How prospective grades 6 – 8 teachers use two definitions of ratio

Sybilla Beckmann\textsuperscript{1} and Andrew Izsák\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, University of Georgia
\textsuperscript{2}Department of Mathematics and Science Education, University of Georgia

Joint AMS MAA meetings, January 2013
Other members of our research team

Erik Jacobson
Eun Jung
Eun Kyung Kang
Burak Ölmez
Muhammet Arican
There is a substantial body of research on children’s thinking and learning about ratio and proportional relationships.

- Much is known about difficulties children have, e.g.,
  - attending to only one quantity,
  - comparing additively rather than multiplicatively,
  - applying proportions inappropriately,
  - not understanding ratios as measures of intensive quantities, ...  

- Productive ways that children can reason are known, e.g.,
  - reasoning about coordinated values in tables, on double number lines
Research on ratio and proportional relationships

There is a substantial body of research on children’s thinking and learning about ratio and proportional relationships.

- Much is known about difficulties children have, e.g.,
  - attending to only one quantity,
  - comparing additively rather than multiplicatively,
  - applying proportions inappropriately,
  - not understanding ratios as measures of intensive quantities, . . .

- Productive ways that children can reason are known, e.g.,
  - reasoning about coordinated values in tables, on double number lines
Research on ratio and proportional relationships

Much less is known about pre-service and in-service teachers’ thinking and learning about ratio and proportional relationships.

Different researchers have viewed ratio and rate in different ways.
A framework for ratio and proportional relationships

We propose a mathematical analysis of ratio and proportional relationships that reveals two perspectives/definitions for ratio:

- “variable numbers of fixed amounts” or “replicating batches” — well-studied
- “fixed numbers of variable parts” or “dilating parts” — largely overlooked

These perspectives/definitions parallel the two quantitative perspectives/definitions for division:

- “how many in each group?” (partitive division)
- “how many groups?” (measurement division)
A framework for ratio and proportional relationships

We propose a mathematical analysis of ratio and proportional relationships that reveals two perspectives/definitions for ratio:

- “variable numbers of fixed amounts” or “replicating batches” well-studied
- “fixed numbers of variable parts” or “dilating parts” largely overlooked

These perspectives/definitions parallel the two quantitative perspectives/definitions for division:

- “how many in each group?” (partitive division)
- “how many groups?” (measurement division)
We propose a mathematical analysis of ratio and proportional relationships that reveals two perspectives/definitions for ratio:

- “variable numbers of fixed amounts” or “replicating batches” — well-studied
- “fixed numbers of variable parts” or “dilating parts” — largely overlooked

These perspectives/definitions parallel the two quantitative perspectives/definitions for division:

- “how many in each group?” (partitive division)
- “how many groups?” (measurement division)
Peach and grape juice mixed in a 3 to 2 ratio

To make Peach Punch, mix 3 cups peach juice with every 2 cups grape juice.

<table>
<thead>
<tr>
<th>batches</th>
<th>cups peach</th>
<th>cups grape</th>
<th>cups total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>6</td>
<td>15</td>
</tr>
</tbody>
</table>
Peach and grape juice mixed in a 3 to 2 ratio

To make Peach Punch, mix 3 parts peach juice with 2 parts grape juice.

<table>
<thead>
<tr>
<th>cups per part</th>
<th>cups peach</th>
<th>cups grape</th>
<th>cups total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>6</td>
<td>15</td>
</tr>
</tbody>
</table>
The mathematical analysis

\[ M \cdot N = P \]

\((\# \text{ of groups}) \cdot (\# \text{ of units in each/one group}) = (\# \text{ of units in } M \text{ groups})\)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M \cdot N = x)</td>
<td>Multiplication&lt;br&gt;Unknown product</td>
</tr>
<tr>
<td>(M \cdot x = P)</td>
<td>Division “How many in one group?”</td>
</tr>
<tr>
<td>(x \cdot N = P)</td>
<td>Division “How many groups?”</td>
</tr>
<tr>
<td>(x \cdot y = P)</td>
<td>Inversely proportional relationship</td>
</tr>
<tr>
<td>(x \cdot N = y)</td>
<td>Proportional relationship&lt;br&gt;“Variable numbers of fixed amounts”</td>
</tr>
<tr>
<td>(M \cdot x = y)</td>
<td>Proportional relationship&lt;br&gt;“Fixed numbers of variable parts”</td>
</tr>
</tbody>
</table>
The mathematical analysis

\[ M \cdot N = P \]

