}_R^{[2]} - \varepsilon;\theta) > 0$ and $F(\tilde{p}_R^{[1]} + \nu;\theta) < 0$. Then by the Intermediate Value Theorem there exists an equilibrium $\tilde{p}'_R$ such that

$$\tilde{p}_R^{[1]} + \nu < \tilde{p}'_R < \tilde{p}_R^{[2]} - \varepsilon.$$

But this contradicts the assumption that $\tilde{p}_R^{[2]}$ is the second smallest equilibrium. Thus, we conclude $\left|\delta(\tilde{p}_R^{[2]};\theta)\right| > 1$. As such, $\tilde{p}_R^{[2]}$ is unstable. □

*Proof of Result 3(3, stable).* We now prove that, if there are multiple equilibria, then there exists at least two stable equilibria. First, we claim that if two equilibria exist, then a third exists as well. To see this, note that we showed that $\delta(\tilde{p}_R^{[2]};\theta) > 1$ in the proof above. Then $D_{p_R}F(\tilde{p}_R^{[2]};\theta) > 0$, so there exists a $p'_R > \tilde{p}_R^{[2]}$ such that $F(p'_R;\theta) > 0$. By Lemma 1, there exists a $\nu$ close to 1 such that $F(\nu;\theta) < 0$. Then the Intermediate Value Theorem implies that there exists a third equilibrium, between $p'_R$ and $\nu$.

Let $\tilde{p}_R^{[3]}$ satisfy the following:

$$\min\left\{\tilde{p}_R \in [0,1] \mid F(\tilde{p}_R;\theta) = 0\right\} \setminus \left\{\tilde{p}_R^{[1]}, \tilde{p}_R^{[2]}\right\},$$

where $\tilde{p}_R^{[1]}$ and $\tilde{p}_R^{[2]}$ are defined above. A solution exists because at least three equilibria exist.

We claim that $\tilde{p}_R^{[3]}$ is best-response stable. If not, then $\delta(\tilde{p}_R^{[3]};\theta) > 1$ identical arguments from the proof of Result 3(1). Then there exists (arbitrarily small) $\varepsilon > 0$ such that $F(\tilde{p}_R^{[3]} - \varepsilon;\theta) < 0$. However, $\delta(\tilde{p}_R^{[2]};\theta) > 1$, so there exists (arbitrarily small) $\mu > 0$ such that $F(\tilde{p}_R^{[2]} + \mu;\theta) > 0$. Then the Intermediate Value Theorem implies there exists an equilibrium strictly between $\tilde{p}_R^{[2]}$ and $\tilde{p}_R^{[3]}$, a contradiction. Thus, $\tilde{p}_R^{[3]}$ is best-response stable. The proof of Result 3(2) shows that $\tilde{p}_R^{[1]}$ is best-response stable as well. Hence, there are two stable equilibria. □

# B Further Monte Carlo Results

## B.1 Multiple Equilibria

This appendix contains additional results from the Monte Carlo experiment where the data are generated under parameters that are consistent with multiple equilibria. A single covariate determines the equilibirum selection. The parameter values used to generate the data can be found in Table 1. Here we consider the estimators' bias, variance, and rate of convergence. Root mean squared error and computation time are both presented in the main text.

**Figure 8:** Bias in signaling estimators with multiple equilibria.

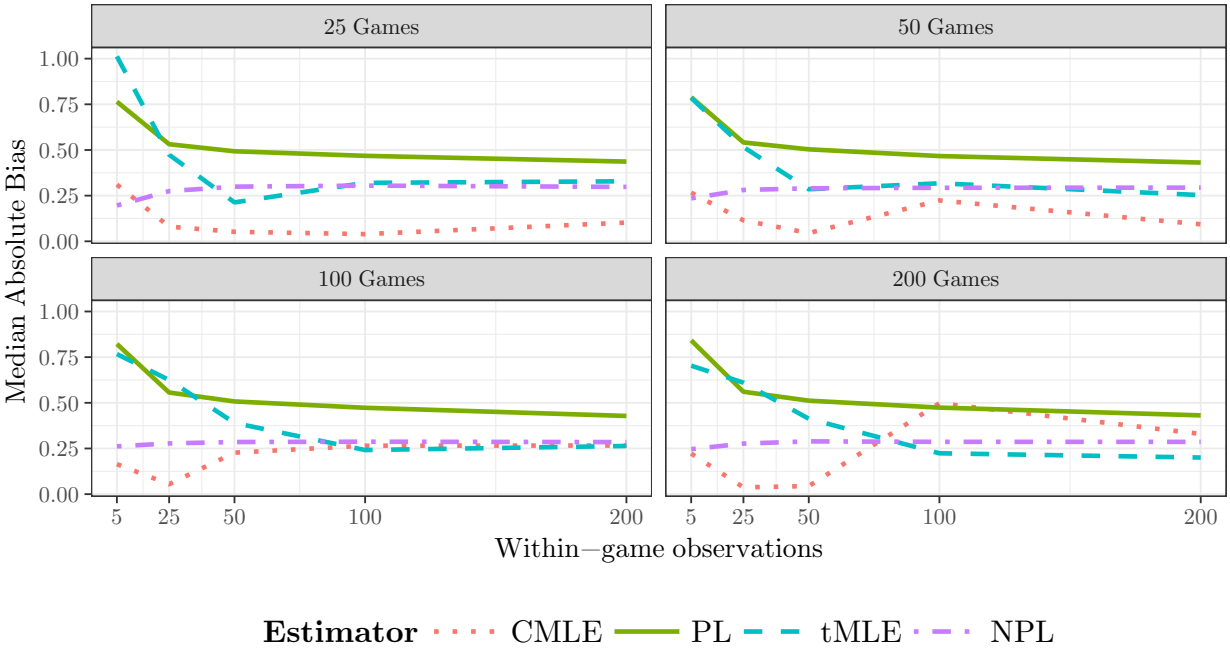



**Figure 9:** Variance in signaling estimators with multiple equilibria.

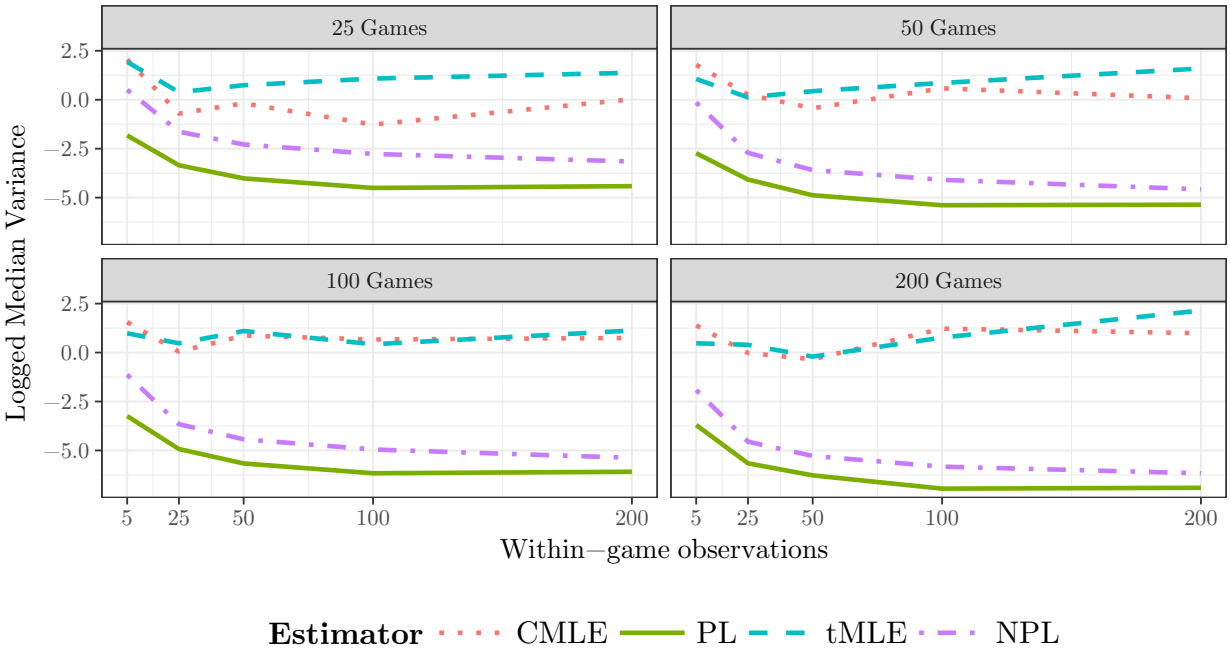


**Figure 10:** Convergence rates in signaling estimators with multiple equilibria.

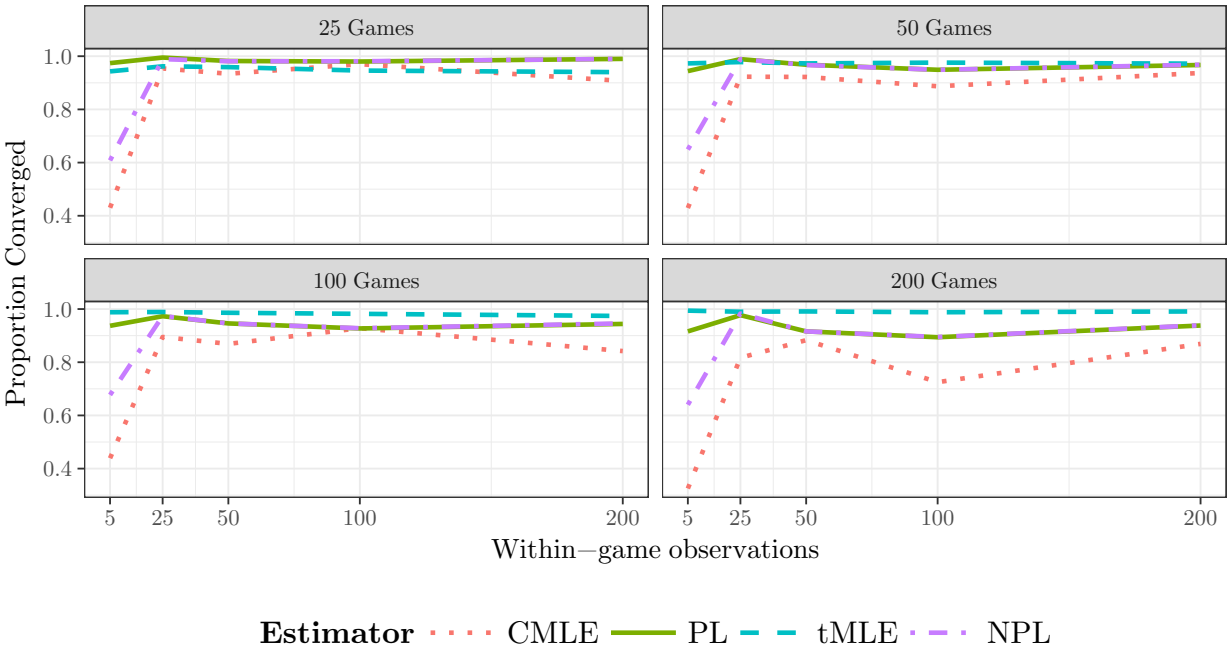
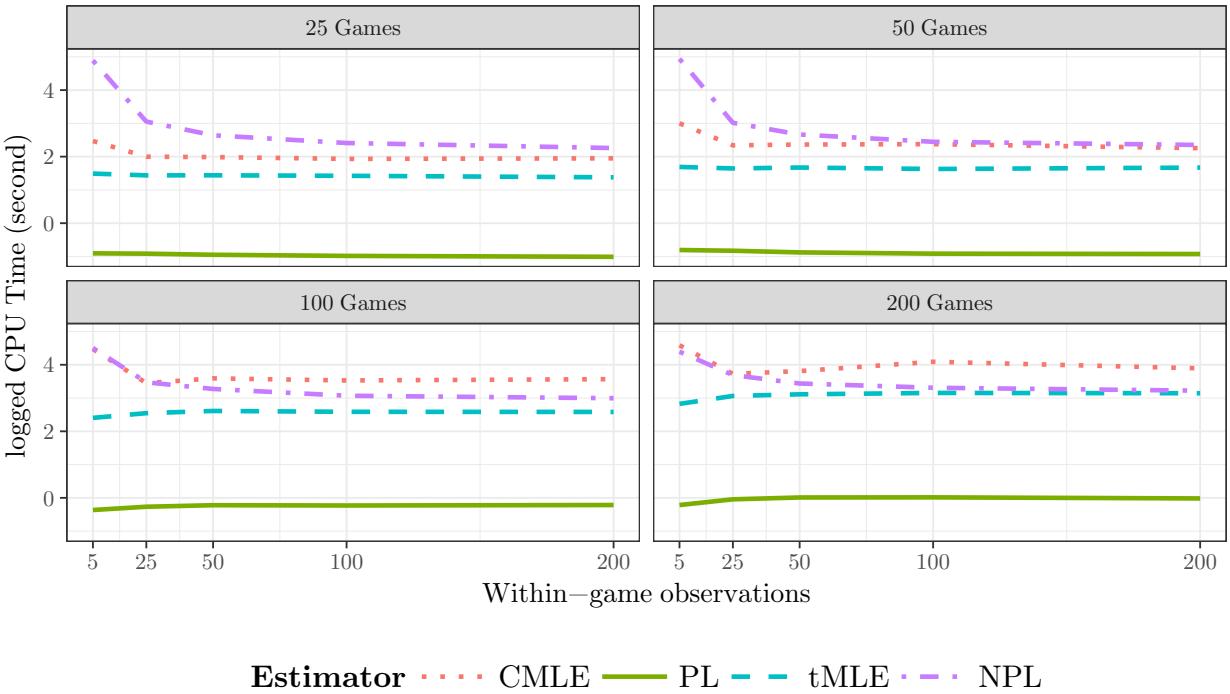



**Figure 11:** Computational time in signaling estimators with multiple equilibria.



## B.2 Unique Equilibrium

This appendix contains additional results from the Monte Carlo experiment where the data are generated from a version of the game with a unique equilibrium. The parameter values used to generate the data can be found in the final column of Table 1. Here we consider the estimators' bias, variance, computation time, and rate of convergence.

