Modeling Uncertainty in Linear Programs: Stochastic and Robust Programming

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Introduction and motivations

In real life, Linear Programs are uncertain for several reasons:

- **Estimation errors**: parameters calculated from statistic estimations.
- **Measure errors**: for measured parameters.
- **A posteriori known parameters**: (prices, demands, . . .)
- **Execution errors**: limited precision to execute an optimal solution.

⇒ Goal: Predict or limit the influence of these noises. Integrate these uncertainty in the decisional process.
Decisional context

Several decisional contexts are possible:

- **Static context**: all the uncertainty is raised in the same time.
  2 stages: before and after knowing the realization

- **Dynamic context**: decisions have to be reactualized when a random event is realized.

We can decide a first solution value, and we modify it after the realization to make it feasible or to improve the objective cost.

First decision must be taken:

- In order to be fitted to most of the cases, minimizing an average cost: *Stochastic approach*.

- To be covered against the worst realization: *Robust approach*. 

Example: scheduling nuclear plan outages

\( d_{i,k,w} \): decisions on the dates of the outages, decisional values.
\( x_{i,k,w} \): decisions on the production, to fulfill the demand after decisions of outages are taken.
Deterministic linear problem

\[
\min \sum_{i,k} C_{\text{reload}}^{i,k} d_{i,k,w} + \sum_{w} C_{\text{flexP}}^{w} \left( \text{DEM}^{w} - \sum_{i,k} \text{PMAX}_{i} x_{i,k,w} \right)
\]

\[
\forall i, k, w, \quad d_{i,k,w} \geq d_{i,k,w-1} \tag{1}
\]

\[
\forall i, k, w, \quad x_{i,k,w} \leq d_{i,k,w-D\text{A}_{i,k}} - d_{i,k+1,w} \tag{2}
\]

\[
\forall i, k, \quad L\text{min}_{i,k} \leq \sum_{w} x_{i,k,w} \leq L\text{max}_{i,k} \tag{3}
\]

\[
\forall w, \quad 0 \leq \text{DEM}^{w} - \sum_{i,k} \text{PMAX}_{i} x_{i,k,w} \leq \text{Pflex}_{w} \tag{4}
\]

\[
\forall c, w \in [\text{ID}_{c}, \text{IF}_{c}], \quad \sum_{i \in A_{c}} (d_{i,k,w-L_{c,i}} - d_{i,k,w-L_{c,i-TU_{c,i}}}) \leq Q_{c} \tag{5}
\]

\(d_{i,k,w}\): decisions on the dates of the outages, decisonal values.
\(x_{i,k,w}\): decisions on the production.
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2. Stochastic Linear Programming
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   - Resolution: L-shaped method
   - Dynamic Stochastic Programming
   - Multistage stochastic programming

3. Robust Linear Programming
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   - Uncertainty in the objective function
   - Uncertainty in the matrix coefficients
   - Uncertainty in the RHS
   - 2-stage Robust Programming

4. Conclusion
A general formulation for these problems are:

$$\min c^T x + \mathbb{E}_{\xi} \left[ \min_y q(\xi)^T y(\xi) \right]$$

s.t. 
$$Ax = b$$
$$T(\xi)x + W(\xi)y(\xi) = h(\xi)$$
$$x \geq 0, y(\xi) \geq 0,$$

Datas \(A, b, c\) are deterministic, while \(q(\xi), T(\xi), W(\xi)\) depends on random events \(\xi\).
First stage decisions, "here and now" variables \(x\), independant of \(\xi\).
Second stage decisions, "wait and see" recourse variables \(y(\xi)\), dependant of \(\xi\).
For discrete stochasticity and we can enumerate the set of scenarios $S$, this leads to a big Linear Program:

$$\min_{x,y^s} \ c^T x + \sum_{s \in S} \pi_s q_s^T y^s$$

subject to:
$$Ax = b$$
$$T^s x + W^s y^s = h^s, \quad s \in S$$
$$x \geq 0, \ y^s \geq 0, \ s \in S$$

