The surprising depth and complexity of elementary- and middle-grades mathematics and the mathematical education of teachers

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Key points

- Elementary math is surprisingly deep and interesting;

- Seeing elementary math as a coherent sense-making enterprise rather than as a collection of isolated tricks is a real challenge;

- We know useful things from detailed studies of students’ and teachers’ thinking;

- To be well-prepared, elementary- and middle-grades teachers need to know a lot.

- To improve mathematics teaching, we need the right kinds of professional environments.
What is there to know about counting at the Kindergarten level?

If a child can correctly say the first five counting numbers, “one, two, three, four, five,” will the child necessarily be able to determine how many blocks there are in this collection?
Counting and Cardinality

Child 1:

“1” “2” “3” “4”

Child 2:

“1” “2” “3” “4”

Child 3:

“1” “2” “3” “4” “5” “6”

Child 4:

“1” “2” “3” “4” “5” “6”
Teacher: “How many blocks are there?”

Child 1:

Child 2:

Teacher: “So how many blocks are there?”

Child 1:

Child 2:

“Five all together!”
Fractions

How to define fractions in elementary school?

Hint: Not as equivalence classes of ordered pairs.
What about defining $\frac{A}{B}$ as "A out of B"?

Good idea or not?
Errors when fractions are viewed as pairs of numbers, not a single number

Common error

\[
\frac{1}{3} + \frac{1}{4} = \frac{2}{7}
\]
Errors when fractions are viewed as pairs of numbers, not a single number

Claire says that \( \frac{4}{9} > \frac{3}{8} \)

because

\[ 4 > 3 \text{ and } 9 > 8 \]
Defining fractions

1 whole or unit amount:

Partitioned into 8 equal parts:

The size of 1 part is a new unit, the unit fraction $\frac{1}{8}$:

5 parts, each of size $\frac{1}{8}$ of the unit amount:

9 parts, each of size $\frac{1}{8}$ of the unit amount:
Adding fractions with like denominators

\[ \frac{2}{9} + \frac{5}{9} = \frac{2+5}{9} = \frac{7}{9} \quad \text{2 ninths + 5 ninths = 7 ninths} \]

\[ \frac{8}{9} - \frac{3}{9} = \frac{8-3}{9} = \frac{5}{9} \quad \text{8 ninths - 3 ninths = 5 ninths} \]
Equivalent fractions

What’s a 4th grade explanation for why

$$\frac{2}{3} = \frac{8}{12}$$

Why is the following equation true?

$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4}$$
Equivalent fractions

In 4th grade students have not yet studied fraction multiplication, so the teacher must know a different explanation from this one that uses multiplication by 1 in the form $\frac{4}{4}$:

$$
\frac{2}{3} = \frac{2}{3} \cdot 1 = \frac{2}{3} \cdot \frac{4}{4} = \frac{2 \cdot 4}{3 \cdot 4}
$$
Equivalent fractions

\[
\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}
\]

split each part into 4 parts
then there are 4 times as many shaded parts
and 4 times as many parts in all
but each part is only a fourth as big
Is it a subtraction problem?

Is this a word problem for $\frac{2}{3} - \frac{1}{2}$?

There was $\frac{2}{3}$ of a cake left over. Denise ate $\frac{1}{2}$ of the cake that was left. Then how much cake was left?

First I showed $\frac{2}{3}$. Then when you take away half of that you have $\frac{1}{3}$ left.
How can students decide if a word problem is solved by an operation?

Known issue: students guess at which operation to use to solve a word problem.
Which problems are solved by multiplication?

Sometimes students are given keywords or principles saying certain types of problems are solved by multiplication:

- Area of rectangles
- Combinatorial counting problems
- Probability
- “Of” means multiply

But:

- Where do the principles come from?
- Why is multiplication a single coherent operation across different types of numbers and different topics?

To construct mathematically valid arguments we need definitions of the operations.
How to define multiplication?

How would you define multiplication?

Could you use (a simplified version of) that definition to explain what $3 \times 5$ means at a third grade level?
How to define multiplication?

- Repeated addition?
  What about fraction multiplication?

- Area of a rectangle?
  How does that help with identifying word problems?

- Equal groups?
Defining multiplication

\[ M \cdot N = P \]

**Multiplier:** number of equal groups

**Multiplicand:** number of units in 1 group

**Product:** number of units in \( M \) groups

3 plates, each will get the same number of cookies

5 cookies on 1 plate

15 cookies on 3 plates
What about the commutative property of multiplication?

It’s not obvious!

\[ 3 \times 5 \]

\[ 5 \times 3 \]
Where does the commutative property of multiplication come from?
Where does the commutative property of multiplication come from?

$3 \times 5$
Where does the commutative property of multiplication come from?

\[ 5 \times 3 \]
Courses for future teachers at UGA

We use definitions of multiplication and fractions to organize the “multiplicative conceptual field”

- Multiplication
- Division
  - How many groups division
  - How many units in 1 group division
- Ratio; fractions
- Proportional and linear relationships

Andrew Izsák has developed a semester-long course for future secondary teachers focusing on multiplicative reasoning.
Current research

Joint work with Andrew Izsák

Investigating Proportional Relationships from Two Perspectives
InPReP₂

We are studying future middle grades mathematics teachers’ reasoning around ideas of multiplication, division, fractions, ratio, proportional relationships, and linear functions.
The multiplicative conceptual field is critically important

Report from the National Center on Education and the Economy (NCEE), 2013:

What does it really mean to be college and work ready?

