ALGEBRAIC TEST FOR THE HURWITZ STABILITY OF A GIVEN SEGMENT OF POLYNOMIALS

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Abstract. For the robust stability analysis of a linear system, due to the nonconvexity of the set of Hurwitz stable polynomials, it is important to have available computational methods to verify the stability of a convex combination of polynomials. In this paper, given two Hurwitz stable polynomials \( p_0 \) and \( p_1 \), a simple algebraic test (a matrix inequality) for the stability of the segment of polynomials determined by \( p_0 \) and \( p_1 \) is proposed. Based on this result the problem of estimating of the minimum left extreme is addressed.

1. Introduction

Motivated by the robustness analysis of systems with uncertain parameters, different approaches to study the stability of segments of polynomials have been proposed ([4], [5], [8], [9], [16]). The question is to find conditions on the stable polynomials \( p_0(t) \) and \( p_1(t) \) such that the segment of polynomials described by \( p(t, \lambda) = \lambda p_0(t) + (1 - \lambda)p_1(t) \) is stable for all \( \lambda \in [0, 1] \). The first result where necessary and sufficient conditions were obtained was Bialas’s Theorem which establishes that if \( p_0 \) is Hurwitz stable and \( \deg(p_0) > \deg(p_1) \) then \( p(t, \lambda) \) is Hurwitz stable for all \( \lambda \in [0, 1] \) if and only if the matrix \( H^{-1}(p_0)H(p_1) \) has no eigenvalues in \( (-\infty, 0) \), where \( H(p) \) is the Hurwitz matrix of the polynomial \( p \) (see [2], [4] and [11]). A different approach in terms of the frequency domain which is known as the Segment Lemma was established by Chapellat and Bhattacharyya (see [3] and [9]). In this lemma the stability of \( p(t, \lambda) \) is equivalent to certain conditions that must be satisfied by the odd and even degree polynomials associated with the polynomials \( p_0(t) \) and \( p_1(t) \). On the other hand, a method to determine the stability of segments of complex polynomials was obtained by N. Bose and is known as Bose’s Test [6].

Based on the above criteria, several algorithms have been developed to test efficiently the stability of segments of polynomials. The Segment Lemma has been used to develop an algorithm in [8]. In the same direction, more recently, in [14] there was obtained a procedure to check the Hurwitz stability of convex combinations of polynomials in a finite number of operations. Related to Bose’s work [7], in [5] there is a test that can be used to determine the stability of segments of complex polynomials. Furthermore, in [16] there were obtained the well-known Rantzer conditions (see also [13]).


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Following the ideas exposed in [1] this work address the problem of obtaining simple algebraic conditions for checking the stability of a segment of polynomials. It is important to note that the approach proposed in this paper provides sufficient conditions used when deg \( p_0 = n \) and deg \( p_1 = n - 1, n - 2 \) in contrast to the Segment Lemma where it is supposed that deg \( p_0 = \text{deg} \ p_1 \). As can be seen in [13], it is not necessary to study the cases when deg\( (p_1(t)) < n - 2 \).

Our approach for the case deg \( p_0 = \text{deg} \ p_1 \) is as follows: Given a Hurwitz stable polynomial \( p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n \) which is the nominal polynomial, let \( p_1(t) = c_1 t^n + c_2 t^{n-1} + \cdots + c_{n+1} \) be an arbitrary polynomial of degree \( n \). Define the matrix \( E_{(n,n)} \in \mathbb{M}_{(n+1) \times (n+1)} \) by

\[
E_{(n,n)} = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-a_2 & a_1 & -1 & 0 & \cdots & 0 & 0 \\
a_4 & -a_3 & a_2 & -a_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{n-1} & -a_{n-2} \\
0 & 0 & 0 & 0 & \cdots & 0 & a_n \\
\end{pmatrix}.
\]

If the polynomials \( p_0(t) \) and \( p_1(t) \) are Hurwitz stable and the vector \( c = (c_1, c_2, \ldots, c_{n+1})^T \succeq 0 \) satisfies the system of linear inequalities

\[
E_{(n,n)} c \succeq 0,
\]

then the convex combination \( \lambda p_0(t) + (1 - \lambda) p_1(t) \) is Hurwitz stable for every \( \lambda \in [0, 1] \). Here the symbol \( \succeq 0 \) means that every component of a given vector is nonnegative (nonpositive) and the symbol \( \preceq 0 \) means that every component of a given vector is nonnegative but there is at least one positive component.