(7)
Stochastic formulation

\[
\min \sum_{i,k} C_{\text{reload}}^{i,k} d_{i,k,w} + \sum_{w,s} \pi^s C_{\text{flexP}}^{w,s} P_{w,s}
\]

\(\forall i, k, w, \quad d_{i,k,w} \geq d_{i,k,w-1}\)  
(8)

\(\forall i, k, w, s, \quad x_{i,k,w,s} \leq d_{i,k,w-DA_{i,k,s}} - d_{i,k+1,w}\)  
(9)

\(\forall i, k, s, \quad L_{\text{min}}_{i,k,s} \leq \sum_{w} x_{i,k,w,s} \leq L_{\text{max}}_{i,k,s}\)  
(10)

\(\forall w, s \quad 0 \leq P_{w,s} \leq P_{\text{flex}}_{w,s}\)  
(11)

\(\forall w, s, \quad \sum_{i,k} P_{\text{MAX}}_{i,s} x_{i,k,w,s} + P_{w,s} = DEM^{w,s}_{w,s}\)  
(12)

\(\forall c, w \in [ID_c, IF_c], \quad \sum_{i \in A_c} (d_{i,k,w-L_{c,i,s}} - d_{i,k,w-L_{c,i,s}-TU_{c,i,s}}) \leq Q_{c}\)  
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**Notations**

\[
\begin{align*}
\min & \quad c^T x + Q(x) \\
\text{s.t.} & \quad Ax \geq b, \quad x \in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \\
\end{align*}
\]

where \[ Q(x) = \sum_{k=1}^{K} p_k Q_k(x) \]

and \[ Q_k(x) = \min \quad q_k^T y \]
\[ W_k y \geq h_k - T_k x \]
\[ y \in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{p_2} \]
Basic ideas

- $Q(x)$ is convex.
- We approximate $Q(x)$ with cutting planes of its convex hull: it gives lower bounds.
- We generate cutting planes in a Benders’ decomposition scheme.
- Master problem with the cutting plane approximation, slave problem to generate feasibility and optimality cuts.
General Benders’ decomposition scheme

Solve
\[
\begin{align*}
\min & \quad c^T x + \hat{Q}^i(x) \\
\text{s.t.} & \quad Ax \geq b, \quad x \in \mathbb{R}^{n_1-p_1} \times \mathbb{Z}^{p_1}
\end{align*}
\]

\(LB\) and \(x^i\)

Evaluate \(c^T x^i + Q(x^i)\)

\(UB\) and \(\hat{Q}^{i+1}(x)\)

Refinement

\(i \leftarrow i + 1\)

\(\hat{Q}(x)\) is computationally tractable lower bounding approximation of \(Q(x)\)

Evaluation is via decomposition

\(UB - LB < \epsilon\)?

Yes

STOP

No
Here \( X = \{x^1, \ldots, x^M\} \subset \{0, 1\}^{n_1} \)

Let \( L \leq Q(x) \quad \forall \ x \in X \)

Denote \( J^m = \{j : x^m_j = 1\} \quad m = 1, \ldots, M. \)

Let
\[
\phi_m(x) = (Q(x^m) - L) \left( \sum_{j \in J^m} x_j - \sum_{j \not\in J^m} x_j \right) - (Q(x^m) - L)(|J^m| - 1) + L
\]

Note
\[
\phi_m(x) \begin{cases} 
= Q(x^m) & \text{if } x = x^m \\
\leq L & \text{if } x \in X \setminus \{x^m\}
\end{cases}
\]

Then \( Q(x) = \max_{m=1,\ldots,M} \{\phi_m(x)\} \quad \forall \ x \in X \)

and \( \hat{Q}^i(x) = \max_{m=1,\ldots,i-1} \{\phi_m(x)\} \)
Integer L-shaped decomposition scheme

\[
\begin{align*}
\min & \quad \mathbf{c}^T \mathbf{x} + \hat{Q}^i(\mathbf{x}) \\
\text{s.t.} & \quad \mathbf{x} \in X \\
\end{align*}
\]

\[
\begin{align*}
\min & \quad \mathbf{c}^T \mathbf{x} + \theta \\
\text{s.t.} & \quad \mathbf{x} \in X \\
\quad & \quad \theta \geq \phi_m(\mathbf{x}) \quad m = 1, \ldots, i - 1
\end{align*}
\]