“A very high priority should be given to the improvement of the teaching of proportional relationships including percent, graphical representations, functions, and expressions and equations . . .” (p. 2)
Jewelry-gold is made by mixing pure gold with copper in the ratio 7 to 5. How much pure gold will you need to mix with 23 grams of copper to make jewelry-gold?
Reasoning about proportional relationships

Jewelry-gold is made by mixing pure gold with copper in the ratio 7 to 5. How much pure gold will you need to mix with 23 grams of copper to make jewelry-gold?

One solution method: set up and solve the following proportion

\[
\frac{7}{5} = \frac{X}{23}
\]
A long fence needs to be painted. If 3 people paint the fence, it will take 24 hours. How long will it take to paint the fence if 12 people paint? Assume that all the painters work at the same time and work at the same steady rate.

Is it legitimate to set up and solve the following proportion?

\[
\frac{3}{24} = \frac{12}{X}
\]
Develop students’ conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.

Encourage students to use visual representations to solve ratio, rate, and proportion problems.
Jewelry-gold is made by mixing pure gold with copper in the ratio 7 to 5. How much pure gold will you need to mix with 23 grams of copper to make jewelry-gold?

How can students reason to solve the problem *without* setting up and solving the proportion

\[
\frac{7}{5} = \frac{X}{23}
\]

How can students solve the problem by reasoning about relationships among quantities?
You might have thought about making batches of jewelry gold:

- 1 batch has 7 grams of gold and 5 grams of copper. You need $\frac{23}{5}$ batches . . .

$$\frac{23}{5} \cdot 7$$

- 1 batch has $\frac{7}{5}$ grams of gold and 1 gram of copper. You need 23 of those “unit-rate batches” . . .

$$23 \cdot \frac{7}{5}$$
Variable parts methods

You might have thought about the jewelry gold as 7 parts gold, 5 parts copper.

\[
\frac{23}{5} \quad \frac{23}{5} \quad \frac{23}{5} \quad \frac{23}{5} \quad \frac{23}{5} \quad \frac{23}{5} \quad \frac{23}{5}
\]

\[\text{? grams}\]

- 23 grams ÷ 5 parts = \( \frac{23}{5} \) grams per part

This approach has been largely overlooked in mathematics education research.
See if you can use this approach to get these solutions:

\[ 7 \cdot \frac{23}{5} \quad \frac{7}{5} \cdot 23 \]
Variable parts methods

- 23 grams ÷ 5 parts = 23/5 grams per part

23/5 grams per part

pure gold:

23 grams

23/5 grams

23/5 grams

23/5 grams

23/5 grams

23/5 grams

23/5 grams

23/5 grams

23/5 grams

23/5 grams

23/5 grams

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23/5 grams
Variable parts methods

7 parts \cdot \frac{23}{5} \text{ grams per part} = 7 \cdot \frac{23}{5} \text{ grams}

pure gold:  
\begin{align*}
\frac{23}{5} & \quad \frac{23}{5} & \quad \frac{23}{5} & \quad \frac{23}{5} & \quad \frac{23}{5} & \quad \frac{23}{5} & \quad \frac{23}{5}
\end{align*}

copper:  
\begin{align*}
\frac{23}{5} & \quad \frac{23}{5} & \quad \frac{23}{5} & \quad \frac{23}{5} & \quad \frac{23}{5}
\end{align*}

23 grams ÷ 5 parts

= \frac{23}{5} \text{ grams per part}
Is there another way?

23 grams

pure gold:

? grams

copper:
Variable parts methods

23 grams

pure gold: ? grams

copper: 23 grams

7/5
Variable parts methods

7/5 \cdot 23 \text{ grams}

pure gold:

23 \text{ grams}

copper:
Slope from a variable-parts perspective
Slope from a variable-parts perspective

\[ \text{cups peach juice} \]
\[ \text{cups grape juice} \]
\[ x \]
\[ y \]
\[ 3/2 \cdot \]

\[ \text{cups peach juice} \]
\[ \text{cups grape juice} \]
\[ x \]
\[ y \]
\[ 3/2 \cdot \]
Slope from a more traditional perspective

cups peach juice

1

3/2

cups grape juice

1

3/2

3/2

1
Slope from a more traditional perspective

$y = x \cdot \frac{3}{2}$
The *variable parts* perspective as a foundation for understanding the derivative?

*Variable parts:*

slope of secant line is 1.25 because

$\Delta y$ is 1.25 of $\Delta x$
What about the mathematics teaching profession?

Many of us share a vision where mathematical reasoning and sense-making play a central role in teaching and learning. Where:

- students engage with mathematical ideas and use ideas in developing skills;

- students make sense of problems, persevere in solving them, and recognize that problems can often be solved correctly in different ways;

- students think and reason, discuss their ideas, construct arguments, and examine their arguments critically.
What about the mathematics teaching profession?

But:

- There is a lot to know to teach math well;

- We know a lot from mathematics education research about how to teach math effectively, especially at the early grades;

- But currently, we often do not implement what we know. Why not?
We need a broader sense of community around mathematics teaching

All of us who teach math should see ourselves as a *professional* community. We should

- own and control the mathematics teaching profession;
- take collective responsibility for math teaching at all levels;
- seek to develop consensus on our practices;
- demand high standards for entry into *our* community.
Thank you!

Comments or questions?

Research group webpage:

http://temrrg.wix.com/temrrg

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References