(\# of groups) \cdot (\# of units in each/one group) = (\# of units in \( M \) groups)

\[
\begin{align*}
M \cdot N &= x \\
\text{Multiplication} & \quad \text{Unknown product} \\
M \cdot x &= P \\
\text{Division "How many in one group?"} \\
x \cdot N &= P \\
\text{Division "How many groups?"} \\
x \cdot y &= P \\
\text{Inversely proportional relationship} \\
x \cdot N &= y \\
\text{Proportional relationship} & \quad \text{“Variable numbers of fixed amounts”} \\
M \cdot x &= y \\
\text{Proportional relationship} & \quad \text{“Fixed numbers of variable parts”}
\end{align*}
\]
The mathematical analysis

\[ M \cdot N = P \]

\((\# \text{ of groups}) \cdot (\# \text{ of units in each/one group}) = (\# \text{ of units in } M \text{ groups})\)

<table>
<thead>
<tr>
<th>(M \cdot N = x)</th>
<th>(M \cdot x = P)</th>
<th>(x \cdot N = P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplication</td>
<td>Division “How many in one group?”</td>
<td>Division “How many groups?”</td>
</tr>
<tr>
<td>Unknown product</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(x \cdot y = P)</th>
<th>(x \cdot N = y)</th>
<th>(M \cdot x = y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inversely proportional relationship</td>
<td>Proportional relationship “Variable numbers of fixed amounts”</td>
<td>Proportional relationship “Fixed numbers of variable parts”</td>
</tr>
</tbody>
</table>
When are two quantities in the ratio $A$ to $B$?

The variable numbers of fixed amounts perspective

View $A$ units of the first quantity and $B$ units (possibly different) of the second quantity as forming a “composed unit” or a “batch.”

$$A \text{ units} \quad B \text{ units}$$

For every $A$ units present of the first quantity, there are $B$ units present of the second quantity.

$$r \cdot A \text{ units} \quad r \cdot B \text{ units}$$

$$x \cdot N = y$$

$$(r \cdot A) \cdot (B/A) = (r \cdot B)$$

The constant of proportionality has units “units of the second quantity per 1 unit of the first quantity”
When are two quantities in the ratio $A$ to $B$?

The variable numbers of fixed amounts perspective

View $A$ units of the first quantity and $B$ units (possibly different) of the second quantity as forming a “composed unit” or a “batch.”

\[
\begin{align*}
A \text{ units} & \quad B \text{ units} \\
\end{align*}
\]

For every $A$ units present of the first quantity, there are $B$ units present of the second quantity.

\[
\begin{align*}
r \cdot A \text{ units} & \quad r \cdot B \text{ units} \\
\end{align*}
\]

\[
x \cdot N = y \\
(r \cdot A) \cdot (B/A) = (r \cdot B)
\]

The constant of proportionality has units “units of the second quantity per 1 unit of the first quantity.”
When are two quantities in the ratio $A$ to $B$?

The variable numbers of fixed amounts perspective

View $A$ units of the first quantity and $B$ units (possibly different) of the second quantity as forming a “composed unit” or a “batch.”

\[
\begin{align*}
A \text{ units} & \quad B \text{ units} \\
(x \cdot N) &= y \\
(r \cdot A) \cdot (B/A) &= (r \cdot B)
\end{align*}
\]

The constant of proportionality has units “units of the second quantity per 1 unit of the first quantity”
When are two quantities in the ratio $A$ to $B$?

The fixed numbers of variable parts perspective

Consider a “part” to be of variable size.

$$A \text{ parts} \quad B \text{ parts}$$

A parts of the first quantity and $B$ parts of the second quantity are in the ratio $A$ to $B$.

$$A \cdot r \text{ units} \quad B \cdot r \text{ units}$$

$$M \cdot x = y$$

$$(B/A) \cdot (A \cdot r) = (B \cdot r)$$

The constant of proportionality can be seen as a unit-less scale factor.
When are two quantities in the ratio $A$ to $B$?