Master Problem

\[LB\text{ and }x^i\]

\[
\text{Evaluate } \mathbf{c}^T x^i + Q(x^i)
\]

\[UB\text{ and }\phi_i(\mathbf{x})\]

Refinement

\[i \leftarrow i + 1\]

Cuts

Evaluation is via decomposition

\[UB - LB < \epsilon\]

No

Yes

STOP
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Dynamic case: non anticipativity constraints

- Let two scenarios $s_1$ and $s_2$ equal on the first $t_0$ time steps.
- Decisions $x_{i,k,t,s_1}$ and $x_{i,k,t,s_2}$ could be different for $t < t_0$ in the last static approach.
- How could we have decided this at time $t$, when it was impossible to make the difference between these two scenarios? It is anticipative.
- *Non anticipativity constraints:* $x_{i,k,t,s_1} = x_{i,k,t,s_2}$ for $t \leq t_0$.

$\implies$ It is equivalent to consider a scenario tree to group these decisions. Decisions are indexed on nodes of this tree.
Scenario trees
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4. Conclusion
These problems have the following dynamic structure:

\[
\begin{align*}
\min_{x_0, \ldots, x_N} & \quad c_0 \cdot x_0 + \mathbb{E}_{x_1} [c_1 \cdot x_1 + \mathbb{E}_{x_2} [\cdots + \mathbb{E}_{x_N} [c_N \cdot x_N]]] \\
\text{s.t.} & \quad Ax_0 \leq b \\
& \quad T_i \cdot x_{i-1} + W_i \cdot x_i \leq h_i, \forall i \in \mathcal{N}^* \\
& \quad x_i \geq 0 
\end{align*}
\]

(14)

\(\mathcal{N}\): set of nodes of the scenario tree (the root is 0)
\(x_0\): decision before every realization.
\(x_i\) decision before knowing node \(i\) but assuming \(i - 1\).
These problems have the following structure:

\[
\begin{align*}
\min_{x_0, \ldots, x_N} & \quad c_0 x_0 + \sum_{i \in \mathcal{N}^*} \pi_i c_i x_i \\
\text{s.t.} & \quad Ax_0 \leq b \\
& \quad T_i x_{i-1} + W_i x_i \leq h_i, \forall i \in \mathcal{N}^* \\
& \quad x_i \geq 0
\end{align*}
\] (15)

\(\mathcal{N}\): set of nodes of the scenario tree (the root is 0)
\(x_0\): decision before every realization.
\(x_i\) decision before knowing node \(i\) but assuming \(i - 1\).
Typical examples

- Finance portfolio problems for purchase/sales strategies.
- Can be applied for our problem to model dynamicity, but too many recourses and variables. Difficulty because decision on a whole planning, not only on a ponctual decision at time steps.

⇒ The result is not a single solution, but a strategy, set of solution depending on the random realization.
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What is robustness?

Classical definitions for robust solutions:

- Solution with margins? Adding slacks. (”light robustness”)
- Solution resistant to ”approximations” or ”ignorance zones” to avoid bad impacts.
- Solution which do not change ”in a significant way” when the vector of decision is slightly perturbed.
- Solution ”acceptable” in a large number of scenarios and which is never ”too bad”.