**Figure 12:** Bias in signaling estimators with a unique equilibrium.

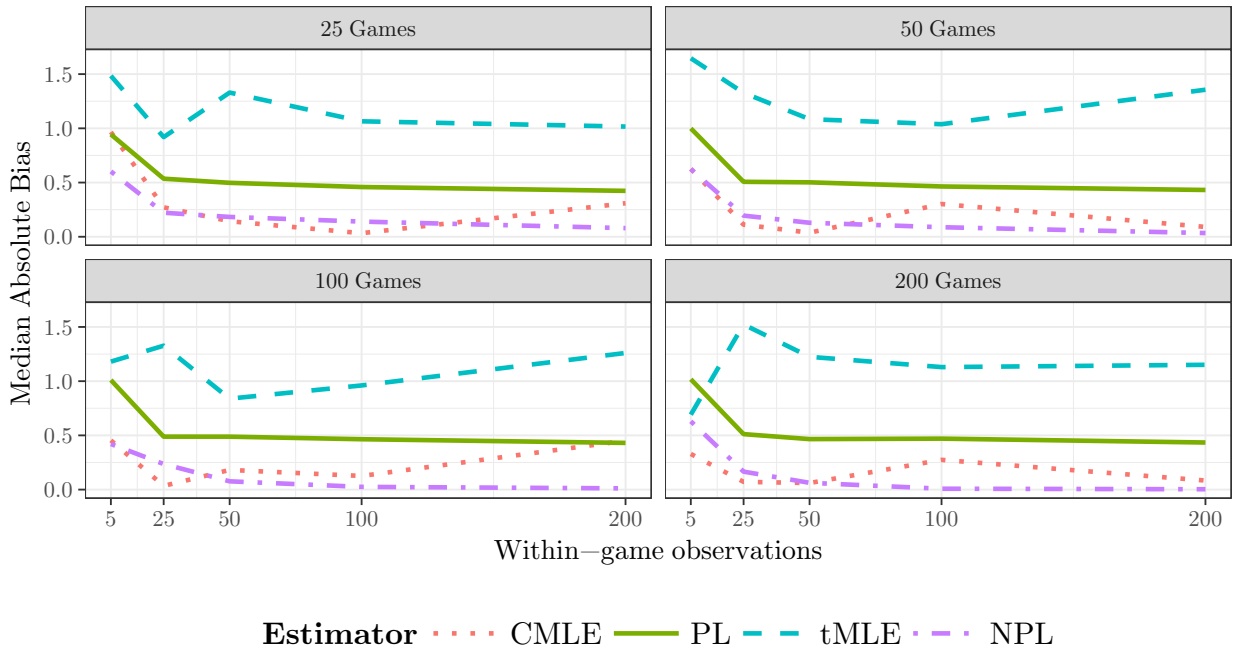


**Figure 13:** Variance in signaling estimators with a unique equilibrium.

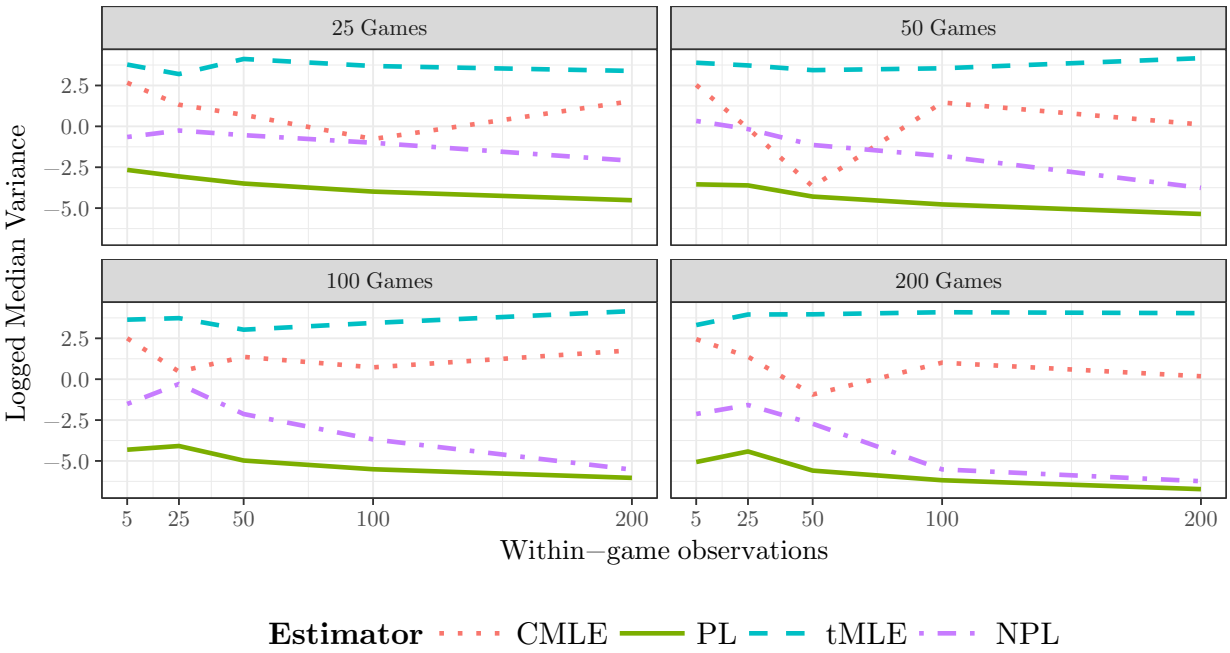



**Figure 14:** Computational time in signaling estimators with a unique equilibrium.

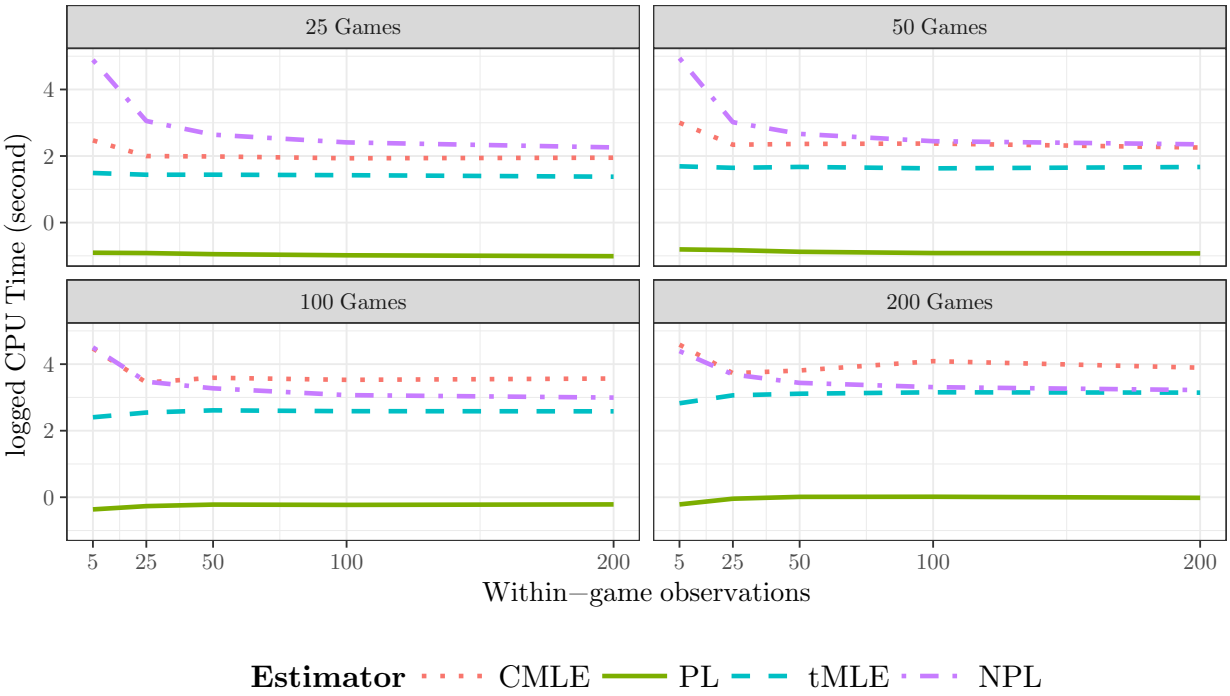
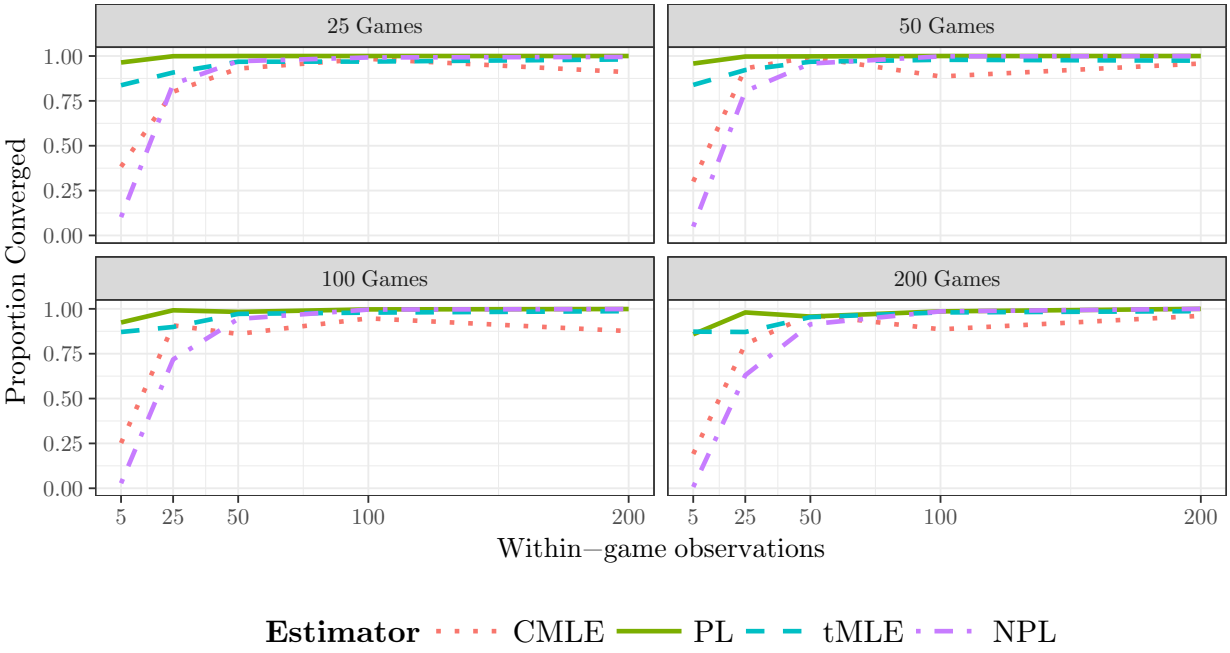


**Figure 15:** Convergence rates in signaling estimators with a unqiue equilibrium.



## B.3 Best-Response Stability

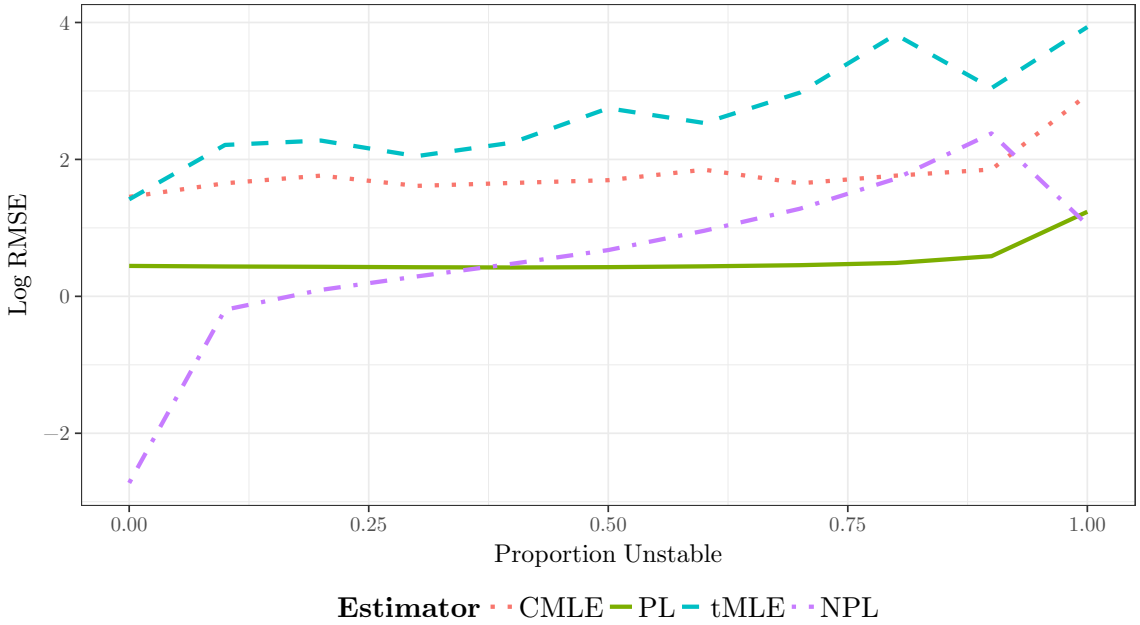
The best performing solutions make use of best response functions, which begs the question: How sensitive are the estimators to best-response unstable equilibria? To answer this question, we conduct another Monte Carlo experiment. Here, we assume payoffs are generated as in the multiple setting in Table 1, and the equilibrium selection rule follows the left-hand graph in Figure 2. Let $q \in [0, 1]$ denote the percentage of unstable equilibria. For $q \cdot D$ dyads, $x_d$ is draw from a uniform distribution over the interval $(\frac{1}{3}, \frac{2}{3})$. For the remaining $D - q \cdot D$ observations, $x_d$ is drawn uniformly from the intervals $(0, \frac{1}{3})$ or $(\frac{2}{3}, 1)$ with equal probability. From the formal arguments in the Appendix, the middle equilibrium, i.e., the one selected when $x_d \in (\frac{1}{3}, \frac{2}{3})$, is unstable.[20] As we vary $q$ from 0 to 1, we analyze how the estimators' performance varies as the data are generated with a larger proportion of best-response unstable equilibria. In this experiment, we set $D = 500$ and $T = 500$, which means there is a large amount of data as to better isolate the affects of unstable equilibria. For all values of $q$, we draw $x_d$, select the corresponding equilibria, and estimate the model $1,000$ times. We expect the PL and NPL to perform worse as $q$ approaches 1.

Figure 16 summarizes the results, where we vary the percentage of unstable equilibria along the horizontal axis and plot log RMSE along the vertical axis. Unsurprisingly, the PL and the NPL have larger RMSE as the data are generated with larger proportions of unstable equilibria, and this trend appears in the remaining estimators as well. For data with less than 40% unstable equilibria, the NPL outperforms the PL, and *vice versa* for data with more than 40% unstable equilibria. More surprisingly, we find that the PL and NPL perform better than the remaining estimators *even when more than 75% of observations are drawn from unstable equilibria*. Thus, best-response instability is a problem for all estimators that we consider as more unstable equilibria generally increase RMSE. The two estimators that explicitly rely on best-response iteration, however, have the best performance.

What explains this unintuitive finding? We note two possibilities. First, with $T = 500$,

[20] See the Proof of Result 3(3). Unstable equilibria can also be identified numerically using Theorem 2.

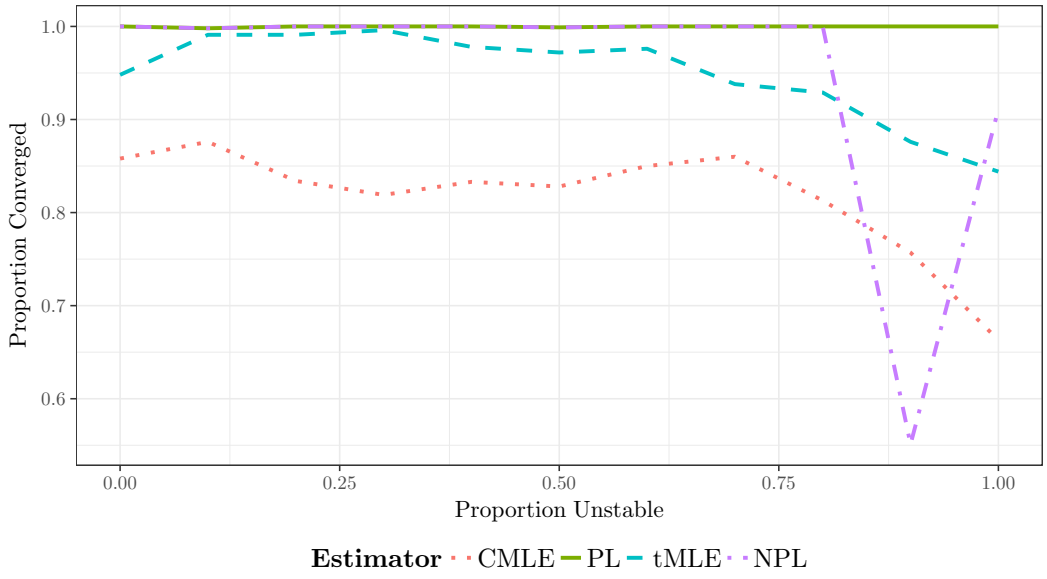
**Figure 16:** RMSE in signaling estimators with more unstable equilibria.




there is a large amount of within game observations, leading to precise first-stage estimates. Thus, even if an estimation routine relies on best-response iteration, the first-stage estimates may be precise enough where only minimal problems arise. Second, Figure 16 only reports results that have converged, and we find that when unstable equilibria dominate in the data, convergence of all estimators decreases. This trend is illustrated in Figure 17, where horizontal axis is the proportion of observations with unstable equilibria and the vertical axis is the proportion of bootstraps converged. Notice, convergence rates of all estimators, besides the PL, decrease once the proportion of unstable equilibria approaches 60–80%.[21] Thus, conditional on converging, the estimators return results with fairly reasonable RMSE even with a large proportion of unstable equilibria. They are all generally less likely to converge when unstable equilibria permeate the data, however.

[21] We find that the convergence rate of the CMLE method is decreasing in the number of dyads, $D$. This is not surprising as the method requires estimating $D$ equilibrium constraints in a highly nonlinear problem. When $D \leq 200$, the CMLE's convergence rate is between 80–100%. See Figures 10 and 15 in Appendices B.1 and B.2, respectively, for more information.

**Figure 17:** Convergence rates in signaling estimators with unstable equilibria.



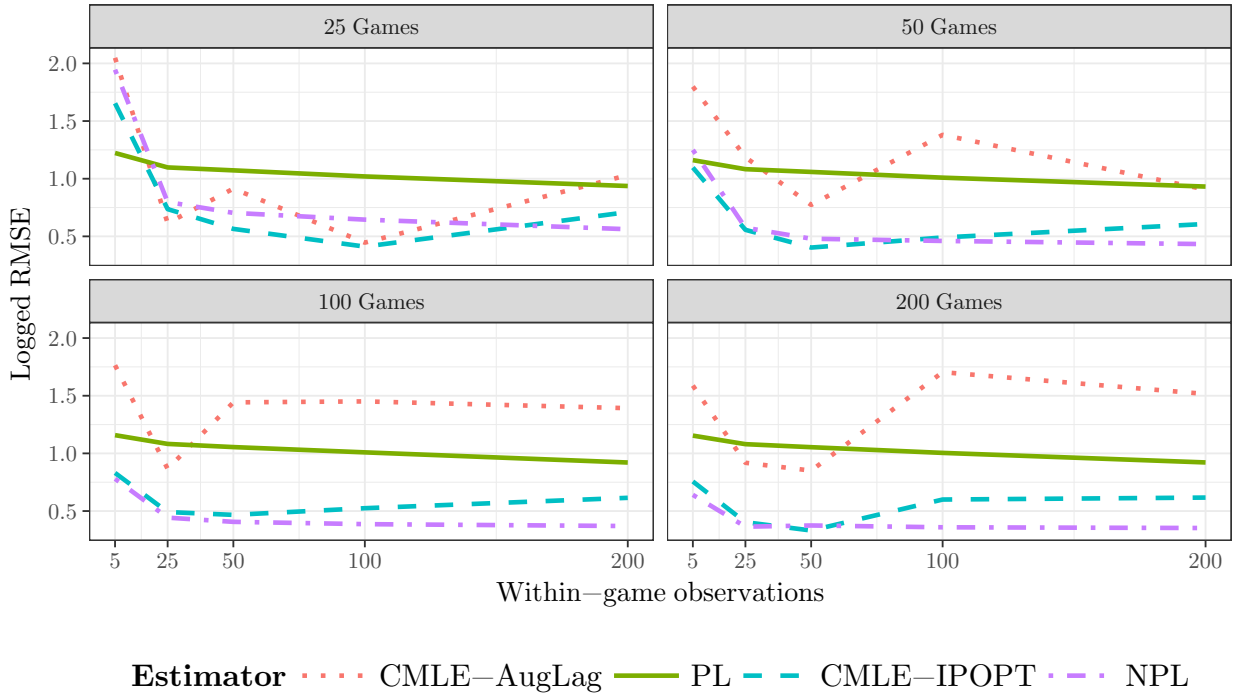
# C CMLE Implementation

In our economic sanctions application we fit the CMLE using the program IPOPT (Interior Point OPTimizer), which is an open-source optimizer designed to handle large scale problems (Wächter and Biegler 2006). In trials, IPOPT had better performance properties than other optimizers such as sequential quadratic solvers (found in Python's `scipy.optimize` module and MATLAB) and a version of the Augmented Lagrangian Method (from R's `alabama` package, used for Monte Carlo analysis).

The main difficulty in using interior-point methods is that they require an accurate second derivative of the Lagrangian associated with the problem in Equation 10. We find that finite difference approximations are insufficient. As such, we use the program ADOL-C, software for algorithmic differentiation (AD) (Griewank, Juedes and Utke 1996), to precisely compute the Hessian. The AD software allows us to only supply the log-likelihood and constraint function from Equation 10. The AD program repeatedly applies the chain rule to our functions to compute first- and second-order derivatives. In our economic sanctions example, we use IPOPT and ADOL-C within Python 2.7.13 on UbuntuGNOME 17.04 by calling the `pyipopt` module developed by Xu (2014) and the `pyadolc` module developed by Walter (2014), respectively. Finally, we estimate standard errors using Silvey (1959, Lemma 6, p. 401).

To demonstrate the trade offs of these two approaches we conduct an additional Monte Carlo comparing IPOPT-AD (Python) to the Augmented Lagrangian (R) in terms of statistical performance (RMSE), computation time, and convergence rate. Also included are the pseudo-likelihood methods, which serve as a benchmark. The data generating process uses the multiple equilibrium setup (first column of Table 1). In Figure 18, we see that while the Augmented Lagrangian (R) of the CMLE is sometimes better or worse than the pseudo-likelihood, the interior point method (Python) is almost always superior. This performance enhancement comes from the inclusion of high quality first and second order derivatives through, which greatly decreases the variance of the estimates. These improvements are not free, however, as we observe in Figure 19 that the interior point method suffers from a frequent failure to

**Figure 18:** Comparing CMLE implementations: RMSE




converge. On the other hand, we note in Figure 20 that the two approaches require roughly similar amounts of time, with the interior point method sometimes taking an average of 2-3 additional minutes longer. This trade-off between statistical and computational performance should be considered on a case-by-case basis when fitting these models.