A similar result can be obtained for the case deg\( (p_1(t)) = n - 1 \). In this case the matrix \( E_{(n,n-1)} \in \mathbb{M}_{n \times n} \) is defined by

\[
E_{(n,n-1)} = \begin{pmatrix}
a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
-a_3 & a_2 & -a_1 & 1 & \cdots & 0 & 0 \\
a_5 & -a_4 & a_3 & -a_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{n-1} & -a_{n-2} \\
0 & 0 & 0 & 0 & \cdots & 0 & a_n \\
\end{pmatrix}
\]

and the corresponding inequality is

\[
E_{(n,n-1)} c \preceq 0.
\]

We also we study the situation when only one Hurwitz polynomial, say \( p_0(t) \), is known and the problem is to find all possible \( p_1(t) \) such that \( \lambda p_0(t) + (1 - \lambda) p_1(t) \) is Hurwitz for every \( \lambda \in [0, 1] \).

Finally, we use the same approach to estimate the minimum left extreme of a stable segment, that is, given the Hurwitz stable polynomials \( p_0(t) \) and \( p_1(t) \) such that the vector of coefficients of \( p_1 \) satisfies (1.2) or (1.4) then we find a number \( k_0 < 0 \) such that \( p_0(t) + k p_1(t) \) is Hurwitz stable for every \( k > k_0 \). The problem of calculating the minimum left extreme was solved by Bialas [4]. Although \( k_0 \) is only an estimate of \( k_{\text{min}} \), the novelty of our approach is that \( k_0 \) is obtained by a simple algebraic calculation. Contrary to stability of segments
where a good deal of work has been reported about the minimum left extreme we can only mention Biallas’ work, hence Section 5 might be interesting.

The paper is organized as follows: in Section 2 sufficient conditions assuring that a segment of polynomials consists of Hurwitz stable polynomials are given when it is known that the extremes \( p_0(t) \) and \( p_1(t) \) are Hurwitz stable. In Section 3 we compare our approach with other known sufficient conditions and two computational methods. In Section 4 we suppose that \( p_0(t) \) is Hurwitz stable and we see that the matrix inequality (1.2) is a sufficient condition on the vector of coefficients of \( p_1(t) = c_1 t^n + c_2 t^{n-1} + \cdots + c_{n+1} \) to establish that \( [p_0, p_1] \) is a segment of Hurwitz polynomials and we characterize the solution set of (1.2). Finally, in Section 5 the minimum left extreme of a stable segment is estimated.

2. Hurwitz Stable segments

The aim of this section is to obtain conditions for the stability of segments of polynomials. The main results are based on the following lemma where sufficient conditions are given for a real polynomial to be Hurwitz stable.

**Lemma (2.1).** Let \( F(t) \) and \( f(t) \) be real polynomials of degree \( n \), such that \( f(t) \) has positive coefficients, \( f(0) \neq 0 \) and the roots of \( F(t) \) are contained in \( \mathbb{C}^+ \). Consider the polynomial of degree \( 2n \) given by \( F(t)f(t) \). If \( F(i\omega)f(i\omega) \neq 0 \) and \( F(i\omega)f(i\omega) \) does not intersect \( \mathcal{L} \) for all \( \omega > 0 \), where \( \mathcal{L} \) is a straight line in the complex plane that passes through the origin, then all the roots of \( f(t) \) are in \( \mathbb{C}^- \).

**Proof.** Suppose \( n \) is even (the odd case is analogous). Let \( n = 2m \) and let \( F(t), f(t) \) be given by

\[
F(t) = b_0 t^{2m} + b_1 t^{2m-1} + \cdots + b_{2m}, \quad f(t) = d_0 t^{2m} + d_1 t^{2m-1} + \cdots + d_{2m}.
\]

Without loss of generality we may suppose that \( b_0 > 0 \), and then \( b_{2m} > 0 \) also since the roots of \( F(t) \) are in \( \mathbb{C}^+ \). Let \( l \) and \( r \) be the number of roots of \( F(t)f(t) \) contained in \( \mathbb{C}^- \) and \( \mathbb{C}^+ \), respectively. Let \( \theta(\omega) \) be the argument of \( F(i\omega)f(i\omega) \). Denote by \( \Delta_0^\infty \theta(\omega) = \theta(\infty) - \theta(0) \) the net change in the argument. Since \( F(t)f(t) \) does not have roots on the imaginary axis we get that \( \Delta_0^\infty \theta(\omega) = \frac{\pi}{2} (l - r) \) ([15], p. 406; [12], p. 174). The fact that \( F(i\omega)f(i\omega) \) does not intersect \( \mathcal{L} \) for \( \omega > 0 \) implies \( |\Delta_0^\infty \theta(\omega)| \leq \pi \).