⇒ The decisional context has to precise the definition.
Uncertainty sets

**Definition**: predefinite set of random events which we want to be covered from.

**Classical uncertainty sets**:

- Cartesian product of intervals
- Polyhedrons
- Ellipsoids
Uncertainty sets for our example

We assume that outage prolongations are independent random events. The uncertainty of the outage \((i, k)\) prolongation is in \([0, DA_{i,k}]\).

\[
\Omega_{\text{worstCase}} = \prod_{i,k} [0, DA_{i,k}]
\]

\[
\Omega^\Gamma_{\text{budget}} = \left\{ (da_{i,k}) \in \prod_{i,k} [0, DA_{i,k}] \mid \sum_{i,k} da_{i,k} \leq \Gamma \right\}
\]

We can decompose prolongations into nominal prolongation in \([0, DA^0_{i,k}]\), extreme prolongations in \([1 + DA^0_{i,k}, DA_{i,k}]\):

\[
\Omega^N_{\text{ext}} = \left\{ (da_{i,k}) \mid \exists (\varepsilon_{i,k}) \in \{0, 1\}, \forall i, k, da_{i,k} \leq DA^0_{i,k} + (DA_{i,k} - DA^0_{i,k})\varepsilon_{i,k} \text{ et } \sum_{i,k} \varepsilon_{i,k} \leq N \right\}
\]
Robust framework

Considering a linear program with uncertainty $\omega \in \Omega$:

$$\min_{d,x} c_D(\omega)d + c_X(\omega)x$$

$$s.t.: \quad Ad \leq B$$
$$\quad T(\omega)d + W(\omega)x \leq h(\omega)$$
$$\quad d, x \geq 0$$

(16)

We consider following class or problems

$$\min_{d,x} \max_{\omega \in \Omega} f(d, x, \omega)$$

$$s.t.: \quad Ad \leq B$$
$$\quad T(\omega)d + W(\omega)x \leq h(\omega)$$
$$\quad d, x \geq 0$$

(17)

where $f(d, x, \omega)$ is an objective to define, various robust criteria.
**Game theory interpretation**

- Min-max problem
- We are the first player, we play on the decisions $x, d$.
- Randomness is the opponent, it plays on the random realization $\omega$ to penalize the decision $x, d$.
- Our goal is to anticipate the worst case for us, the best choice of the opponent, after our choice on $x, d$. 

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Easiest case: uncertainty does not influence feasibility

\[
\min_x C(\omega)x \\
\text{s.t.} \quad Ax = b \\
x \geq 0
\]  
(18)
Worst case objective

Minimization problem of the worst case which can happen, \( f(X, \omega) = C(\omega).X \):

\[
\min_x \max_{\omega \in \Omega} C(\omega)x \\
\text{s.t.: } Ax = b \\
x \geq 0
\]  

(19)

Very conservative approach.
Maximal regret

Minimization problem of the gap between the robust solution and the deterministic solution assuming an alea.

\[ f(X, \omega) = C(\omega)X - C(\omega)X^*(\omega). \]

\[
\min_x \max_{\omega \in \Omega} \min_{x^*(\omega)} c(\omega)x - c(\omega)x^*(\omega) \\
\text{s.t.} \quad Ax = b \\
Ax(\omega) = b, \forall \omega \\
X, X^*(\omega) \geq 0
\]

Advantage: favorite objective for risk averse managers.

Drawback: This leads to difficult problems. (Cutting planes algorithm of Shimizu and Aiyoshi)
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Robust Linear Programming

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Soyster’s approach: column uncertainty

Every column $A_j \in K_j$, with $K_j$ convex in $\mathbb{R}^m$

$$\min_x c x$$
$$s.t: \sum_j A_j x_j \leq b$$
$$x \geq 0$$ (21)

The robust problem is equivalent to:

$$\min_x c x$$
$$s.t: \sum_j \bar{A}_j x_j \leq b$$
$$x \geq 0$$ (22)

with $\bar{A}_{i,j} = \max_{A_j \in K_j} A_{i,j}$
Soyster’s approach with interval uncertainty

If $K_j$ is an interval, it is equivalent to:

$$\min_x cx \quad s.t. \quad \sum_j \min A_{i,j} x_j \leq b_i, \forall i$$

(23)

⇒ Worst case conservative approach.

⇒ There is often a too big gap between these conservative solutions and nominal solution.

⇒ This lead to parametric approaches: we do not want anymore to be covered against ALL the risks in the uncertainty set, but having a parametrization to ponderate this gap.
Parametric approaches

Idea: extreme scenario for all random event are unusual.