**Figure 19:** Comparing CMLE implementations: Convergence

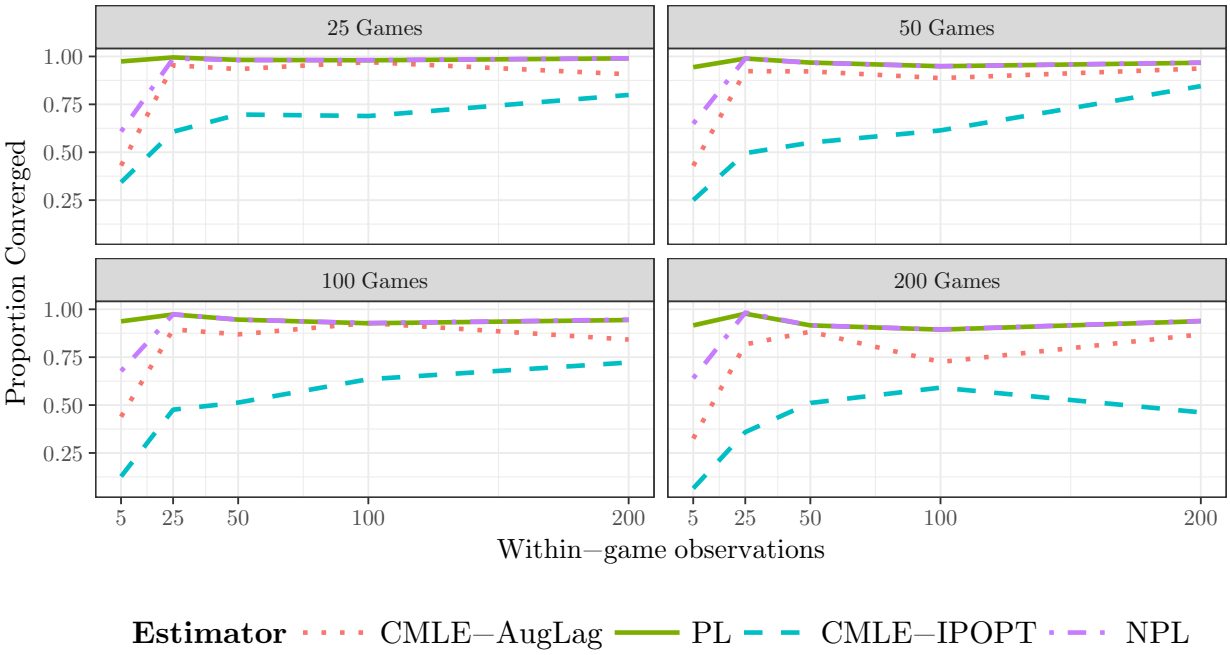
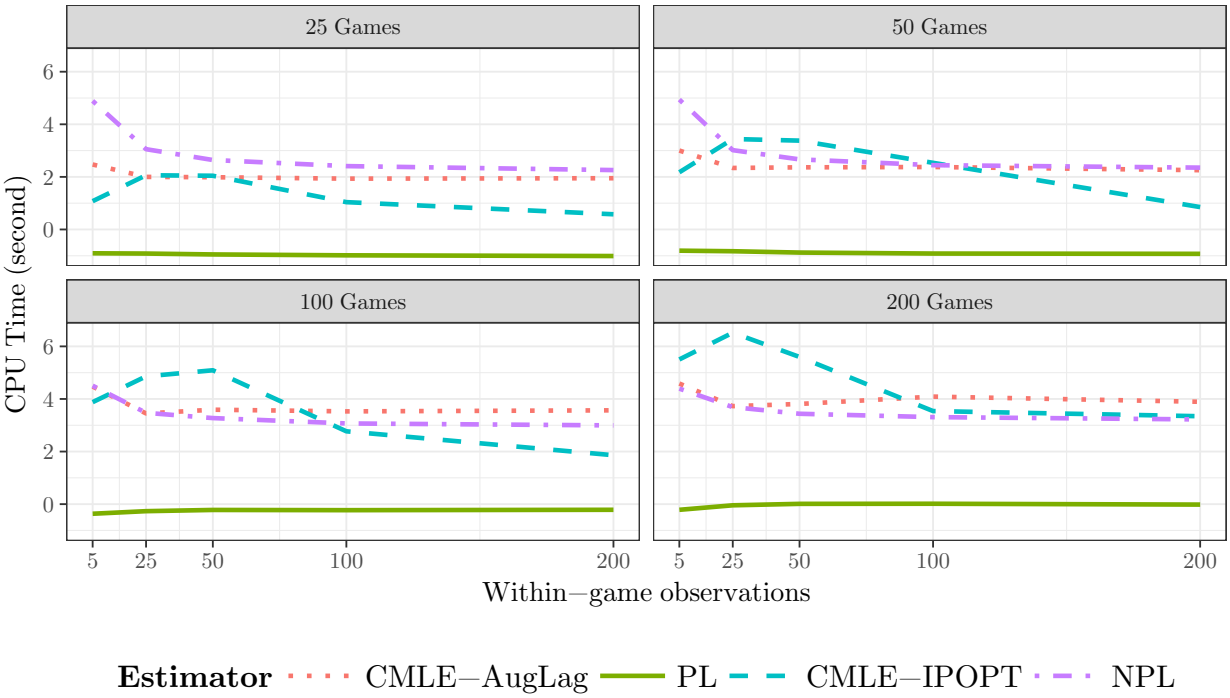



**Figure 20:** Comparing CMLE Computation time



# D Traditional ML and Starting Values

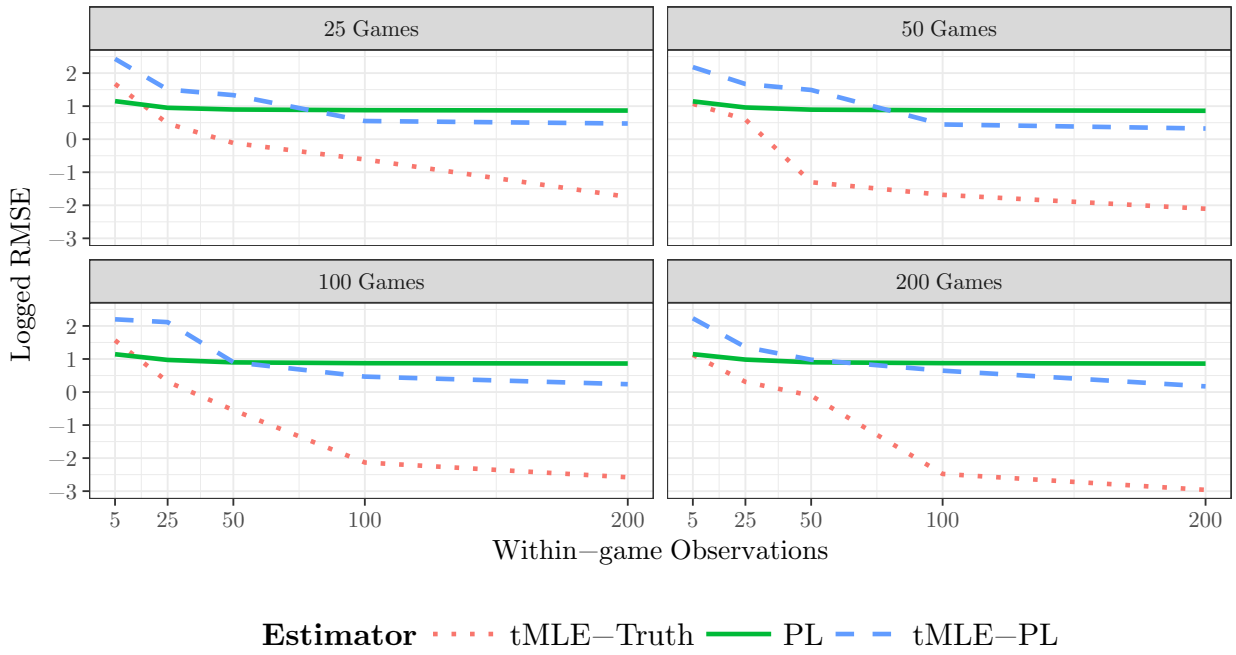
Our Monte Carlo experiments demonstrate that the tMLE may not be consistent even when there is a unique equilibrium in the signaling game from the data generating process. The reason such problems arise is that the maximization routine will oftentimes evaluate the likelihood function at a guess of the parameters where multiple equilibria arise. In this case, the traditional approach will select an equilibrium in an ad-hoc fashion, which may encourage the maximization routine to move away from the correct parameters. This may be surprising as both Jo (2011*a*) and Bas, Signorino and Whang (2014) conduct similar Monte Carlo experiments and conclude that the tMLE performs well when the data were generated with parameters that admit a unique equilibrium.[22]

To the best of our knowledge, the differences arise from starting values. In our study, starting values for $\theta$ were drawn from a standard uniform distribution, and we the starting values for all four estimators in each simulation. In Jo (2011*a*), they were the values from the data generating process (Jo 2011*b*). Although we were not able to locate replication materials from Bas, Signorino and Whang (2014), we do conduct an additional Monte Carlo experiment to investigate the possibility that differences in starting values lead to different results. To do this, we replicate our unique equilibrium Monte Carlo experiment. We estimate the model using the pseudo-likelihood routine, a tMLE routine with starting values from the data generating process, i.e., the parameters listed in the unique column of Table 1, and a tMLE routine with starting values from the pseudo-likelihood output.

Figure 21 graphs the median (over the parameters) logged RMSE of the three estimation procedures as we vary the number of dyads $D$ and the number of observations $T$. In a similar manner, Figures 22 and 23 report the estimators' bias and variance, respectively. There are two major takeaways. First, if the tMLE has privileged starting values, either from the

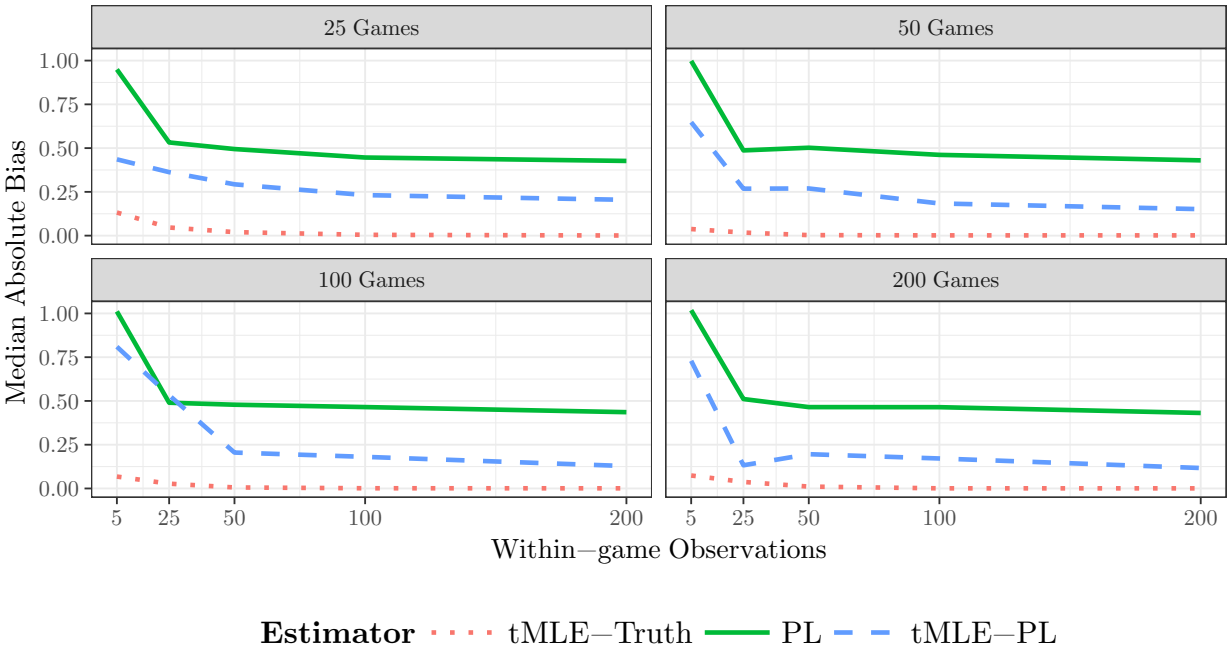
[22] Jo (2011*a*, p. 357) writes "It is easy to see that when there is a unique equilibrium, the estimates get closer to their true values as the number of observations increases." Bas, Signorino and Whang (2014, p. 26) write "All coefficients on average are estimated very close to the true parameter values, and the accuracy of the estimates increases as the sample size increases."

**Figure 21:** RMSE with a unique equilibrium and different starting values.




**Caption:** Graphs of the estimators' RMSE as a function of the number of games $D$ and the number of within game observations $T$. The figure graphs the median log-RMSE over the parameters in the unique column from Table 1.

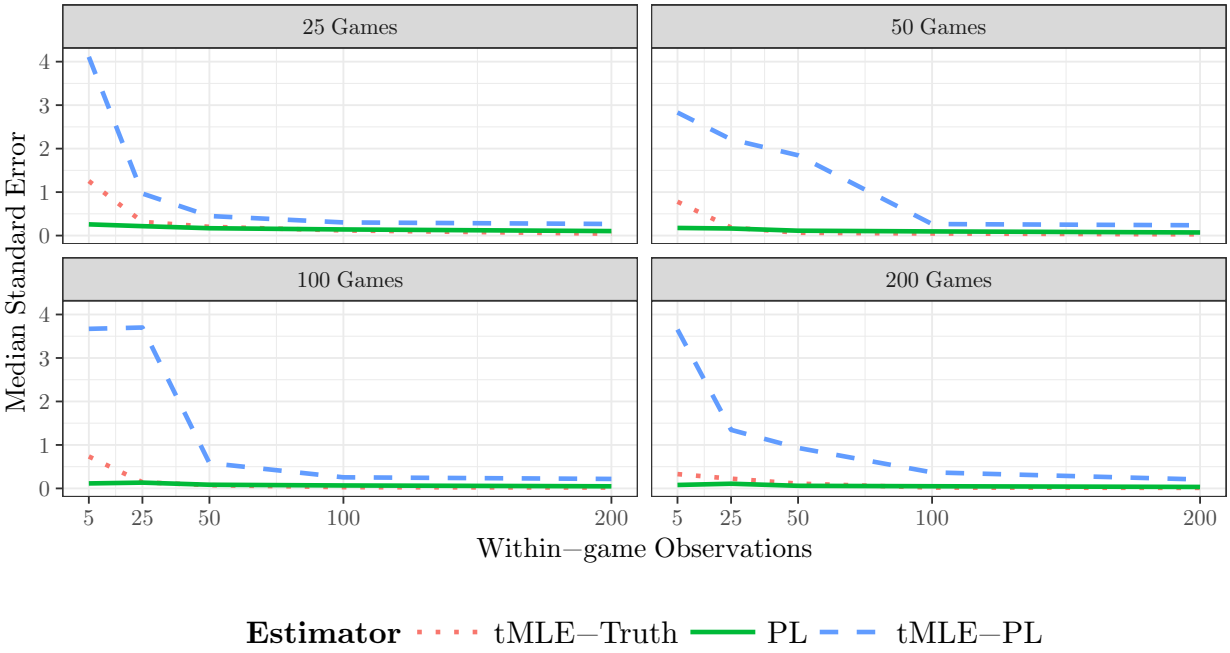
**Figure 22:** Bias with a unique equilibrium and different starting values.




**Caption:** Graphs of the estimators' bias as a function of the number of games $D$ and the number of within game observations $T$. The figure graphs the median bias over the parameters in the unique column from Table 1.

truthful values or the PL estimates, then it generally performs as well as (if not better than) the pseudo-likelihood estimator. This stands in stark contrast to the results in Figure 5, where the tMLE performs substantially worse than the PL routine. Second, if $T = 5$, then the pseudo-likelihood estimator appears to perform better than either of the tMLE routines even though one of these routines was provided the correct estimates as starting values. This improvement is mostly driven by the relatively smaller variance in the PL method. Given these results, we conclude different choices in starting values generate the contradictory findings in previous work.

**Figure 23:** Variance with a unique equilibrium and different starting values.




**Caption:** Graphs of the estimators' variance as a function of the number of games $D$ and the number of within game observations $T$. The figure graphs the median variance over the parameters in the unique column from Table 1.