Now we will analyze \( \theta(\omega) - \theta(0) \) when \( \omega \) is large. First, we have that for large \( \omega \), \( F(i\omega)f(i\omega) \approx b_0 d_0 \omega^{4m} - i(b_1 d_0 + b_0 d_1) \omega^{4m-1} \). Therefore \( \text{Re} \left[ F(i\omega)f(i\omega) \right] > 0 \) and \( \frac{\text{Im} F(i\omega)f(i\omega)}{\text{Re} F(i\omega)f(i\omega)} \to 0 \) when \( \omega \to \infty \). Since \( F(0)f(0) = b_{2m} d_{2m} > 0 \) it follows that \( \Delta_0^\infty \theta(\omega) = \theta(\infty) - \theta(0) = 2s\pi \), where \( s \) is an integer. Since \( F(i\omega)f(i\omega) \) does not intersect \( \mathcal{L} \) for \( \omega > 0 \) then \( |\Delta_0^\infty \theta(\omega)| \leq \pi \), and therefore we get that \( \Delta_0^\infty \theta(\omega) = 0 \).

Consequently, the polynomial \( F(t)f(t) \) has as many roots in \( \mathbb{C}^- \) as in \( \mathbb{C}^+ \). Since such a polynomial has degree \( 2n \), there are \( n \) roots in \( \mathbb{C}^- \). In fact the roots in \( \mathbb{C}^+ \) correspond to the roots of \( F(t) \). Hence, the \( n \) roots in \( \mathbb{C}^- \) correspond to the roots of \( f(t) \), which means that \( f(t) \) is Hurwitz stable. \( \square \)
Remark (2.2). Particular cases of Lemma (2.1) are the situations in which $L$ is the real or the imaginary axis. When $L$ is one of the axes, the associated matrices are easy to calculate. Our main results are based on these two cases.

In the following theorem we apply Lemma (2.1) when $L$ is the imaginary axis.

**Theorem (2.3).** Consider the Hurwitz stable polynomials $p_0(t) = t^n + a_1t^{n-1} + \ldots + a_n$ and $p_1(t) = c_1t^n + c_2t^{n-1} + \ldots + c_{n+1}$. If $c = (c_1, c_2, \ldots, c_{n+1})^T \succeq 0$ is a solution to (1.2), then, for all $\lambda \in [0, 1]$, the polynomial $\lambda p_0(t) + (1 - \lambda)p_1(t)$ is Hurwitz stable.

**Proof.** Suppose $n$ is even (the odd case is analogous). Let $n = 2m$ and $\lambda \in [0, 1]$. Let $p, q, P, Q$ denote the polynomials

$$
\begin{align*}
p(L) &= c_{2m+1} - c_{2m-1}L + c_{2m-3}L^2 + \ldots + (-1)^m c_1 L^m, \\
q(L) &= c_{2m} - c_{2m-2}L + \ldots + (-1)^{m-1}c_1 L^{m-1}, \\
P(L) &= a_{2m} - a_{2(m-1)}L + \ldots + (-1)^{m-1}a_2 L^{m-1} + (-1)^m L^m, \\
Q(L) &= a_{2m-1} - a_{2m-3}L + \ldots + (-1)^{m-1}a_1 L^{m-1}.
\end{align*}
$$

Then it holds that

$$
\begin{align*}
\lambda p_0 + (1 - \lambda)p_1(i\omega) &= \lambda P(i\omega^2) + \lambda Q(i\omega^2), \\
p_0(i\omega) &= P(\omega^2) + i\omega Q(\omega^2).
\end{align*}
$$

Consider the polynomial $p_0(-t) [\lambda p_0(t) + (1 - \lambda)p_1(t)].$ Thus we get

$$
p_0(-i\omega)\lambda p_0 + (1 - \lambda)p_1(i\omega) = P(\omega^2) \left[ \lambda P(\omega^2) + (1 - \lambda)p(\omega^2) \right] + \alpha^2 Q(\omega^2) \left[ \lambda Q(\omega^2) + (1 - \lambda)q(\omega^2) \right] + i\omega(1 - \lambda) \left[ P(\omega^2)q(\omega^2) - Q(\omega^2)p(\omega^2) \right].
$$

That is,

$$
p_0(-i\omega)[\lambda p_0 + (1 - \lambda)p_1(i\omega) = \lambda \left[ P^2(\omega^2) + \alpha^2 Q(\omega^2)^2 \right] + (1 - \lambda) \left[ P(\omega^2)p(\omega^2) + \alpha^2 Q^2(\omega^2)q(\omega^2)^2 \right] + i\omega(1 - \lambda) \left[ P(\omega^2)q(\omega^2) - Q(\omega^2)p(\omega^2) \right].
$$

Since $P(\omega^2)p(\omega^2) + \alpha^2 Q(\omega^2)^2q(\omega^2) = \sum_{i=1}^{n+1} (E(c, c))^{2n+1-i}$ and the vector $c \succeq 0$ is a solution to the system of the linear inequalities (1.2), the polynomial $P(\omega^2)p(\omega^2) + \alpha^2 Q(\omega^2)^2q(\omega^2)$ does not have positive roots. Consequently, for all $\omega > 0$, $p_0(-i\omega)[\lambda p_0 + (1 - \lambda)p_1(i\omega)$ does not intersect the imaginary axis. Finally, since $p_0(-t)$ and $\lambda p_0(t) + (1 - \lambda)p_1(t)$ satisfy the hypothesis of Lemma (2.1) we have that the polynomial $\lambda p_0(t) + (1 - \lambda)p_1(t)$ is Hurwitz stable for all $\lambda \in [0, 1].$