Ben Tal and Nimerovski approach: maximal deviations on a constraint $i$ is in an ellipsoid defined with par:

$$
\Xi_i(\Omega_i) = \{\xi_{i,j} \in [1, 1]^n | \sqrt{\sum_j \xi_{ij}^2} \leq \Omega_i\}.
$$

Bertsimas and Sim approach: $\Gamma_i$ for a constraint $i$ is the total deviation to nominal values in constraint $i$. Uncertainty set is $\Phi_i(\Gamma_i)$ is thus:

$$
\Phi_i(\Gamma_i) = \{\xi_{i,j} \in [1, 1]^n | \sum_j |\xi_{ij}| \leq \Gamma_i\}.
$$

It leads to linear programs. $\Gamma_i = n$ is Soyster’s approach.
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4. Conclusion
A special case of column uncertainty

\[
\begin{aligned}
\min_x c.x \\
\text{s.t.:} & \quad \sum_{j=1}^n A_j x_j - B x_{n+1} \leq 0 \\
& \quad x_{n+1} = 1 \\
& \quad x \geq 0
\end{aligned}
\]

(24)

Last general approaches are used for this particular cases.
Is it possible to use the special structure?

- Not a lot of papers on this field.
- The dual problem of an uncertain RHS Linear Program is a Linear Program with costs uncertainty.
- Remli PhD thesis: results for inequality constraints.
- Equality constraints: needs to a penalty model.
- Minoux: results on PERT scheduling with uncertain RHS.
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4. Conclusion
Analogously to the first stage and second stage decision of Stochastic Programming.

\[
\begin{align*}
\min_d \max_{\omega \in \Omega} \min_{x(\omega)} \quad & c_D(\omega) \cdot d + c_X(\omega) \cdot x(\omega) \\
\text{s.c.:} \quad & Ad \leq b \\
& T(\omega) \cdot d + W(\omega) \cdot x(\omega) \leq h(\omega) \\
& d, x(\omega) \in \{0, 1\}
\end{align*}
\]

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In our game theory illustration, we can play second stage decision after the opponent played the stochastic event. Second stage variable are recourse variables.
Related works

- Minoux, Thiele: several studies on 2 stage robust programming with RHS uncertainty
- Remli PhD thesis: application for 2 stage decision localisation-transportation problem.
- Atamturk and Zhang: Two-Stage Robust Network Flow and Design Under Demand Uncertainty.
- Bertsimas and Litvinov: Security Constrained Unit Commitment
- My future PhD work on scheduling nuclear maintenance outages?
First example

\[
\min_{d_i,k,w} \max_{\alpha \in \Omega} \min_{x_i,k,w,\alpha} \sum_{i,k} C_{\text{reload}}^{i,k} d_{i,k,w} + \sum_{w} C_{\text{flex}}^{w} \left( DEM^{w} - \sum_{i,k} \text{PMA}X_{i,k,w,\alpha}\right)
\]

\[
\forall i, k, w, \quad d_{i,k,w} \geq d_{i,k,w-1}
\]

\[
\forall i, k, w, \alpha, \quad x_{i,k,w,\alpha} \leq d_{i,k,w-\text{DA}_{i,k,\alpha}} - d_{i,k+1,w}
\]

\[
\forall i, k, \alpha \quad L_{\text{min},i,k} \leq \sum_{w} x_{i,k,w,\alpha} \leq L_{\text{max},i,k}
\]

\[
\forall w, \alpha, \quad 0 \leq DEM^{w} - \sum_{i,k} \text{PMA}X_{i,k,w,\alpha} \leq \text{Pflex}_{w}
\]

\[
\forall c, \alpha, w \in [\text{ID}_c, \text{IF}_c], \quad \sum_{i \in A_c} (d_{i,k,w-L_{c,i}} - d_{i,k,w-L_{c,i}} - \text{TU}_{c,i,\alpha}) \leq Q_c
\]
Linearization