# E Traditional ML and Equation Solvers

**Table 4:** Effects of Equation Solver on tMLE

| | tML<br>*Broyden Solver*<br>Model 1 | tML<br>*Select Largest Eq.*<br>Model 2 |
|---|---|---|
| $S_A$: Econ. Dep$_A$ | 0.05 | −0.38 |
| | (0.29) | (0.27) |
| $S_A$: Dem$_A$ | 0.00 | 0.03* |
| | (0.00) | (0.01) |
| $S_A$: Contiguity | 0.27* | 0.28* |
| | (0.10) | (0.02) |
| $S_A$: Alliance | −0.06 | 0.38* |
| | (0.08) | (0.05) |
| $V_A$: Const. | −0.06 | −0.14 |
| | (0.08) | (0.09) |
| $V_A$: Costs$_A$ | −0.04 | −1.71* |
| | (0.03) | (0.26) |
| $C_B$: Const. | 0.81 | 1.08* |
| | (0.91) | (0.31) |
| $C_B$: Econ. Dep$_B$ | −0.21 | 0.23* |
| | (0.16) | (0.10) |
| $C_B$: Costs$_B$ | −0.08* | −0.12* |
| | (0.03) | (0.01) |
| $C_B$: Contiguity | −0.25* | −0.15* |
| | (0.02) | (0.01) |
| $C_B$: Alliance | 0.10 | −0.37* |
| | (0.09) | (0.04) |
| $\bar{W}_A$: Const. | −0.15 | −0.41* |
| | (0.78) | (0.14) |
| $\bar{W}_A$: Econ. Dep$_A$ | 0.07 | 0.3 |
| | (0.75) | (0.32) |
| $\bar{W}_A$: Dem$_A$ | 0.01 | 0.06* |
| | (0.01) | (0.01) |
| $\bar{W}_A$: Cap. Ratio | −0.01 | 0.07* |
| | (0.01) | (0.01) |
| $\bar{W}_B$: Const. | −0.38 | 1.19* |
| | (1.13) | (0.40) |
| $\bar{W}_B$: Dem$_B$ | 0.01* | 0.00* |
| | (0.00) | (0.00) |
| $\bar{W}_B$: Cap. Ratio | 0.01 | −0.05* |
| | (0.01) | (0.02) |
| $\bar{a}$: Const. | −0.56 | −0.76* |
| | (0.77) | (0.14) |
| $\bar{a}$: Dem$_A$ | 0.00 | 0.06* |
| | (0.01) | (0.01) |
| Log $L$ | -4102.76 | -4302.08 |
| $N$ | 418 × 120 | 418 × 120 |

*Notes:* $^*p < 0.05$
Standard Errors in Parenthesis

# F R Code

Below we list the basic code required to use any of the estimation routines found in this paper. The complete code used to replicate this entire paper can be found in the replication archive.

```
##
## This file contains code for the EQ constraint in Jo (2011).
## It also includes functions for generating data and functions
## necessary to implement the four estimators.
## Additional packages: pbivnorm, rootSolve, maxLik
##

################### HELPER FUNCTIONS ###################
vec2U.regr <- function(x,regr){
## Function for converting parameters and regressors to
## utilities over outcomes
## INPUTS:
## x: vector of regression parameters (betas) in the order SA, VA, CB, barWA, barWB, bara, VB
## regr: a list of regressor matrices, one for each utility in the same order as x
## OUTPUTS:
## param: A list of utilities in the same order as regr.
## Each element of this list is a vector of length equal
## to the number of games.


## create indices to appropriately sort the elements of x
## into the correct outcomes.
idx0 <- lapply(regr, ncol)
idx0 <- sapply(idx0, function(x){if(is.null(x)){0}else{x}})
idx1 <- cumsum(idx0)
idx0 <- idx1-idx0+1
idx <- rbind(idx0, idx1)
idx[,apply(idx, 2, function(x){x[1]>x[2]})] <- 0
idx[,apply(idx, 2, function(x){x[1]==x[2]})] <- rbind(0,idx[1,apply(idx, 2, function(x){x[1]==x[2]})] )

indx <- list(idx[1,1]:idx[2,1],
idx[1,2]:idx[2,2],
idx[1,3]:idx[2,3],
idx[1,4]:idx[2,4],
idx[1,5]:idx[2,5],
idx[1,6]:idx[2,6],
idx[1,7]:idx[2,7])
indx <- lapply(indx,
function(x){
if(0 %in% x){
return(x[length(x)])
}else{
```

```
        return(x)
        }
        }
        )


        ## Create the utilities using simple X * beta
        param <- list(barWA = regr[[4]] %*% x[indx[[4]]],
        barWB = regr[[5]] %*% x[indx[[5]]],
        bara = regr[[6]]  %*% x[indx[[6]]],
        VA = regr[[2]] %*% x[indx[[2]]],
        VB =  regr[[7]] %*% x[indx[[7]]],
        SA = regr[[1]] %*% x[indx[[1]]],
        CB = regr[[3]] %*% x[indx[[3]]],
        sig = 1)
        param <- lapply(param, as.numeric)
        return(param)
        }

        ## Functions from Jo (2011)
        cStar.jo <- function(p, U){
        ## returns c*, a value that appears frequently
        ## p are the equilibrium probabilities p_R
        return((U$SA - (1-p)*U$VA)/p)
        }


        g.jo <- function(c,U){
        ## returns p_C for a given value of c (from cStar.jo, above) and U
        v1 <- (c-U$barWA)/U$sig
        v2 <- (c-U$bara)/U$sig
        return(1 - pnorm(v1)*pnorm(v2))
        }


        h.jo <- function(c, U){
        ## returns p_F for a given value of c (from cStar.jo, above) and U
        d1 <- (U$barWA - U$bara)/(U$sig*sqrt(2))
        d2 <- (U$barWA - c)/(U$sig)
        return(pbivnorm(d1, d2,rho=1/sqrt(2)))
        }

        f.jo <- function(p, U){
        ## returns p_R for a given value of p_F (from h.jo, above) and U
        return(pnorm((p*U$barWB + (1-p)*U$VB - U$CB)/(U$sig*p)))
        }

        const.jo <- function(p, U){
        ## Function to compute the equilibirum constraint p_R - f(h(p_R)
        c <- cStar.jo(p,U)
        g.jo <- g.jo(c,U)
        g.jo[g.jo<=.Machine$double.eps] <- .Machine$double.eps ##numeric
            stability
        j <- h.jo(c,U)/g.jo
```

```
        return(p - f.jo(j,U))
        }


        eqProbs <- function(p, U,RemoveZeros=F){
        ## This function generates p_C and p_F from equilibrium
        ## probability p_R
        ## INPUTS:
        ## p: p_R (the equilibrium)
        ## U: Utilities (from vec2U.regr, above)
        ## RemoveZeros: Boolean, should the function check for numeric issues?
        ## OUTPUTS: A matrix of M by 3 (M is the number of games)

        ck <- cStar.jo(p,U)
        pC <- g.jo(ck, U)
        if (RemoveZeros){
        pC[pC <= .Machine$double.eps] <- .Machine$double.eps
        }
        pF <- h.jo(ck, U)/pC
        return(cbind(p, pC, pF))
        }

        ################## Objective functions ##################


        LL.jo <- function(x, Y,regr){
        ## Log-likelihood function for the CMLE
        ## INPUTS:
        ## x: vector of current parameter guesses in order (beta, p)
        ## Y: 4 by M matrix of tabulated outcomes
        ## regr: list of regressors for each utility function
        ## OUTPUTS:
        ## LL: negative of the log-likelihood for this set of parameters

        M <- dim(Y)[2]
        xP <- plogis(x[(length(x)-M+1):length(x)]) #transform p_R to [0,1]
        xT <- x[1:(length(x)-M)] #beta

        U <- vec2U.regr(xT,regr) #convert beta and regr to utilities
        EQ <- eqProbs(xP,U,T)
        OUT <- cbind(1-EQ[,2],
        EQ[,2]*(1-EQ[,1]),
        EQ[,2]*EQ[,1]*EQ[,3],
        EQ[,2]*EQ[,1]*(1-EQ[,3]))
        OUT[OUT<=sqrt(.Machine$double.eps)] <- sqrt(.Machine$double.eps)
        LL <- sum(log(t(OUT))*Y) #likelihood
        return(-LL)
        }


        QLL.jo  <- function(x,PRhat,PFhat,Y,regr){
        ## Pseudo-log-likelihood for two step method
        ## INPUTS:
        ## x: vector of current parameter guesses in order (beta,p)
```

```
## PRhat: First stage estimates of p_R
## PFhat: First stage estimates of p_F
## Y: 4 by M matrix of tabulated outcomes
## regr: list of regressors for each utility function
## OUTPUTS:
## QLL: negative of the PLL for this set of parameters

U <- vec2U.regr(x,regr)
PR <- f.jo(PFhat, U)
PR[PR<=.Machine$double.eps] <- .Machine$double.eps
PC <- g.jo(cStar.jo(PRhat,U),U)
PC[PC<=.Machine$double.eps] <- .Machine$double.eps
PF <- h.jo(cStar.jo(PRhat,U),U)/PC

OUT <- cbind(1-PC,
PC*(1-PR),
PC*PR*PF,
PC*PR*(1-PF))
OUT[OUT<=sqrt(.Machine$double.eps)] <- sqrt(.Machine$double.eps)
QLL <- sum(log(t(OUT))*Y)
return(-QLL)
}


LL.nfxp <- function(x, Y,regr){
## Log-likelihood function for the Nested Fixed Point
## INPUTS:
## x: vector of current parameter guesses in order (beta,p)
## Y: 4 by M matrix of tabulated outcomes
## regr: list of regressors for each utility function
## OUTPUTS:
## LL: negative of the log-likelihood for this set of parameters

M <- dim(Y)[2]
U <- vec2U.regr(x,regr)

## compute AN equlibrium
f <- function(p){const.jo(p,U)}
grf <- function(p){diag(1-eval_gr_fh(p,U))}
out <- multiroot(f, rep(.5, M), jacfunc=grf, jactype="fullusr",
ctol=1e-6,rtol=1e-6,atol=1e-6)

EQ <- eqProbs(out$root,U)
OUT <- cbind(1-EQ[,2],
EQ[,2]*(1-EQ[,1]),
EQ[,2]*EQ[,1]*EQ[,3],
EQ[,2]*EQ[,1]*(1-EQ[,3]))
OUT[OUT<=sqrt(.Machine$double.eps)] <- sqrt(.Machine$double.eps)
LL <- sum(log(t(OUT))*Y)
return(-LL)
}

const.cmle <- function(x,Y,regr){
## Constraint function for the CMLE
```

```
## INPUTS:
## x: vector of current parameter guesses in order (beta, p)
## Y: 4 by M matrix of tabulated outcomes
## regr: list of regressors for each utility function
## OUTPUTS:
## vio: Constraint violation p_R - f(h(p_R; beta))

M <- dim(Y)[2]
xP <- plogis(x[(length(x)-M+1):length(x)]) #convert to [0,1]
xT <- x[1:(length(x)-M)]

U <- vec2U.regr(xT, regr)
vio <- const.jo(xP, U)
return(vio)
}

npl <- function(pl.hat, Phat, Y, regr, maxit=500, tol=1e-5){
## Estimates the NPL model starting at PL estimates.
## INPUTS:
## pl.hat: vector of beta estimates from the PL model
## Phat: length 2 list of first stage estimates, PRhat and PFhat
## Y: 4 by M matrix of tabulated outcomes
## regr: list of regressors for each utility function
## maxit: Maximum number of iterations
## tol: User specified step tolerance for (beta, pR, pF)
## OUTPUTS:
## npl.out: List containing
##            - NPL estimates (beta)
##            - Final best response update of pR
##            - Final best response update of pF
##            - Convergence code
##                * 1: Gradient close to zero at final inner step
##                * 2: Step tolerance statisfied at final inner step
##                * -69: Maximum out iterations exceded
##                * -99: Other error
##            - Number of outer iterations

#Setup
eval <- Inf
iter <- 0
out.NPL <- list(estimate = pl.hat)
fqll <- function(x){ #PL likelihood
-QLL.jo(x, Phat$PRhat, Phat$PFhat, Y, regr)
}
gr.qll <- function(x){ #PL gradient
-eval_gr_qll(x, Phat$PRhat, Phat$PFhat, Y, regr)
}
while(eval > tol & iter < maxit){
Uk <- vec2U.regr(out.NPL$estimate, regr)
Pk.F <- eqProbs(Phat$PRhat, Uk, RemoveZeros = T)[,3]
Pk.R <- pnorm((Phat$PFhat*Uk$barWB + (1-Phat$PFhat)*Uk$VB - Uk$CB)/Phat
    $PFhat)
Phat.k_1 <- Phat
Phat <- list(PRhat = Pk.R, PFhat = Pk.F)
```

```
        #normalize
        Phat$PRhat <-  pmin(pmax(Phat$PRhat, 0.0001), .9999)
        Phat$PFhat <-  pmin(pmax(Phat$PFhat, 0.0001), .9999)

        out.NPL.k <- try(maxLik(start=out.NPL$estimate, logLik=fqll, grad=gr.qll, method="NR"))
        if(class(out.NPL.k[[1]])=="character" || out.NPL.k$code==100){ #maxLik failure
        out.NPL <- out.NPL.k
        break
        }
        out.NPL.k$convergence <- out.NPL.k$code
        eval <- mean((c(out.NPL.k$estimate, unlist(Phat)) -c(out.NPL$estimate, unlist(Phat.k_1)))^2)
        out.NPL <- out.NPL.k
        iter <- iter + 1
        }
        if(class(out.NPL[[1]])=="character"|| out.NPL.k$code==100){ #if there was a failure
        out.NPL$estimate <- rep(NA, 6)
        out.NPL$convergence <- -99
        out.NPL$iter <- -99
        }else{
        out.NPL$convergence <- ifelse(iter==maxit, -69, out.NPL$convergence)
        out.NPL$convergence <- ifelse(eval==0, -99, out.NPL$convergence)
        }
        npl.out <- list(par = out.NPL$estimate,
        PRhat = Phat$PRhat,
        PFhat = Phat$PFhat,
        convergence = out.NPL$convergence,
        iter = out.NPL$iter)
        return(npl.out)
        }
```

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# Supplementary Information: Estimating Signaling Games in International Relations

**Abstract**

This document contains the appendices for the manuscript "Estimating Signaling Games in International Relations: Problems and Solutions." In Appendix A, we state and prove two results mentioned in the manuscript, and in Appendix B we present additional results from the Monte Carlo experiments. In Appendix C, we discuss the CMLE's Python implementation. In Appendix D, we analyze the importance of starting values for tMLE routines. In Appendix E, we compare tMLE results when we change equation solvers used in our economic sanctions application. Finally, Appendix F contains our R code used to implement the estimators.

## A Definitions, Results, and Proofs

This Appendix contains the formal arguments for two additional results discussed in the main manuscript. First we define the regularity refinement from Harsanyi (1973) and van Damme (1996).

**Definition 1** *An equilibrium* $\tilde{p}_R$ *is* regular *if* $\delta(\tilde{p}_R; \theta) \neq 1$.

With this definition we can now state our result concerning the regularity of equilibria.

**Result 2** *For almost all* $\theta$*, all equilibria of the crisis-signaling game are regular.*

To prove the result and subsequent ones, it is more straightforward to work with the function $F : (0,1) \times \mathbb{R}^8 \to \mathbb{R}$ such that

$$F(p_R;\theta) = f \circ h(p_R;\theta) - p_R,$$

where $\tilde{p}_R$ is an equilibrium if and only if $F(\tilde{p}_R;\theta) = 0$. Likewise, we will also use $\delta(p_R;\theta)$ to be the first derivative of $f \circ h$ with respect to $p_R$ given parameters $\theta$. We state two intermediary results before proving result 2. The first is from Jo (2011*a*) and the second is the parameterized Transversality Theorem.

**Lemma 1** *For all $\theta$, $\lim_{p_R \to 0} f \circ h(pR;\theta) > 0$ and $\lim_{p_R \to 1} f \circ h(p_R;\theta) < 1$.*

Thus, there are no equilibria at the boundaries. In addition, for any fixed $\theta$, there exists $\varepsilon > 0$ and $\nu > 0$ such that $F(\varepsilon;\theta) > 0$ and $F(1-\nu;\theta) < 0$

**Theorem 1** (Transversality Theorem) *Consider an open set $X \subseteq \mathbb{R}^n$. Let $L : X \times \mathbb{R}^s \to \mathbb{R}^n$ be continuously differentiable. Assume that the Jacobian $D_{(x,y)}L$ has rank $n$ for all $(x,y) \in X \times \mathbb{R}^s$ such that $L(x,y) = 0$. Then, for all most all $y' \in \mathbb{R}^s$, the Jacobian $D_x L$ has rank $n$ for all $x \in X$ such that $L(x,y') = 0$.*

We now prove Result 2.

*Proof of Result 2.* Note that $\tilde{p}_R$ is a regular equilibrium if and only if $D_{p_R}F(p_R;\theta) \neq 0$. To prove Result 2, we verify the conditions of the Transversality condition, where in our application, $L = F$ and $(x,y) = (p_R;\theta)$, which means $n = 1$ and $s = 8$. First, note that $F$ is continuously differentiable, because $f \circ h$ is the composition of normal cumulative distribution functions and polynomial functions, and $F$ is defined over the open interval $(0,1)$.

Third and finally, we show that $D_{(p_R;\theta)}F(p_R;\theta)$ has at least one non-zero element (i.e., rank 1) when $F(p_R;\theta) = 0$. To do this, we show a stronger result: for all $(p_R;\theta)$, $D_{(p_R;\theta)}F(p_R;\theta) \neq 0$. To see this, consider $D_{\bar{W}_B}F(p_R;\theta)$. By Result 1, the functions $g$ and $h$ are constant in

parameter $\bar{W}_B$, that is, $D_{\bar{W}_B} g(p_R;\theta) = D_{\bar{W}_B} h(p_R;\theta) = 0$. Then we have

$$\begin{aligned}
D_{\bar{W}_B} F(p_R;\theta) &= D_{\bar{W}_B} f \circ h(p_R;\theta) \\
&= D_{\bar{W}_B} \Phi \left( \frac{h(p_R;\theta)\bar{W}_B + (1 - h(p_R;\theta))V_B - C_B}{h(p_R;\theta)} \right) \\
&= D_{\bar{W}_B} \Phi \left( \bar{W}_B + \frac{(1 - h(p_R;\theta))V_B - C_B}{h(p_R;\theta)} \right) \\
&= \phi \left( \bar{W}_B + \frac{(1 - h(p_R;\theta))V_B - C_B}{h(p_R;\theta)} \right) \\
&\neq 0,
\end{aligned}$$

which implies $D_{(p_R;\theta)} F(p_R;\theta) \neq 0$ as required. □

Although the regularity refinement does not generically reduce the number of equilibria, showing that all the equilibria are regular is advantageous for applied empirical research. Regular equilibria can be implicitly expressed as continuous functions of parameters. This property is particularly important in empirical analyses: if we uncover noisy, but sufficiently accurate estimates of $\theta$, then equilibrium choice probabilities will be close to their true values as well. In addition, comparative statics (predicted probabilities) on regular equilibria will be well behaved, i.e., the equilibrium will not vanish if we vary the data or parameters by some small amount.

Our next result focuses on best response iteration. Before stating the result, we define best-response stable and best-response unstable equilibria.

**Definition 2** *An equilibrium $\tilde{p}_R$ is* best-response stable *if there exists $\varepsilon > 0$ such that for all $p_R^0 \in (\tilde{p}_R - \varepsilon, \tilde{p}_R + \varepsilon)$ the sequence*

$$p_R^k = f \circ h(p_R^{k-1};\theta),\ k \in \mathbb{N}$$

*converges to $\tilde{p}_R$.*

The next definition introduces best-response unstable equilibria, which is not simply the negation of Definition 2.

**Definition 3** *An equilibrium $\tilde{p}_R$ is* best-response unstable *if there exists $\varepsilon > 0$ such that for all $p_R^0 \in (\tilde{p}_R - \varepsilon, \tilde{p}_R + \varepsilon)$, with $p_R^0 \neq \tilde{p}_R$, the sequence*

$$p_R^k = f \circ h(p_R^{k-1}; \theta),\ k \in \mathbb{N}$$

*leaves the interval $(\tilde{p}_R - \varepsilon, \tilde{p}_R + \varepsilon)$ at least once. That is, there exists $n \in \mathbb{N}$ such that $p_R^n \notin (\tilde{p}_R - \varepsilon, \tilde{p}_R + \varepsilon)$*

With these definitions, we are now ready to state Results 3.

**Result 3** *In high-stakes games where all equilibria are regular, the following hold:*

1. *There is a finite number of equilibria, and every equilibrium is either best-response stable or best-response unstable.*
2. *If there is a unique equilibrium, then it is best-response stable.*
3. *If there are multiple equilibria, then there exists a best-response unstable equilibrium and at least two best-response stable equilibria.*

To prove Result 3, we need two intermediate results.

**Lemma 2** *Assume $(V_B - C_B)(V_A - S_A) > 0$. Then $f \circ h$ is strictly increasing in $p_R$, i.e., $\delta(p_R; \theta) > 0$.*

*Proof.* The derivative of $f$ with respect to $p_F$ is

$$D_{p_F} f(p_F) = (C_B - V_B) \frac{\phi\left(\bar{W}_B - \frac{(1-p_F)V_B - C_B}{p_F}\right)}{p_F}, \tag{11}$$

where $\phi$ is the probability density function of the standard normal distribution. Notice that if $p_F > 0$, the fraction in Equation 11 divides one positive number by another. Thus, $C_B > V_B$ implies $D_{p_F} f(p_F) > 0$, i.e., $f$ is strictly increasing in $p_F$. Likewise, $C_B < V_B$ implies $D_{p_F} f(p_F) < 0$, i.e., $f$ is strictly decreasing in $p_F$. Similar computations show that $V_A > S_A$ implies $h$ is strictly increasing in $p_R$, and $V_A < S_A$ implies $h$ is strictly decreasing in $p_R$.