The fixed numbers of variable parts perspective

Consider a “part” to be of variable size.

\[ A \text{ parts } B \text{ parts} \]

$A$ parts of the first quantity and $B$ parts of the second quantity are in the ratio $A$ to $B$.

\[ A \cdot r \text{ units } B \cdot r \text{ units} \]

\[ M \cdot x = y \]
\[ (B/A) \cdot (A \cdot r) = (B \cdot r) \]

The constant of proportionality can be seen as a unit-less scale factor.
When are two quantities in the ratio $A$ to $B$?

The fixed numbers of variable parts perspective

Consider a “part” to be of variable size.

$A$ parts $B$ parts

$A$ parts of the first quantity and $B$ parts of the second quantity are in the ratio $A$ to $B$.

$A \cdot r$ units $B \cdot r$ units

$M \cdot x = y$

$(B/A) \cdot (A \cdot r) = (B \cdot r)$

The constant of proportionality can be seen as a unit-less scale factor.
Why use the fixed numbers of variable parts perspective?

Rectangles whose height to width is in a 2 to 3 ratio:
Why use the fixed numbers of variable parts perspective?

Ramps whose height to width is in a 3 to 4 ratio:

All parts are the same size and stretch or shrink together.
Interviewed

- 4 pairs of preservice middle school students
- 4 pairs of preservice secondary teachers
- 2 hour-long interviews with each pair

All were in third semester of 3 semester sequences of content for teachers courses

- number and operation
- geometry
- algebra

Instruction emphasized:

- the two perspectives on ratio
- solving problems
- reasoning with quantities
- using drawn models.
First Interview

Used mixture problems so that both perspectives could be applied

- Which perspectives do preservice teachers opt to use?
- How do number lines and strip diagrams support their reasoning?
- Where are they more and less proficient?
Sample tasks:

**Scenario**: A fragrant oil was made by mixing 3 milliliters lavender oil with 2 milliliters rose oil.

- What other amounts of lavender oil and rose oil can be mixed to make mixture that has exactly the same fragrance?

- How much lavender oil and how much rose oil do you need to make 35 mL of mixture?
Focus on students’ understandings of variable parts and strip diagrams

- Can students think about parts as variable?
- Are students able to use partitive division?
Sample tasks:

**Scenario**: 14-karat gold for jewelry is made by mixing pure gold with a metal alloy in the ratio 7 to 5.

Explain how to reason about a strip diagram to solve the following: How much pure gold and copper will you need to mix 65 grams of jewelry gold?
Preliminary Results

Similarities Across Pairs

- Students could explain and use both perspectives on ratio, at least to some extent

- Most students tended to use the batch perspective in combination with the variable parts perspective: they sometimes entered the number of batches into the parts of their strip diagrams
An Example
Mixing the “parts” and “batches” perspectives

How much lavender oil and how much rose oil do you need to make 35 mL of mixture?

\[
\begin{align*}
35 \text{ mL total} & \quad 7 \text{ mixtures is } 35 \text{ mL} \\
LO & \quad 7 \text{ mL} + 7 \text{ mL} + 7 \text{ mL} = 21 \text{ mL} \\
RO & \quad 7 \text{ mL} + 7 \text{ mL} = 14 \text{ mL} \\
& \quad 21 \text{ mL} + 14 \text{ mL} = 35 \text{ mL total}
\end{align*}
\]
Preliminary Results

Divergence Across Pairs

- Some students tended to focus on additive relationships; others focused on multiplicative relationships.

- Students’ facilities with partitive division constrained their capacity to apply the “parts” perspective.
How much gold and how much copper do you need to make 65 g of jewelry gold mixed in a 7 to 5 ratio?
An Example

Inproficient reasoning with division and the “parts” perspective

How much gold and how much copper do you need to make 65 g of jewelry gold mixed in a 7 to 5 ratio?
The two perspectives on ratio are accessible to preservice teachers

Separating the reasoning associated with the two perspectives takes effort

Applying the two perspectives on ratio pushes and reveals students’ capacities with division