$$\min_{d_{i,k,w}, C} \sum_{i,k} C_{i,k}^{i,k} d_{i,k,w} + C$$

$$\forall i, k, w, \quad d_{i,k,w} - 1 \leq d_{i,k,w}$$

$$\forall \alpha \in \Omega, \quad \sum_{w} C_{\text{flexP}}^{w} \left( \text{DEM}^{w} - \sum_{i,k} \text{PMA}x_{i,k,w,\alpha} \right) \leq C$$

$$\forall i, k, w, \alpha, \quad x_{i,k,w,\alpha} \leq d_{i,k,w} - \text{DA}_{i,k,\alpha} - d_{i,k+1,w}$$

$$\forall i, k, \alpha, \quad \text{Lmin}_{i,k} \leq \sum_{w} x_{i,k,w,\alpha} \leq \text{Lmax}_{i,k}$$

$$\forall w, \alpha, \quad 0 \leq \text{DEM}^{w} - \sum_{i,k} \text{PMA}x_{i,k,w,\alpha} \leq \text{Pflex}^{w}$$

$$\forall c, \alpha, w \in [\text{ID}_{c}, \text{IF}_{c}], \quad \sum_{i \in A_{c}} (d_{i,k,w} - L_{c,i} - d_{i,k,w} - L_{c,i} - \text{TU}_{c,i,\alpha}) \leq Q_{c}$$
### Adding recourse to the first stage decisions

\[
\begin{align*}
\min_{d_{i,k,w}} & \quad \max_{\alpha \in \Omega} \min_{c_{i,k},\Delta} \sum_{i,k} C_{i,k,\text{reload}}^i + \sum_w C_{\text{flexP}}^w \left( DEM^w - \sum_{i,k} \text{PMAX}_i x_{i,k,w,\alpha} \right) + \sum_{i,k} C_{\text{org}} i c_{i,k,\alpha_i,k} \\
& \quad \forall i, k, w, \quad d_{i,k,w} \geq d_{i,k,w-1} \\
& \quad \forall i, k, w, \quad \Delta_{i,k,w,\alpha_i,k} \leq \Delta_{i,k,w-1,\alpha_i,k} \\
& \quad \forall i, k, w, \alpha \in \Omega^O, \quad \Delta_{i,k,w,\alpha} = d_{i,k,w} \\
& \quad \forall i, k, w, w', \quad \Delta_{i,k,w',\alpha_i,k} - d_{i,k,w'} \leq d_{i,k,w} \\
& \quad \forall i, k, w, w', \quad d_{i,k,w'} - \Delta_{i,k,w',\alpha_i,k} \leq d_{i,k,w} \\
& \quad \forall i, k, w, \alpha \in \Omega, \quad x_{i,k,w,\alpha} \leq d_{i,k,w-DA_{i,k,\alpha}} - d_{i,k+1,w} \\
& \quad \forall i, k, \alpha \in \Omega \quad L_{\text{min}i,k} \leq \sum_w x_{i,k,w,\alpha} \leq L_{\text{max}i,k} \\
& \quad \forall w, \alpha, \quad 0 \leq DEM^w - \sum_{i,k} \text{PMAX}_i x_{i,k,w,\alpha} \leq P_{\text{flex}}^w \\
& \quad \forall c, \alpha, w \in [ID_c, IF_c], \quad \sum_{i \in A_c} (d_{i,k,w-L_{c,i}} - d_{i,k,w-L_{c,i}-TU_{c,i,\alpha}}) \leq Q_c
\end{align*}
\]
Conclusion

Stochastic Linear Programming:
- Requires probabilistic distribution of scenarios: unknown in general applications.
- Decision on the average cost: less conservative solutions.
- Big size of Linear Programs. Generic resolution with Bender’s decomposition

Robust Linear Programming:
- Requires no probabilistic distribution of randomness: worst case approach.
- Importance of the choice of uncertainty sets.
- Different approach following the uncertain coefficient: objective / matrix / RHS.
- Min-max problems.
Bibliography