There are two cases to consider. First $(V_B - C_B) > 0$ and $(V_A - S_A) > 0$. If $(V_A - S_A) > 0$, then $h$ is strictly increasing. Because the range of $h$ is positive, $(V_B - C_B) > 0$ and the above

analysis implies $f \circ h$ is strictly increasing because it is the composition of two strictly increasing functions. A similar argument holds for the second case where $(V_B - C_B) < 0$ and $(V_A - S_A) < 0$ because the composition of two strictly decreasing functions is strictly increasing. $\square$

Notice that a high-stakes game satisfies the assumption in Lemma 2 because high-stakes games have $V_B > C_B$ and $V_A > S_A$. This assumption relaxes Condition **??**, and Result 3 still holds under the more general assumption that $(V_B - C_B)(V_A - S_A) > 0$. The next theorem states a standard result in nonlinear dynamics and fixed point iteration. See Theorem 6.5 in Holmgren (1994).

**Theorem 2** *Consider an equilibrium $\tilde{p}_R$. If $|\delta(\tilde{p}_R\theta)| < 1$, then $\tilde{p}_R$ is best-response stable. If $|\delta(\tilde{p}_R;\theta)| > 1$, then $\tilde{p}_R$ is best-response unstable.*

To end this Appendix, we prove Result 3.

*Proof of Result 3(1).* We first prove that all regular equilibria are either best-response stable or best-response unstable in high-stakes games. By regularity, $\delta(\tilde{p}_R;\theta) \neq 1$ for some equilibrium $\tilde{p}_R$ and fixed parameters $\theta$. By Lemma 2, $\delta(\tilde{p}_R;\theta) > 0$, which means $\delta(\tilde{p}_R;\theta) \neq -1$. Then Theorem 2 implies that equilibrium $\tilde{p}$ is either best-response stable or best-response unstable.

Second, we prove that there is a finite number of equilibria. By assumption all equilibria are regular, which implies $D_{p_R}F(\tilde{p}_R;\theta) \neq 0$ for all $\tilde{p}_R$ such that $F(\tilde{p}_R;\theta) = 0$. Then the Implicit Function Theorem implies that every equilibrium $\tilde{p}_R$ is locally isolated. Because $F$ is continuous, it has closed level sets, so the set of equilibria is closed. Because equilibria fall within the interval $(0, 1)$, the set of equilibria is bounded, and therefore compact. As a compact set of locally isolated points, the equilibrium set is finite. $\square$

*Proof of Result 3(2).* Let $\tilde{p}_R^{[1]}$ be the smallest equilibrium, i.e., $\tilde{p}_R^{[1]}$ solves

$$\min \{\tilde{p}_R \in [0,1] \mid F(\tilde{p}_R;\theta) = 0\}.$$

Such a solution exists because the equilibrium set is non-empty (Result 1) and finite (Result 3(1)). We claim that $\tilde{p}_R^{[1]}$ is best-response stable, i.e., $\left|\delta(\tilde{p}_R^{[1]};\theta)\right| < 1$. To see this suppose not. Then $\left|\delta(\tilde{p}_R^{[1]};\theta)\right| \geq 1$. By Lemma 2, $\delta(\tilde{p}_R^{[1]};\theta) > 0$ because the game is high-stakes, implying $\delta(\tilde{p}_R^{[1]};\theta) \geq 1$. Regularity then implies $\delta(\tilde{p}_R^{[1]};\theta) > 1$.

Because $F$ is continuously differentiable and $D_{p_R}F = \delta(\tilde{p}_R^{[1]};\theta) - 1$, there exists $\varepsilon > 0$ such that $F$ is strictly increasing on the interval $(\tilde{p}_R^{[1]} - \varepsilon, \tilde{p}_R^{[1]})$. Because $F(\tilde{p}_R^{[1]};\theta) = 0$, this implies that there exists a $p'_R \in (\tilde{p}_R - \varepsilon, \tilde{p}_R^{[1]})$ such that $F(p'_R;\theta) < 0$. By Lemma 1, there exists $\nu \in (0, p'_R)$ such that $F(\nu;\theta) > 0$. Then the Intermediate Value Theorem Implies that there exists a $\tilde{p}_R \in (\nu, p'_R)$ such that $F(\tilde{p}_R;\theta) = 0$, but this contradicts the assumption that $\tilde{p}_R^{[1]}$ is the smallest equilibrium. Hence, we conclude that if $\tilde{p}_R^{[1]}$ is the smallest equilibrium, then $\left|\delta(p_R^{[1]};\theta)\right| < 1$, which implies it is stable by Theorem 2. If there is a unique equilibrium, then it must be the smallest equilibrium, and it is stable. □

*Proof of Result 3(3, unstable).* We prove that, if there are multiple equilibria, then there exists a best-response unstable equilibrium. Let $\tilde{p}_R^{[2]}$ be the second smallest equilibrium, i.e., $\tilde{p}_R^{[2]}$ solves

$$\min \{\tilde{p}_R \in [0,1] \mid F(\tilde{p}_R;\theta) = 0\} \setminus \left\{\tilde{p}_R^{[1]}\right\},$$

where $\tilde{p}_R^{[1]}$ is the smallest equilibrium defined above. Such a solution exists because there are a finite number of equilibria and, by assumption, more than one. We claim that $\left|\delta(\tilde{p}_R^{[2]};\theta)\right| > 1$. To see this suppose not. Then $\left|\delta(\tilde{p}_R^{[2]};\theta)\right| \leq 1$. Because all equilibria are regular, $\delta(\tilde{p}_R^{[2]};\theta) < 1$, implying $D_{p_R}F(\tilde{p}_R^{[2]};\theta) < 0$. This, along with the facts that $F$ is continuously differentiable and $F(\tilde{p}_R^{[2]};\theta) = 0$, implies there exists (arbitrarily small) $\varepsilon > 0$ such that $F(\tilde{p}_R^{[2]} - \varepsilon;\theta) > 0$.

In the proof of Result 3(2), we showed that $\left|\delta(\tilde{p}_R^{[1]};\theta)\right| < 1$ and $F(\tilde{p}_R^{[1]};\theta) = 0$. Then exists (arbitrarily small) $\nu > 0$ such that $F(\tilde{p}_R^{[1]}+\nu;\theta) < 0$ because $F$ is continuously differentiable. So we have $F(\tilde{p}_R^{[2]}-\varepsilon;\theta) > 0$ and $F(\tilde{p}_R^{[1]}+\nu;\theta) < 0$. Then by the Intermediate Value Theorem there exists an equilibrium $\tilde{p}'_R$ such that

$$\tilde{p}_R^{[1]} + \nu < \tilde{p}'_R < \tilde{p}_R^{[2]} - \varepsilon.$$

But this contradicts the assumption that $\tilde{p}_R^{[2]}$ is the second smallest equilibrium. Thus, we conclude $\left|\delta(\tilde{p}_R^{[2]};\theta)\right| > 1$. As such, $\tilde{p}_R^{[2]}$ is unstable. □

*Proof of Result 3(3, stable).* We now prove that, if there are multiple equilibria, then there exists at least two stable equilibria. First, we claim that if two equilibria exist, then a third exists as well. To see this, note that we showed that $\delta(\tilde{p}_R^{[2]};\theta) > 1$ in the proof above. Then $D_{p_R}F(\tilde{p}_R^{[2]};\theta) > 0$, so there exists a $p'_R > \tilde{p}_R^{[2]}$ such that $F(p'_R;\theta) > 0$. By Lemma 1, there exists a $\nu$ close to 1 such that $F(\nu;\theta) < 0$. Then the Intermediate Value Theorem implies that there exists a third equilibrium, between $p'_R$ and $\nu$.

Let $\tilde{p}_R^{[3]}$ satisfy the following:

$$\min\left\{\tilde{p}_R \in [0,1] \mid F(\tilde{p}_R;\theta) = 0\right\} \setminus \left\{\tilde{p}_R^{[1]}, \tilde{p}_R^{[2]}\right\},$$

where $\tilde{p}_R^{[1]}$ and $\tilde{p}_R^{[2]}$ are defined above. A solution exists because at least three equilibria exist. We claim that $\tilde{p}_R^{[3]}$ is best-response stable. If not, then $\delta(\tilde{p}_R^{[3]};\theta) > 1$ identical arguments from the proof of Result 3(1). Then there exists (arbitrarily small) $\varepsilon > 0$ such that $F(\tilde{p}_R^{[3]}-\varepsilon;\theta) < 0$. However, $\delta(\tilde{p}_R^{[2]};\theta) > 1$, so there exists (arbitrarily small) $\mu > 0$ such that $F(\tilde{p}_R^{[2]}+\mu;\theta) > 0$. Then the Intermediate Value Theorem implies there exists an equilibrium strictly between $\tilde{p}_R^{[2]}$ and $\tilde{p}_R^{[3]}$, a contradiction. Thus, $\tilde{p}_R^{[3]}$ is best-response stable. The proof of Result 3(2) shows that $\tilde{p}_R^{[1]}$ is best-response stable as well. Hence, there are two stable equilibria. □

# B Further Monte Carlo Results

## B.1 Multiple Equilibria

This appendix contains additional results from the Monte Carlo experiment where the data are generated under parameters that are consistent with multiple equilibria. A single covariate determines the equilibirum selection. The parameter values used to generate the data can be found in Table 1. Here we consider the estimators' bias, variance, and rate of convergence. Root mean squared error and computation time are both presented in the main text.

**Figure 8:** Bias in signaling estimators with multiple equilibria.

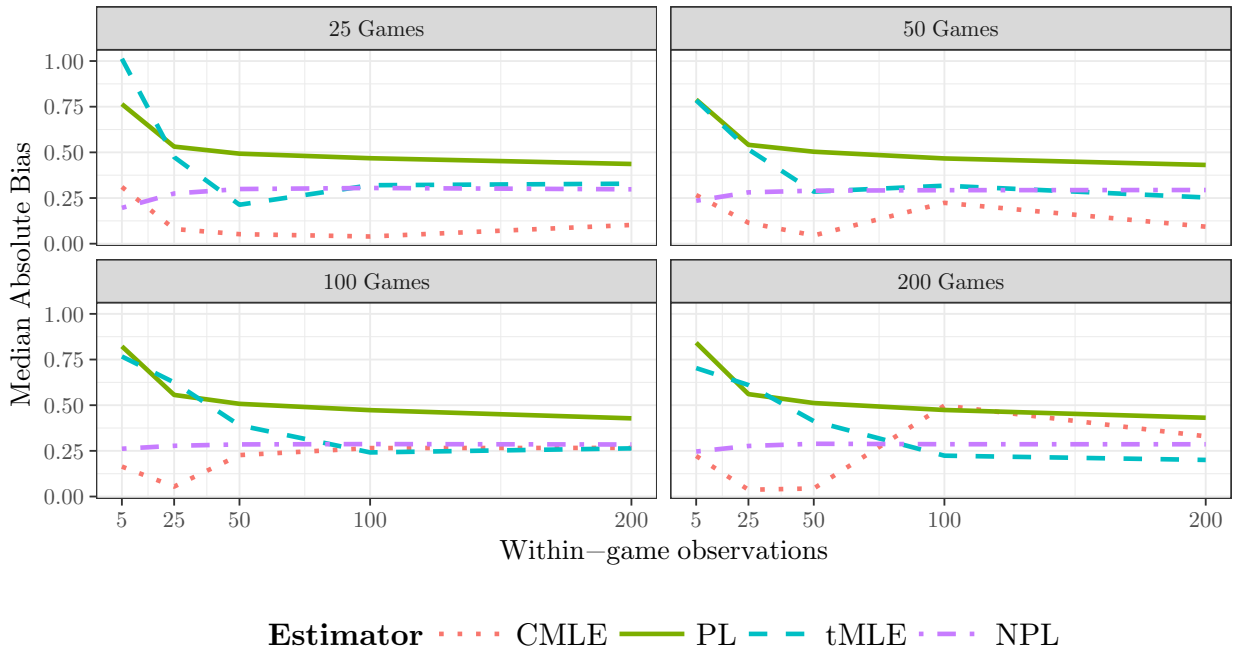


**Figure 9:** Variance in signaling estimators with multiple equilibria.

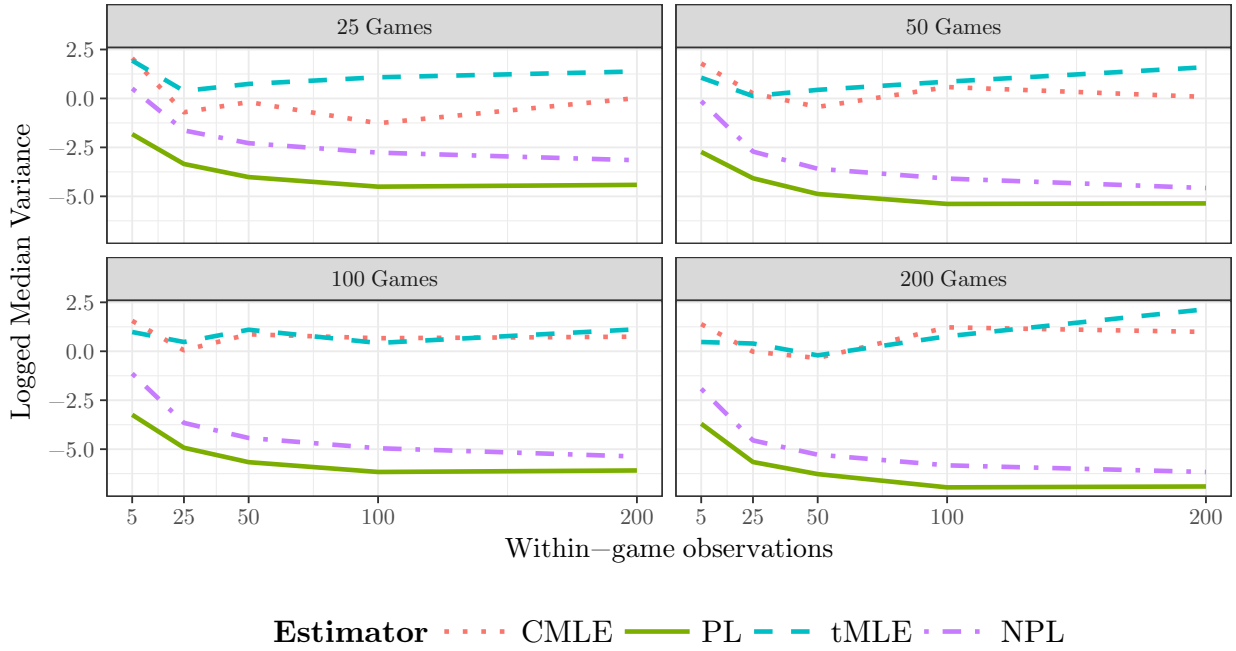



**Figure 10:** Convergence rates in signaling estimators with multiple equilibria.

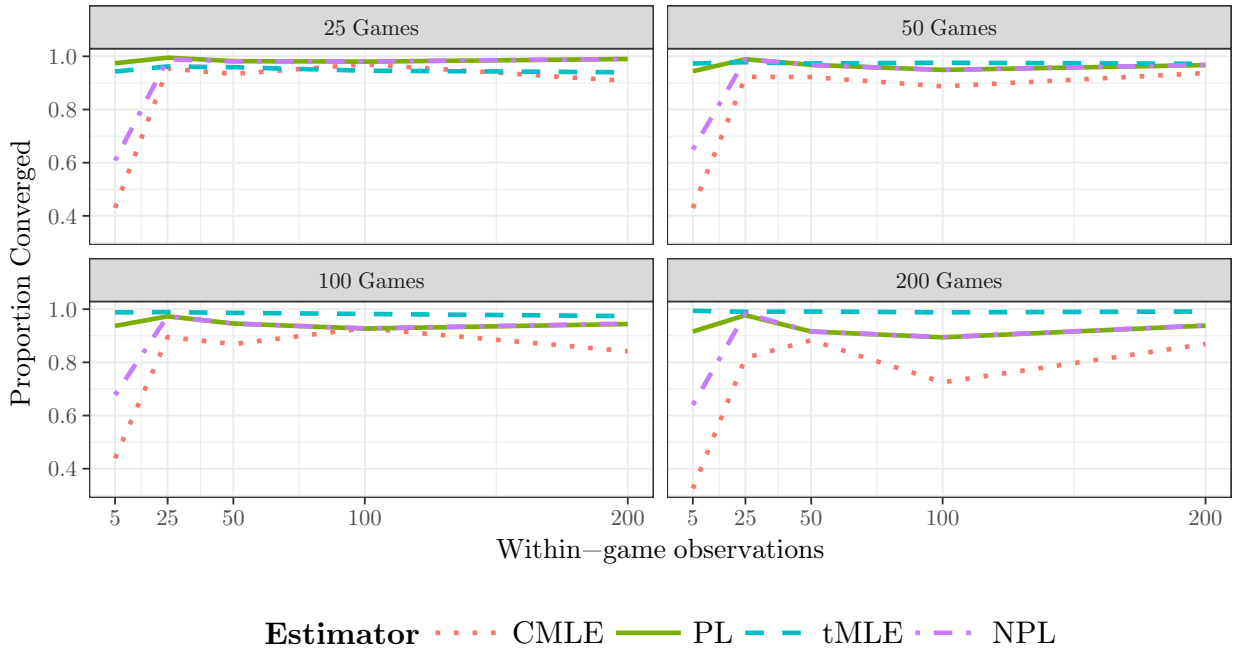
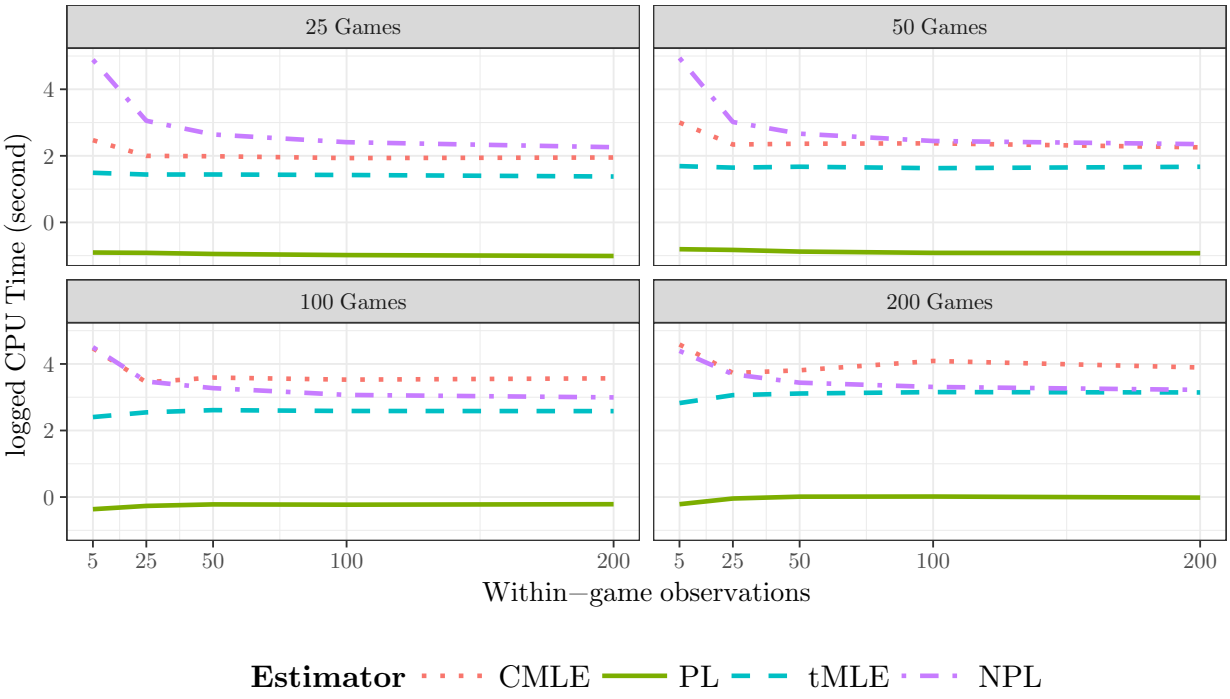


**Figure 11:** Computational time in signaling estimators with multiple equilibria.



## B.2 Unique Equilibrium

This appendix contains additional results from the Monte Carlo experiment where the data are generated from a version of the game with a unique equilibrium. The parameter values used to generate the data can be found in the final column of Table 1. Here we consider the estimators' bias, variance, computation time, and rate of convergence.

