1st Midterm Solution - Mathematical Methods

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1. Give a short answers (three lines) for each of the next questions.

   (a) Give the expression for the scalar product for any coordinate system.

   $$\vec{A}\cdot\vec{B}=\sum_{ij} g_{ij}A^i B^j, \quad \text{with} \quad g_{ij}=\sum_k \frac{\partial x_k}{\partial dq_i}\frac{\partial x_k}{\partial dq_j}$$

   (b) Wich fact does ensure that a transformation matrix is orthogonal?
   If S is real and $SS^T = I \Leftrightarrow S^T = S^{-1} \Rightarrow S$ is orthogonal.

   (c) Put down the Stokes's Theorem.

   $$\int_{\partial S}\vec{B}\cdot d\vec{r}=\int_S \vec{\nabla}\times\vec{B}\cdot d\vec{\sigma}$$

   (d) Put down the metric tensor for a curvilinear transformation.

   $$g_{ij}=\sum_k \frac{\partial x_k}{\partial dq_i}\frac{\partial x_k}{\partial dq_j}, \quad \text{with} \quad x_k = x_k(q_1, q_2, \cdots)$$

   (e) Give the formal definition for a Hilbert Space.
   It is a funtions space with adition (conmute and associate) and multiplication by scalar (conmute, associate and distribute) both closed; generated by an infinite number of numerable funtions and a well defined escalar product.

   $$|f\rangle=\sum_j a_j|\varphi_j\rangle=\sum_j|\varphi_j\rangle\langle\varphi_j|f\rangle=\left(\sum_j|\varphi_j\rangle\langle\varphi_j|\right)|f\rangle$$

2. Shows that:

$$\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}-\vec{B}(\vec{\nabla}\cdot\vec{A})+\vec{A}(\vec{\nabla}\cdot\vec{B})$$

$$\begin{aligned}
[\vec{\nabla}\times(\vec{A}\times\vec{B})]_i &= \varepsilon_{ijk}\partial_j(\varepsilon_{klm}A_l B_m)\\
&= \varepsilon_{ijk}\varepsilon_{klm}(\partial_j(A_l B_m))\\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})(A_l(\partial_j B_m)+B_m(\partial_j A_l))\\
&= \delta_{il}\delta_{jm}(A_l(\partial_j B_m))+\delta_{il}\delta_{jm}(B_m(\partial_j A_l))-\delta_{im}\delta_{jl}(A_l(\partial_j B_m))-\delta_{im}\delta_{jl}(B_m(\partial_j A_l))\\
&= A_i(\partial_j B_j)+(B_j\partial_j)A_i-(A_j\partial_j)B_i-B_i(\partial_j A_j)\\
&= (B_j\partial_j)A_i-(A_j\partial_j)B_i-B_i(\partial_j A_j)+A_i(\partial_j B_j)\\
&= (\vec{B}\cdot\vec{\nabla})A_i-(\vec{A}\cdot\vec{\nabla})B_i-B_i(\vec{\nabla}\cdot\vec{A})+A_i(\vec{\nabla}\cdot\vec{B})
\end{aligned}$$

Scaling both sides by $\hat{e}_i$,

$$\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}-\vec{B}(\vec{\nabla}\cdot\vec{A})+\vec{A}(\vec{\nabla}\cdot\vec{B})$$

Q.E.D.

3. Shows that:

(a) $\vec{\nabla}\cdot(k\phi\vec{A}) = k[(\vec{\nabla}\phi)\cdot\vec{A} + \phi\vec{\nabla}\cdot\vec{A}]$; k constant, ϕ escalar field, $\vec{A}$ vectorial field.

$$\begin{aligned}\vec{\nabla}\cdot(k\phi\vec{A}) &= \sum_i \partial_i(k\phi A_i)\\ &= k\sum_i \big((\partial_i\phi)A_i + \phi\partial_i A_i\big) && k \text{ does not sums over } i \text{ and by the chain rule}\\ &= k\Big[\sum_i(\partial_i\phi)A_i + \phi\sum_i \partial_i A_i\Big]\\ &= k\big[(\vec{\nabla}\phi)\cdot\vec{A} + \phi\nabla\cdot\vec{A}\,\big]\end{aligned}$$

Q.E.D.

(b) Div of gravitation force equals zero.

$$\vec{F}_g = -\frac{GMm}{r^2}\hat{r} \tag{1}$$

But $\hat{r} = \vec{r}/r$ and doing $k = -GMm$,

$$\vec{\nabla}\cdot\vec{F}_g = \vec{\nabla}\cdot\big(k(r^{-3})\vec{r}\big) = k\big[(\vec{\nabla}r^{-3})\cdot\vec{r} + r^{-3}\vec{\nabla}\cdot\vec{r}\,\big], \quad \text{by (a)} \tag{2}$$

Computing $(\vec{\nabla}r^{-3})\cdot\vec{r}$,

$$\begin{aligned}(\vec{\nabla}r^{-3})\cdot\vec{r} &= \Big[(\,\mathbf{i}\partial_x + \mathbf{j}\partial_y + \mathbf{k}\partial_z)(x^2+y^2+z^2)^{-3/2}\Big]\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\\ &= -\frac{3}{2}(2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})(x^2+y^2+z^2)^{-5/2}\\ &= -3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2}\\ &= -3r^{-3}\end{aligned} \tag{3}$$

Computing $r^{-3}\vec{\nabla}\cdot\vec{r}$,

$$\begin{aligned}r^{-3}\vec{\nabla}\cdot\vec{r} &= r^{-3}(\,\mathbf{i}\partial_x + \mathbf{j}\partial_y + \mathbf{k}\partial_z)\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\\ &= 3r^{-3}\end{aligned} \tag{4}$$

Then, putting (3) and (4) into (2),

$$\vec{\nabla}\cdot\vec{F}_g = k[-3r^{-3} + 3r^{-3}] = 0$$

Q.E.D.

4. Shows that:

$$\int_V \vec{\nabla}\phi\cdot\vec{A}\;d\tau = \int_{\partial V}\phi\vec{A}\cdot d\vec{\sigma} - \int_V \phi\vec{\nabla}\cdot\vec{A}\;d\tau$$

By the Gauß Theorem,

$$\int_V \vec{\nabla}\cdot\vec{B}\;d\tau = \int_{\partial V}\vec{B}\cdot d\vec{\sigma}$$

Let $\vec{B} = \phi\vec{A}$. Then,

$$\int_V \vec{\nabla}\cdot(\phi\vec{A})\;d\tau = \int_{\partial V}\phi\vec{A}\cdot d\vec{\sigma}$$

By the chain rule,

$$\int_V (\vec{\nabla}\phi\cdot\vec{A} + \phi\vec{\nabla}\cdot\vec{A})\;d\tau = \int_{\partial V}\phi\vec{A}\cdot d\vec{\sigma}$$

Or,

$$\int_V \vec{\nabla}\phi\cdot\vec{A}\;d\tau = \int_{\partial V}\phi\vec{A}\cdot d\vec{\sigma} - \int_V \phi\vec{\nabla}\cdot\vec{A}\;d\tau$$

Q.E.D.