**Figure 12:** Bias in signaling estimators with a unique equilibrium.

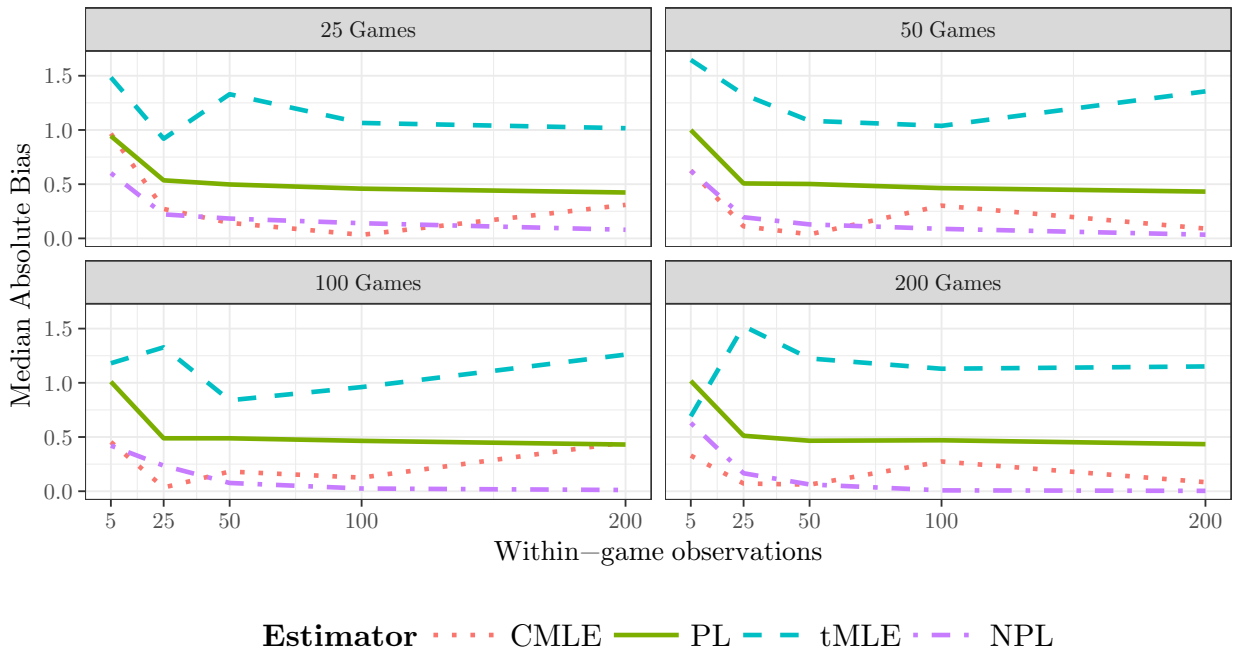


**Figure 13:** Variance in signaling estimators with a unique equilibrium.

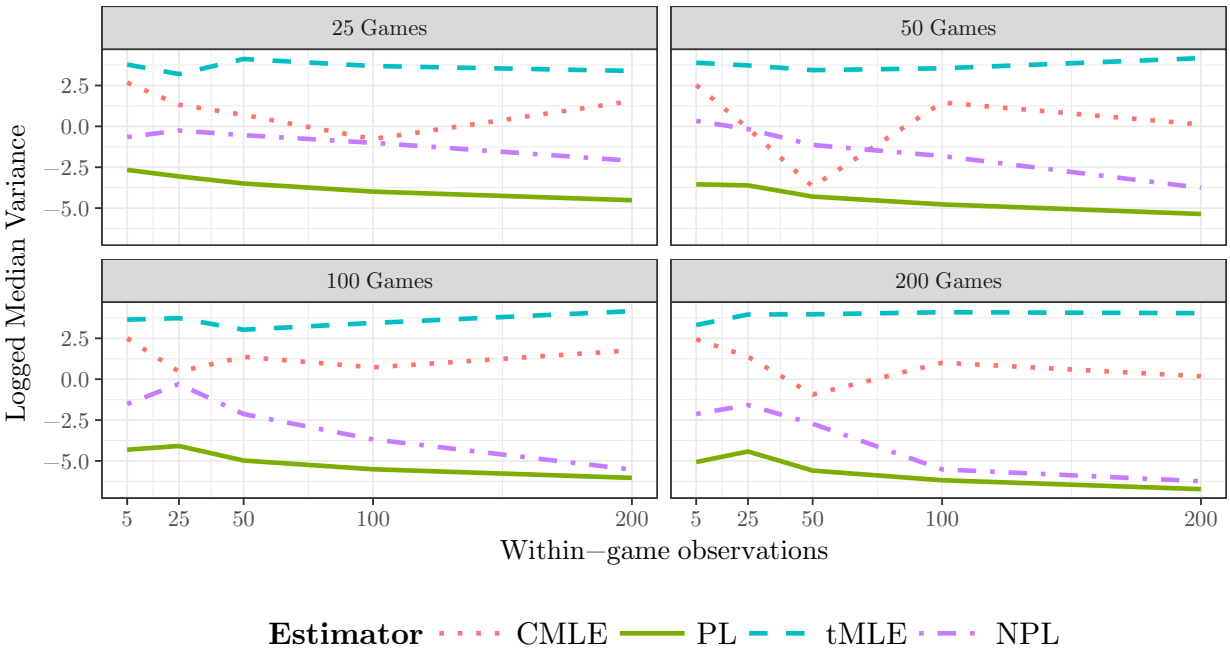



**Figure 14:** Computational time in signaling estimators with a unique equilibrium.

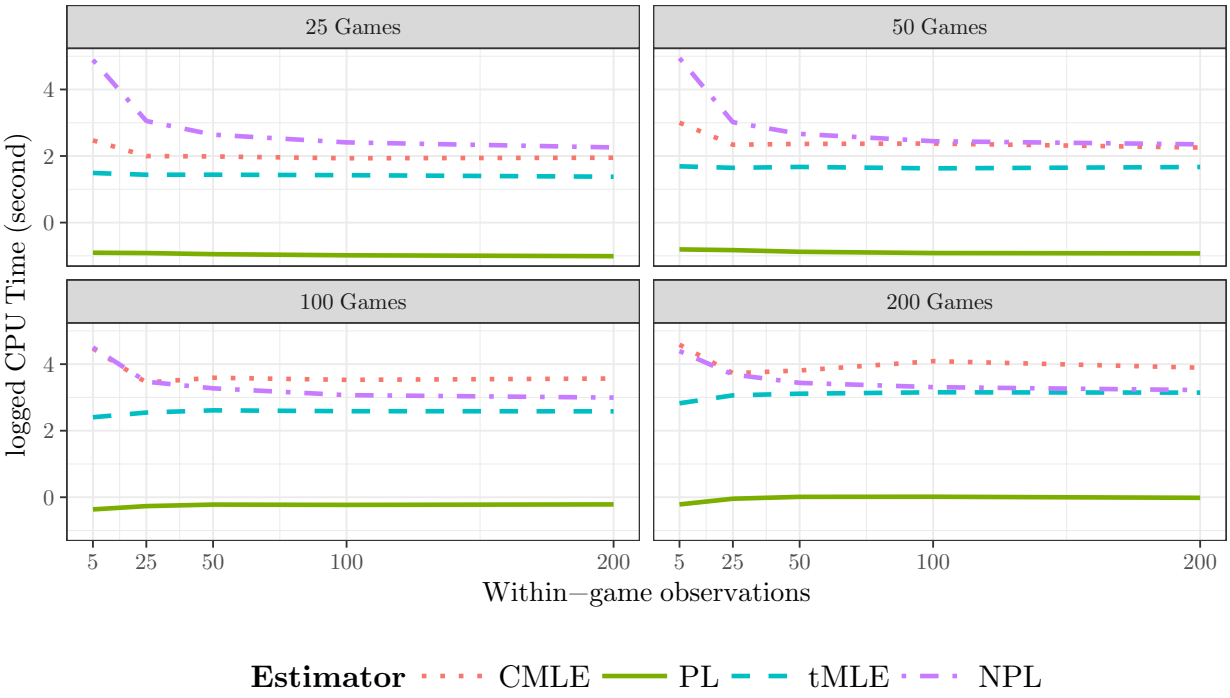
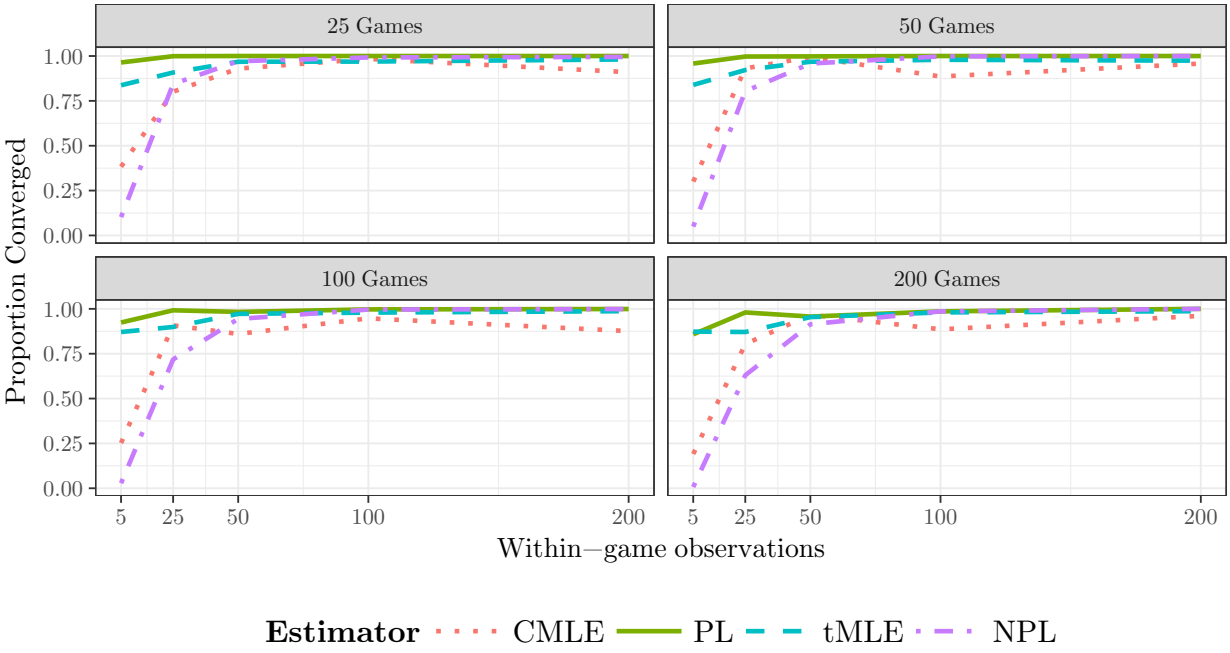


**Figure 15:** Convergence rates in signaling estimators with a unqiue equilibrium.



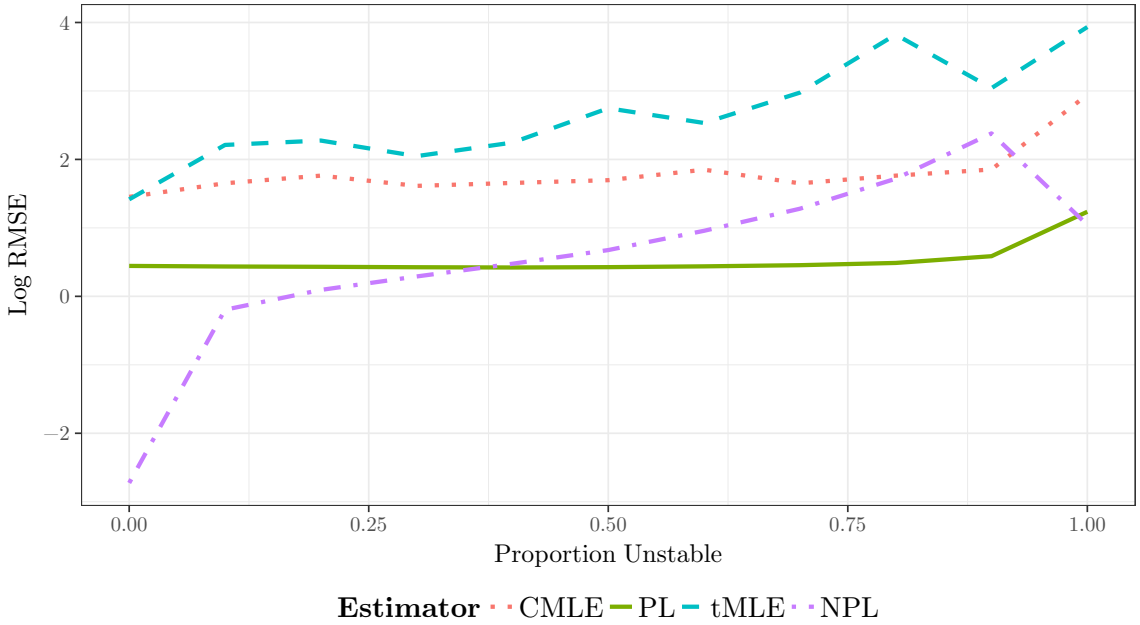
## B.3 Best-Response Stability

The best performing solutions make use of best response functions, which begs the question: How sensitive are the estimators to best-response unstable equilibria? To answer this question, we conduct another Monte Carlo experiment. Here, we assume payoffs are generated as in the multiple setting in Table 1, and the equilibrium selection rule follows the left-hand graph in Figure 2. Let $q \in [0, 1]$ denote the percentage of unstable equilibria. For $q \cdot D$ dyads, $x_d$ is draw from a uniform distribution over the interval $(\frac{1}{3}, \frac{2}{3})$. For the remaining $D - q \cdot D$ observations, $x_d$ is drawn uniformly from the intervals $(0, \frac{1}{3})$ or $(\frac{2}{3}, 1)$ with equal probability. From the formal arguments in the Appendix, the middle equilibrium, i.e., the one selected when $x_d \in (\frac{1}{3}, \frac{2}{3})$, is unstable.[1] As we vary $q$ from 0 to 1, we analyze how the estimators' performance varies as the data are generated with a larger proportion of best-response unstable equilibria. In this experiment, we set $D = 500$ and $T = 500$, which means there is a large amount of data as to better isolate the affects of unstable equilibria. For all values of $q$, we draw $x_d$, select the corresponding equilibria, and estimate the model $1,000$ times. We expect the PL and NPL to perform worse as $q$ approaches 1.

Figure 16 summarizes the results, where we vary the percentage of unstable equilibria along the horizontal axis and plot log RMSE along the vertical axis. Unsurprisingly, the PL and the NPL have larger RMSE as the data are generated with larger proportions of unstable equilibria, and this trend appears in the remaining estimators as well. For data with less than 40% unstable equilibria, the NPL outperforms the PL, and *vice versa* for data with more than 40% unstable equilibria. More surprisingly, we find that the PL and NPL perform better than the remaining estimators *even when more than 75% of observations are drawn from unstable equilibria*. Thus, best-response instability is a problem for all estimators that we consider as more unstable equilibria generally increase RMSE. The two estimators that explicitly rely on best-response iteration, however, have the best performance.

[1] See the Proof of Result 3(3). Unstable equilibria can also be identified numerically using Theorem 2.

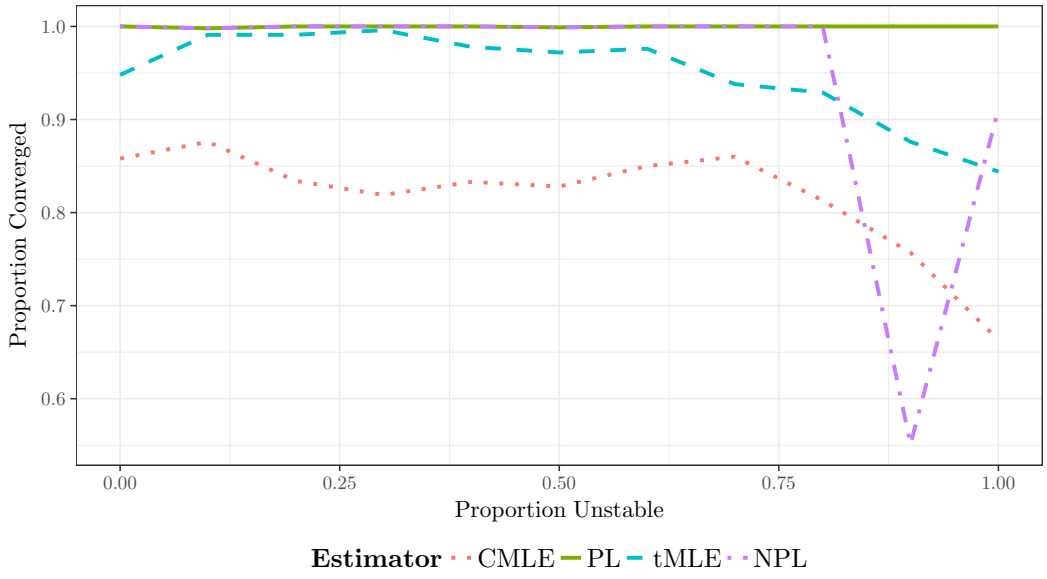
**Figure 16:** RMSE in signaling estimators with more unstable equilibria.




What explains this unintuitive finding? We note two possibilities. First, with $T = 500$, there is a large amount of within game observations, leading to precise first-stage estimates. Thus, even if an estimation routine relies on best-response iteration, the first-stage estimates may be precise enough where only minimal problems arise. Second, Figure 16 only reports results that have converged, and we find that when unstable equilibria dominate in the data, convergence of all estimators decreases. This trend is illustrated in Figure 17, where horizontal axis is the proportion of observations with unstable equilibria and the vertical axis is the proportion of bootstraps converged. Notice, convergence rates of all estimators, besides the PL, decrease once the proportion of unstable equilibria approaches 60–80%.[2] Thus, conditional on converging, the estimators return results with fairly reasonable RMSE even with a large proportion of unstable equilibria. They are all generally less likely to converge when unstable equilibria permeate the data, however.

[2]We find that the convergence rate of the CMLE method is decreasing in the number of dyads, $D$. This is not surprising as the method requires estimating $D$ equilibrium constraints in a highly nonlinear problem. When $D \leq 200$, the CMLE's convergence rate is between 80–100%. See Figures 10 and 15 in Appendices B.1 and B.2, respectively, for more information.

**Figure 17:** Convergence rates in signaling estimators with unstable equilibria.



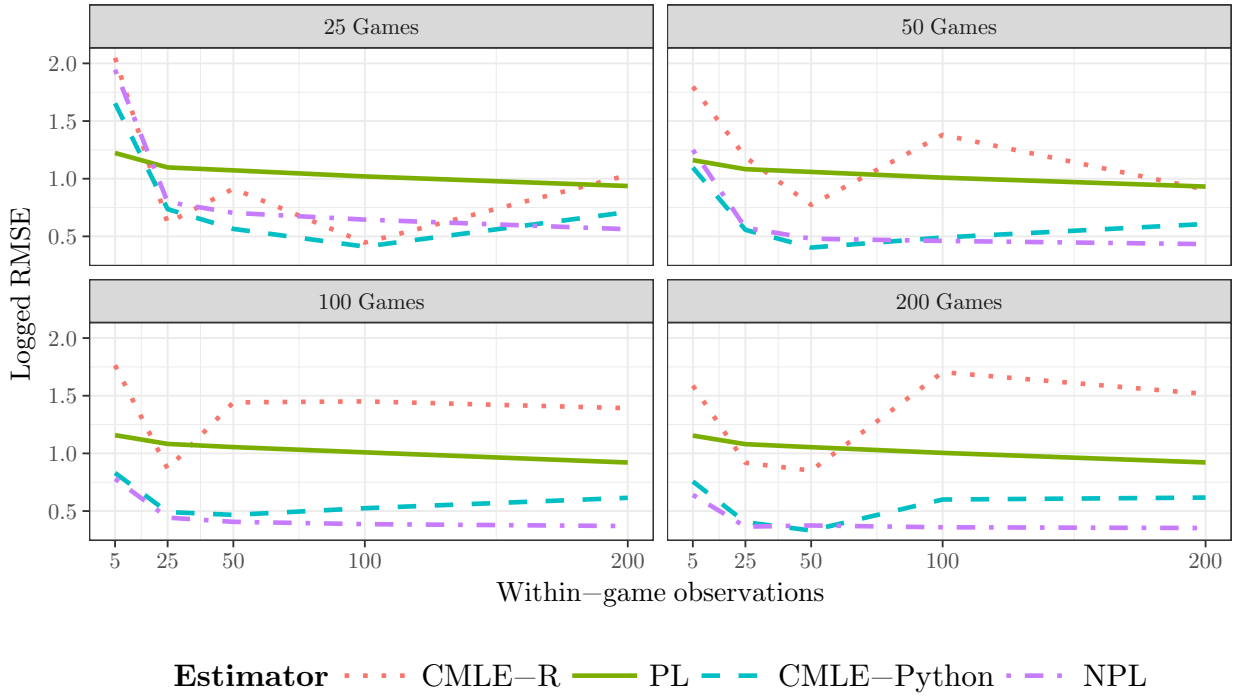
# C CMLE Implementation

In our economic sanctions application we fit the CMLE using the program IPOPT (Interior Point OPTimizer), which is an open-source optimizer designed to handle large scale problems (Wächter and Biegler 2006). In trials, IPOPT had better performance properties than other optimizers such as sequential quadratic solvers (found in Python's `scipy.optimize` module and MATLAB) and a version of the Augmented Lagrangian Method (from R's `alabama` package, used for Monte Carlo analysis).

The main difficulty in using interior-point methods is that they require an accurate second derivative of the Lagrangian associated with the problem in Equation 10. We find that finite difference approximations are insufficient. As such, we use the program ADOL-C, software for algorithmic differentiation (AD) (Griewank, Juedes and Utke 1996), to precisely compute the Hessian. The AD software allows us to only supply the log-likelihood and constraint function from Equation 10. The AD program repeatedly applies the chain rule to our functions to compute first- and second-order derivatives. In our economic sanctions example, we use IPOPT and ADOL-C within Python 2.7.13 on UbuntuGNOME 17.04 by calling the `pyipopt` module developed by Xu (2014) and the `pyadolc` module developed by Walter (2014), respectively. Finally, we estimate standard errors using Silvey (1959, Lemma 6, p. 401).

To demonstrate the trade offs of these two approaches we conduct an additional Monte Carlo comparing IPOPT-AD (Python) to the Augmented Lagrangian (R) in terms of statistical performance (RMSE), computation time, and convergence rate. Also included are the pseudo-likelihood methods, which serve as a benchmark. The data generating process uses the multiple equilibrium setup (first column of Table 1). In Figure 18, we see that while the Augmented Lagrangian (R) of the CMLE is sometimes better or worse than the pseudo-likelihood, the interior point method (Python) is almost always superior. This performance

**Figure 18:** Comparing CMLE implementations: RMSE




enhancement comes from the inclusion of high quality first and second order derivatives through, which greatly decreases the variance of the estimates. These improvements are not free, however, as we observe in Figure 19 that the interior point method suffers from a frequent failure to converge. On the other hand, we note in Figure 20 that the two approaches require roughly similar amounts of time, with the interior point method sometimes taking an average of 2-3 additional minutes longer. This trade-off between statistical and computational performance should be considered on a case-by-case basis when fitting these models.

**Figure 19:** Comparing CMLE implementations: Convergence

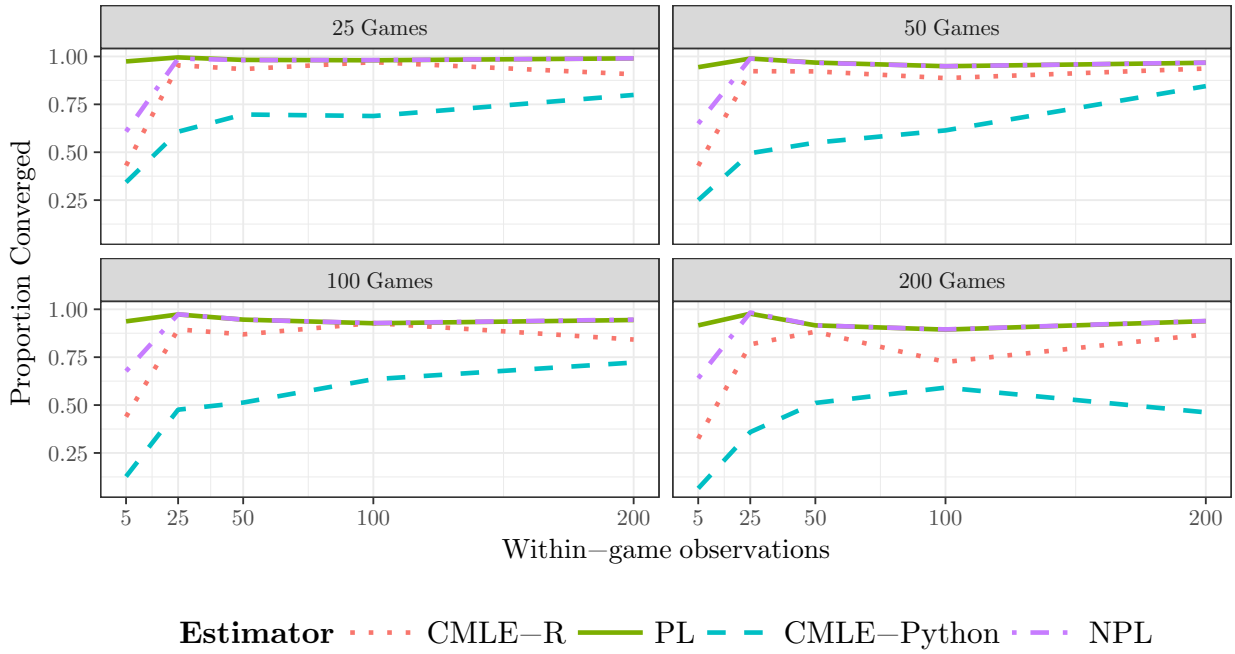
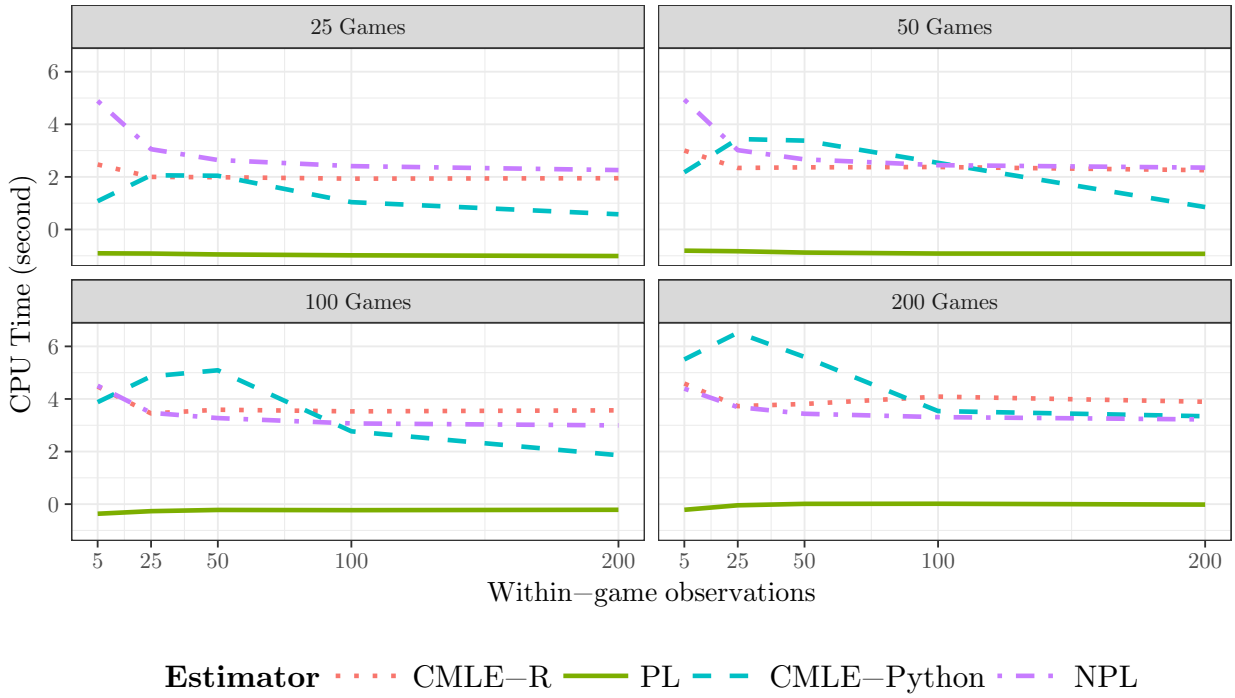



**Figure 20:** Comparing CMLE Computation time



# D Traditional ML and Starting Values

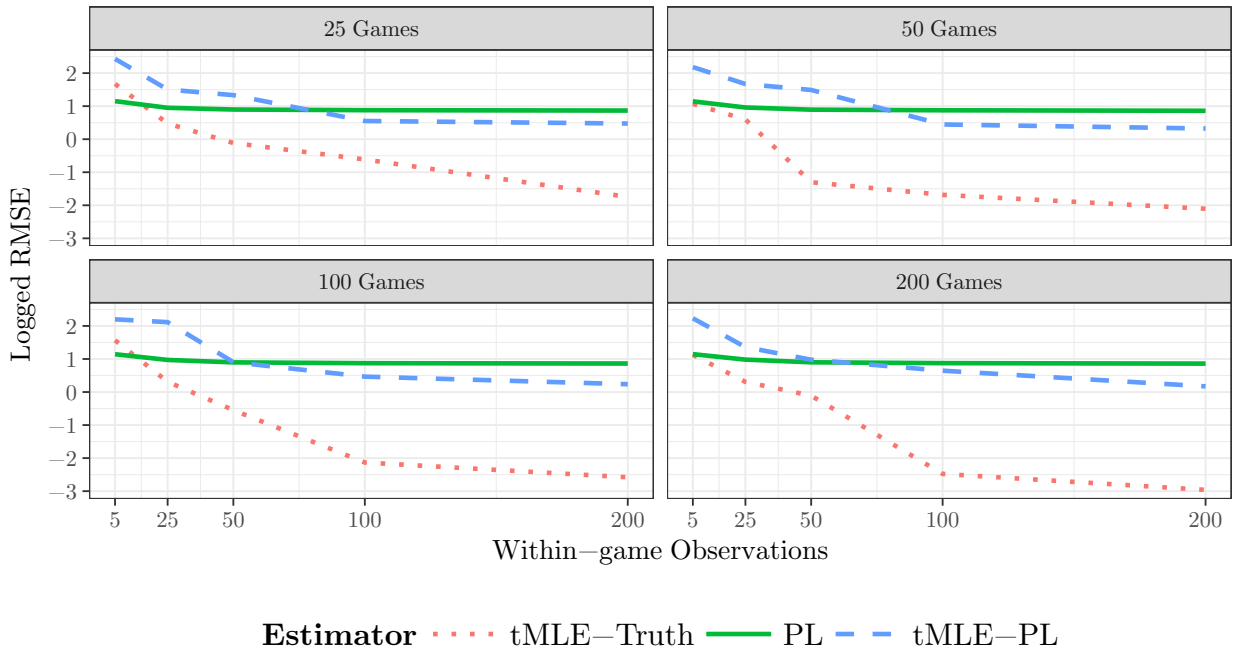
Our Monte Carlo experiments demonstrate that the tMLE may not be consistent even when there is a unique equilibrium in the signaling game from the data generating process. The reason such problems arise is that the maximization routine will oftentimes evaluate the likelihood function at a guess of the parameters where multiple equilibria arise. In this case, the traditional approach will select an equilibrium in an ad-hoc fashion, which may encourage the maximization routine to move away from the correct parameters. This may be surprising as both Jo (2011*a*) and Bas, Signorino and Whang (2014) conduct similar Monte Carlo experiments and conclude that the tMLE performs well when the data were generated with parameters that admit a unique equilibrium.[3]

To the best of our knowledge, the differences arise from starting values. In our study, starting values for $\theta$ were drawn from a standard uniform distribution, and we the starting values for all four estimators in each simulation. In Jo (2011*a*), they were the values from the data generating process (Jo 2011*b*). Although we were not able to locate replication materials from Bas, Signorino and Whang (2014), we do conduct an additional Monte Carlo experiment to investigate the possibility that differences in starting values lead to different results. To do this, we replicate our unique equilibrium Monte Carlo experiment. We estimate the model using the pseudo-likelihood routine, a tMLE routine with starting values from the data generating process, i.e., the parameters listed in the unique column of Table 1, and a tMLE routine with starting values from the pseudo-likelihood output.

Figure 21 graphs the median (over the parameters) logged RMSE of the three estimation procedures as we vary the number of dyads $D$ and the number of observations $T$. In a similar manner, Figures 22 and 23 report the estimators' bias and variance, respectively.

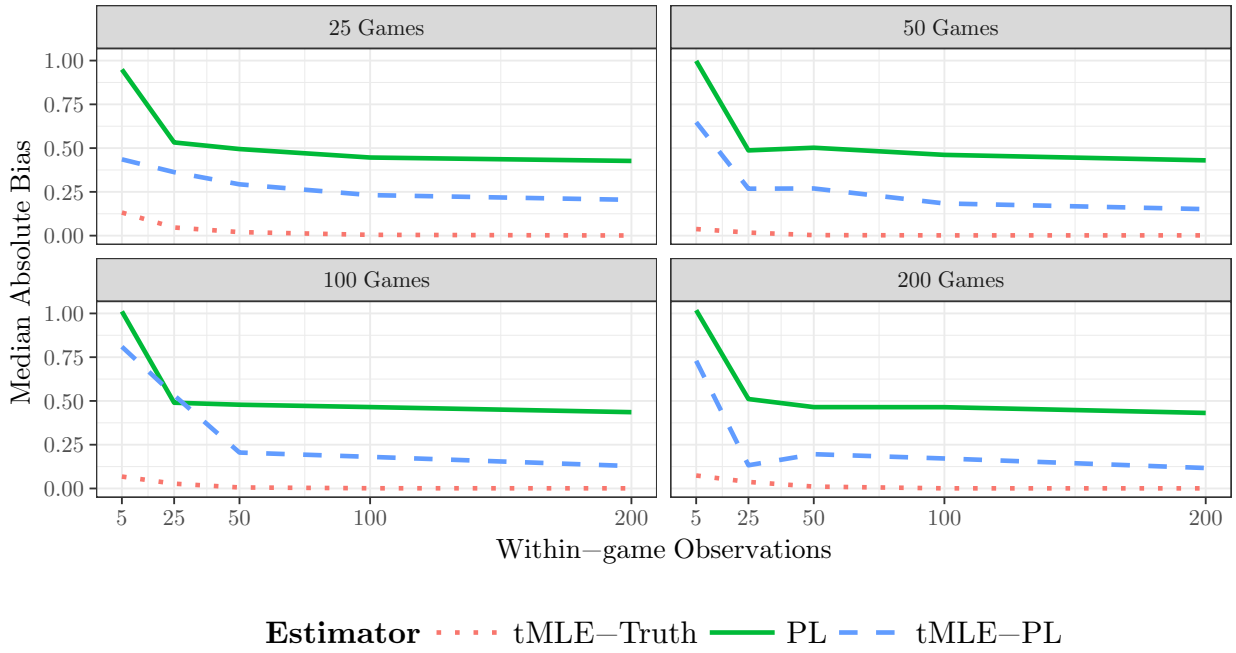
[3] Jo (2011*a*, p. 357) writes "It is easy to see that when there is a unique equilibrium, the estimates get closer to their true values as the number of observations increases." Bas, Signorino and Whang (2014, p. 26) write "All coefficients on average are estimated very close to the true parameter values, and the accuracy of the estimates increases as the sample size increases."

**Figure 21:** RMSE with a unique equilibrium and different starting values.




**Caption:** Graphs of the estimators' RMSE as a function of the number of games $D$ and the number of within game observations $T$. The figure graphs the median log-RMSE over the parameters in the unique column from Table 1.

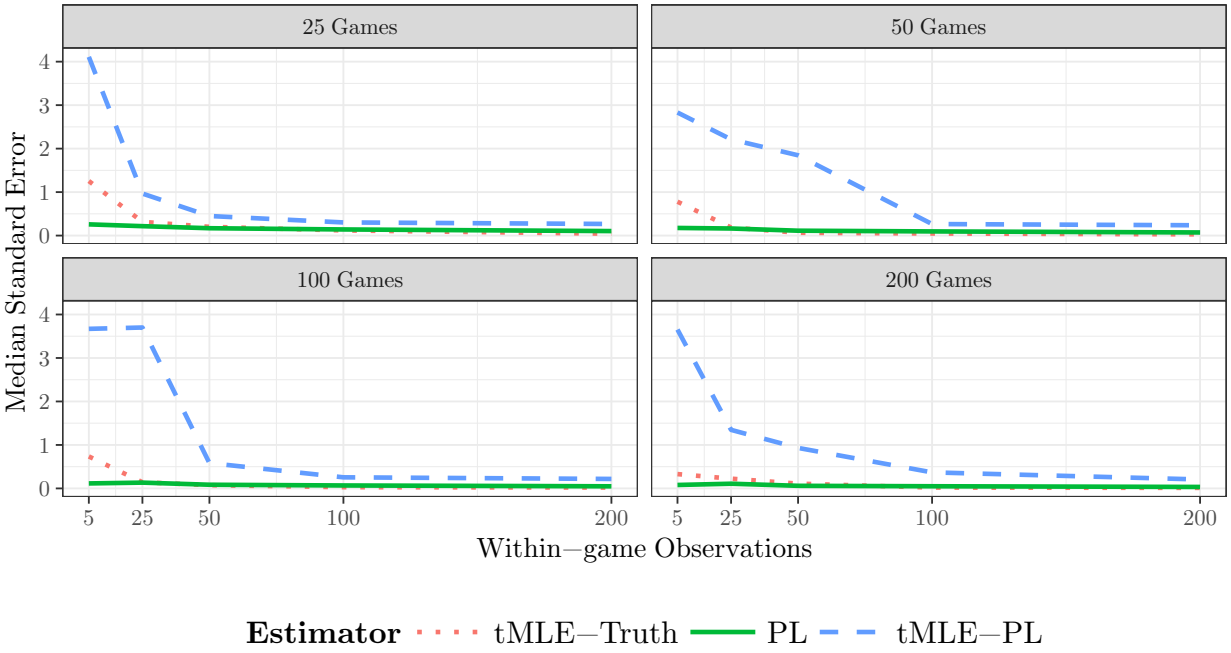
**Figure 22:** Bias with a unique equilibrium and different starting values.




**Caption:** Graphs of the estimators' bias as a function of the number of games $D$ and the number of within game observations $T$. The figure graphs the median bias over the parameters in the unique column from Table 1.

There are two major takeaways. First, if the tMLE has privileged starting values, either from the truthful values or the PL estimates, then it generally performs as well as (if not better than) the pseudo-likelihood estimator. This stands in stark contrast to the results in Figure 5, where the tMLE performs substantially worse than the PL routine. Second, if $T = 5$, then the pseudo-likelihood estimator appears to perform better than either of the tMLE routines even though one of these routines was provided the correct estimates as starting values. This improvement is mostly driven by the relatively smaller variance in the PL method. Given these results, we conclude different choices in starting values generate the contradictory findings in previous work.

**Figure 23:** Variance with a unique equilibrium and different starting values.




**Caption:** Graphs of the estimators' variance as a function of the number of games $D$ and the number of within game observations $T$. The figure graphs the median variance over the parameters in the unique column from Table 1.

# E Traditional ML and Equation Solvers

**Table 4:** Effects of Equation Solver on tMLE

| | tML<br>*Broyden Solver*<br>Model 1 | tML<br>*Select Largest Eq.*<br>Model 2 |
|---|---|---|
| $S_A$: Econ. Dep$_A$ | 0.05 | −0.38 |
| | (0.29) | (0.27) |
| $S_A$: Dem$_A$ | 0.00 | 0.03* |
| | (0.00) | (0.01) |
| $S_A$: Contiguity | 0.27* | 0.28* |
| | (0.10) | (0.02) |
| $S_A$: Alliance | −0.06 | 0.38* |
| | (0.08) | (0.05) |
| $V_A$: Const. | −0.06 | −0.14 |
| | (0.08) | (0.09) |
| $V_A$: Costs$_A$ | −0.04 | −1.71* |
| | (0.03) | (0.26) |
| $C_B$: Const. | 0.81 | 1.08* |
| | (0.91) | (0.31) |
| $C_B$: Econ. Dep$_B$ | −0.21 | 0.23* |
| | (0.16) | (0.10) |
| $C_B$: Costs$_B$ | −0.08* | −0.12* |
| | (0.03) | (0.01) |
| $C_B$: Contiguity | −0.25* | −0.15* |
| | (0.02) | (0.01) |
| $C_B$: Alliance | 0.10 | −0.37* |
| | (0.09) | (0.04) |
| $\bar{W}_A$: Const. | −0.15 | −0.41* |
| | (0.78) | (0.14) |
| $\bar{W}_A$: Econ. Dep$_A$ | 0.07 | 0.3 |
| | (0.75) | (0.32) |
| $\bar{W}_A$: Dem$_A$ | 0.01 | 0.06* |
| | (0.01) | (0.01) |
| $\bar{W}_A$: Cap. Ratio | −0.01 | 0.07* |
| | (0.01) | (0.01) |
| $\bar{W}_B$: Const. | −0.38 | 1.19* |
| | (1.13) | (0.40) |
| $\bar{W}_B$: Dem$_B$ | 0.01* | 0.00* |
| | (0.00) | (0.00) |
| $\bar{W}_B$: Cap. Ratio | 0.01 | −0.05* |
| | (0.01) | (0.02) |
| $\bar{a}$: Const. | −0.56 | −0.76* |
| | (0.77) | (0.14) |
| $\bar{a}$: Dem$_A$ | 0.00 | 0.06* |
| | (0.01) | (0.01) |
| Log $L$ | -4102.76 | -4302.08 |
| $N$ | $418 \times 120$ | $418 \times 120$ |

*Notes:* $^*p < 0.05$
Standard Errors in Parenthesis

# F R Code

Below we list the basic code required to use any of the estimation routines found in this paper. The complete code used to replicate this entire paper can be found in the replication archive.

```
##
## This file contains code for the EQ constraint in Jo (2011).
## It also includes functions for generating data and functions
## necessary to implement the four estimators.
## Additional packages: pbivnorm, rootSolve, maxLik
##

################## HELPER FUNCTIONS ##################
vec2U.regr <- function(x,regr){
    ## Function for converting parameters and regressors to
    ## utilities over outcomes
    ## INPUTS:
    ## x: vector of regression parameters (betas) in the order SA, VA, CB, barWA, barWB, bara, VB
    ## regr: a list of regressor matrices, one for each utility in the same order as x
    ## OUTPUTS:
    ## param: A list of utilities in the same order as regr.
    ## Each element of this list is a vector of length equal
    ## to the number of games.


    ## create indices to appropriately sort the elements of x
    ## into the correct outcomes.
    idx0 <- lapply(regr, ncol)
    idx0 <- sapply(idx0, function(x){if(is.null(x)){0}else{x}})
    idx1 <- cumsum(idx0)
    idx0 <- idx1-idx0+1
    idx <- rbind(idx0, idx1)
    idx[,apply(idx, 2, function(x){x[1]>x[2]})] <- 0
    idx[,apply(idx, 2, function(x){x[1]==x[2]})] <- rbind(0,idx[1,apply(idx, 2, function(x){x[1]==x[2]})] )

    indx <- list(idx[1,1]:idx[2,1],
    idx[1,2]:idx[2,2],
    idx[1,3]:idx[2,3],
    idx[1,4]:idx[2,4],
    idx[1,5]:idx[2,5],
    idx[1,6]:idx[2,6],
    idx[1,7]:idx[2,7])
    indx <- lapply(indx,
                   function(x){
                     if(0 %in% x){
                       return(x[length(x)])
```

```
                      }else{
                        return(x)
                      }
                    }
                   )


    ## Create the utilities using simple X * beta
    param <- list(barWA = regr[[4]] %*% x[indx[[4]]],
                  barWB = regr[[5]] %*% x[indx[[5]]],
                  bara = regr[[6]]  %*% x[indx[[6]]],
                  VA = regr[[2]] %*% x[indx[[2]]],
                  VB =  regr[[7]] %*% x[indx[[7]]],
                  SA = regr[[1]] %*% x[indx[[1]]],
                  CB = regr[[3]] %*% x[indx[[3]]],
                  sig = 1)
    param <- lapply(param, as.numeric)
    return(param)
}

## Functions from Jo (2011)
cStar.jo <- function(p, U){
    ## returns c*, a value that appears frequently
    ## p are the equilibrium probabilities p_R
    return((U$SA - (1-p)*U$VA)/p)
}


g.jo <- function(c,U){
    ## returns p_C for a given value of c (from cStar.jo, above) and U
    v1 <- (c-U$barWA)/U$sig
    v2 <- (c-U$bara)/U$sig
    return(1 - pnorm(v1)*pnorm(v2))
}


h.jo <- function(c, U){
    ## returns p_F for a given value of c (from cStar.jo, above) and U
    d1 <- (U$barWA - U$bara)/(U$sig*sqrt(2))
    d2 <- (U$barWA - c)/(U$sig)
    return(pbivnorm(d1, d2,rho=1/sqrt(2)))
}

f.jo <- function(p, U){
    ## returns p_R for a given value of p_F (from h.jo, above) and U
    return(pnorm((p*U$barWB + (1-p)*U$VB - U$CB)/(U$sig*p)))
}

const.jo <- function(p, U){
    ## Function to compute the equilibirum constraint p_R - f(h(p_R)
    c <- cStar.jo(p,U)
    g.jo <- g.jo(c,U)
    g.jo[g.jo<=.Machine$double.eps] <- .Machine$double.eps ##numeric stability
    j <- h.jo(c,U)/g.jo
```

```
    return(p - f.jo(j,U))
}


eqProbs <- function(p, U,RemoveZeros=F){
    ## This function generates p_C and p_F from equilibrium
    ## probability p_R
    ## INPUTS:
    ## p: p_R (the equilibrium)
    ## U: Utilities (from vec2U.regr, above)
    ## RemoveZeros: Boolean, should the function check for numeric issues?
    ## OUTPUTS: A matrix of M by 3 (M is the number of games)

    ck <- cStar.jo(p,U)
    pC <- g.jo(ck, U)
    if (RemoveZeros){
      pC[pC <= .Machine$double.eps] <- .Machine$double.eps
    }
    pF <- h.jo(ck, U)/pC
    return(cbind(p, pC, pF))
}

################## Objective functions ##################


LL.jo <- function(x, Y,regr){
    ## Log-likelihood function for the CMLE
    ## INPUTS:
    ## x: vector of current parameter guesses in order (beta, p)
    ## Y: 4 by M matrix of tabulated outcomes
    ## regr: list of regressors for each utility function
    ## OUTPUTS:
    ## LL: negative of the log-likelihood for this set of parameters

    M <- dim(Y)[2]
    xP <- plogis(x[(length(x)-M+1):length(x)]) #transform p_R to [0,1]
    xT <- x[1:(length(x)-M)] #beta

    U <- vec2U.regr(xT,regr) #convert beta and regr to utilities
    EQ <- eqProbs(xP,U,T)
    OUT <- cbind(1-EQ[,2],
    EQ[,2]*(1-EQ[,1]),
    EQ[,2]*EQ[,1]*EQ[,3],
    EQ[,2]*EQ[,1]*(1-EQ[,3]))
    OUT[OUT<=sqrt(.Machine$double.eps)] <- sqrt(.Machine$double.eps)
    LL <- sum(log(t(OUT))*Y) #likelihood
    return(-LL)
}


QLL.jo  <- function(x,PRhat,PFhat,Y,regr){
    ## Pseudo-log-likelihood for two step method
    ## INPUTS:
    ## x: vector of current parameter guesses in order (beta,p)
```

```
    ## PRhat: First stage estimates of p_R
    ## PFhat: First stage estimates of p_F
    ## Y: 4 by M matrix of tabulated outcomes
    ## regr: list of regressors for each utility function
    ## OUTPUTS:
    ## QLL: negative of the PLL for this set of parameters

    U <- vec2U.regr(x,regr)
    PR <- f.jo(PFhat, U)
    PR[PR<=.Machine$double.eps] <- .Machine$double.eps
    PC <- g.jo(cStar.jo(PRhat,U),U)
    PC[PC<=.Machine$double.eps] <- .Machine$double.eps
    PF <- h.jo(cStar.jo(PRhat,U),U)/PC

    OUT <- cbind(1-PC,
    PC*(1-PR),
    PC*PR*PF,
    PC*PR*(1-PF))
    OUT[OUT<=sqrt(.Machine$double.eps)] <- sqrt(.Machine$double.eps)
    QLL <- sum(log(t(OUT))*Y)
    return(-QLL)
}


LL.nfxp <- function(x, Y,regr){
    ## Log-likelihood function for the Nested Fixed Point
    ## INPUTS:
    ## x: vector of current parameter guesses in order (beta,p)
    ## Y: 4 by M matrix of tabulated outcomes
    ## regr: list of regressors for each utility function
    ## OUTPUTS:
    ## LL: negative of the log-likelihood for this set of parameters

    M <- dim(Y)[2]
    U <- vec2U.regr(x,regr)

    ## compute AN equlibrium
    f <- function(p){const.jo(p,U)}
    grf <- function(p){diag(1-eval_gr_fh(p,U))}
    out <- multiroot(f, rep(.5, M), jacfunc=grf, jactype="fullusr",
    ctol=1e-6,rtol=1e-6,atol=1e-6)

    EQ <- eqProbs(out$root,U)
    OUT <- cbind(1-EQ[,2],
    EQ[,2]*(1-EQ[,1]),
    EQ[,2]*EQ[,1]*EQ[,3],
    EQ[,2]*EQ[,1]*(1-EQ[,3]))
    OUT[OUT<=sqrt(.Machine$double.eps)] <- sqrt(.Machine$double.eps)
    LL <- sum(log(t(OUT))*Y)
    return(-LL)
}

const.cmle <- function(x,Y,regr){
    ## Constraint function for the CMLE
```

```
    ## INPUTS:
    ## x: vector of current parameter guesses in order (beta, p)
    ## Y: 4 by M matrix of tabulated outcomes
    ## regr: list of regressors for each utility function
    ## OUTPUTS:
    ## vio: Constraint violation p_R - f(h(p_R; beta))

    M <- dim(Y)[2]
    xP <- plogis(x[(length(x)-M+1):length(x)]) #convert to [0,1]
    xT <- x[1:(length(x)-M)]

    U <- vec2U.regr(xT, regr)
    vio <- const.jo(xP, U)
    return(vio)
}

npl <- function(pl.hat, Phat, Y, regr, maxit=500, tol=1e-5){
    ## Estimates the NPL model starting at PL estimates.
    ## INPUTS:
    ## pl.hat: vector of beta estimates from the PL model
    ## Phat: length 2 list of first stage estimates, PRhat and PFhat
    ## Y: 4 by M matrix of tabulated outcomes
    ## regr: list of regressors for each utility function
    ## maxit: Maximum number of iterations
    ## tol: User specified step tolerance for (beta, pR, pF)
    ## OUTPUTS:
    ## npl.out: List containing
    ##            - NPL estimates (beta)
    ##            - Final best response update of pR
    ##            - Final best response update of pF
    ##            - Convergence code
    ##                * 1: Gradient close to zero at final inner step
    ##                * 2: Step tolerance statisfied at final inner step
    ##                * -69: Maximum out iterations exceded
    ##                * -99: Other error
    ##            - Number of outer iterations

    #Setup
    eval <- Inf
    iter <- 0
    out.NPL <- list(estimate = pl.hat)
    fqll <- function(x){ #PL likelihood
        -QLL.jo(x, Phat$PRhat, Phat$PFhat, Y, regr)
    }
    gr.qll <- function(x){ #PL gradient
        -eval_gr_qll(x, Phat$PRhat, Phat$PFhat, Y, regr)
    }
    while(eval > tol & iter < maxit){
        Uk <- vec2U.regr(out.NPL$estimate, regr)
        Pk.F <- eqProbs(Phat$PRhat, Uk, RemoveZeros = T)[,3]
        Pk.R <- pnorm((Phat$PFhat*Uk$barWB + (1-Phat$PFhat)*Uk$VB - Uk$CB)/
            Phat$PFhat)
        Phat.k_1 <- Phat
        Phat <- list(PRhat = Pk.R, PFhat = Pk.F)
```

```r

        #normalize
        Phat$PRhat <-  pmin(pmax(Phat$PRhat, 0.0001), .9999)
        Phat$PFhat <-  pmin(pmax(Phat$PFhat, 0.0001), .9999)

        out.NPL.k <- try(maxLik(start=out.NPL$estimate, logLik=fqll, grad=gr.qll, method="NR"))
        if(class(out.NPL.k[[1]])=="character" || out.NPL.k$code==100){ #maxLik failure
            out.NPL <- out.NPL.k
            break
        }
        out.NPL.k$convergence <- out.NPL.k$code
        eval <- mean((c(out.NPL.k$estimate, unlist(Phat)) -c(out.NPL$estimate, unlist(Phat.k_1)))^2)
        out.NPL <- out.NPL.k
        iter <- iter + 1
    }
    if(class(out.NPL[[1]])=="character"|| out.NPL.k$code==100){ #if there was a failure
        out.NPL$estimate <- rep(NA, 6)
        out.NPL$convergence <- -99
        out.NPL$iter <- -99
    }else{
        out.NPL$convergence <- ifelse(iter==maxit, -69, out.NPL$convergence)
        out.NPL$convergence <- ifelse(eval==0, -99, out.NPL$convergence)
    }
    npl.out <- list(par = out.NPL$estimate,
                    PRhat = Phat$PRhat,
                    PFhat = Phat$PFhat,
                    convergence = out.NPL$convergence,
                    iter = out.NPL$iter)
    return(npl.out)
}
```

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