**Remark (2.6).** Theorem (2.12) can be extended to the case when $\deg p_1(t) = n - 1$. To prove this result we need to redefine the polynomials $p(L)$ and $q(L)$ by

$$
p(L) = c_{2m} - c_{2(m-1)}L + \ldots + (-1)^{m-1}c_2 L^{m-1}, \\
q(L) = c_{2m-1} - c_{2m-3}L + \ldots + (-1)^{m-1}c_1 L^{m-1},
$$

and the proof follows the same steps as the proof of Theorem (2.3).
On the other hand, using the same method, Theorem (2.3) cannot be extended to the case when \( \deg(p(t)) = n - 2 \) since the corresponding matrix \( E_{(n,n-2)} \) in \( M_{n \times (n-1)} \) is given by

\[
E_{(n,n-2)} = \begin{pmatrix}
-1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
-2 & -a_1 & 1 & 0 & \ldots & 0 & 0 \\
-\alpha_4 & a_3 & -a_2 & a_1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & a_{n-1} & -a_{n-2} \\
0 & 0 & 0 & 0 & \ldots & 0 & a_n
\end{pmatrix}
\]

but the first inequality implies that \(-c_1 \geq 0\) which is not satisfied since \(c_1 > 0\).

Remark (2.8). In [1] we obtain a condition like (1.2) for the stability of rays of polynomials. There is an obvious relation between stable rays and stable segments of polynomials: if \( p_0(t) + kg(t) \) is a Hurwitz stable polynomial for every \( k \geq 0 \) then \( \left( \frac{1}{1+k} \right) p_0(t) + \left( \frac{1}{1+k} \right) g(t) \) is a Hurwitz stable polynomial for every \( k \geq 0 \), which means that the stability of the ray \( p_0(t) + kg(t) \) is equivalent to the stability of the open segment \([p_0(t), g(t)]\). Observe that for \( p_1(t) \) Hurwitz stable we get the stability of the closed segment \([p_0(t), g(t)]\).

In the proof of Theorem (2.3), when we analyze the complex function \( p_0(-i\omega) \) \( [\lambda p_0 + (1-\lambda)p_1]i\omega \) defined in (2.5), the straight line \( \mathcal{L} \) was the imaginary axis. A different possibility is to consider \( \mathcal{L} \) as the real axis. Such an analysis was done in [1] and the results were given in terms of a similar inequality \( Dc \geq 0 \) given by the following matrices:

\[
D_{(n,n)} = \begin{pmatrix}
a_1 & -1 & 0 & 0 & \ldots & 0 & 0 \\
-\alpha_3 & a_2 & -a_1 & 1 & \ldots & 0 & 0 \\
a_3 & -\alpha_4 & a_3 & -a_1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & a_{n-2} & a_{n-3} \\
0 & 0 & 0 & 0 & \ldots & 0 & a_n & -a_{n-1}
\end{pmatrix}
\]

for the case \( \deg(p_1(t)) = n \), while

\[
D_{(n,n-1)} = \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
-\alpha_2 & a_1 & -1 & 0 & \ldots & 0 & 0 \\
\alpha_4 & -\alpha_3 & a_2 & -a_1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & a_{n-2} & a_{n-3} \\
0 & 0 & 0 & 0 & \ldots & 0 & a_n & a_{n-1}
\end{pmatrix}
\]

for the case \( \deg(p_1(t)) = n - 1 \), and for \( \deg(p_1(t)) = n - 2 \), the matrix \( D_{(n,n-2)} \) is

\[
D_{(n,n-2)} = \begin{pmatrix}
a_1 & -1 & 0 & 0 & \ldots & 0 & 0 \\
-\alpha_3 & a_2 & -a_1 & 1 & \ldots & 0 & 0 \\
a_5 & -\alpha_4 & a_3 & -a_2 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & a_{n-2} & a_{n-3} \\
0 & 0 & 0 & 0 & \ldots & 0 & a_n & a_{n-1}
\end{pmatrix}
\]
Rewriting the results in [1] for segments of polynomials instead of rays, we obtain the next result.

**Theorem (2.12).** Consider the Hurwitz stable polynomial \( p_0(t) = t^n + a_1t^{n-1} + \cdots + a_n \). If \( p_1(t) \) is a Hurwitz stable polynomial with \( \deg(p_1(t)) = n, n - 1, \) or \( n - 2 \), and its vector of coefficients \( c \) satisfies the system of linear inequalities
\[
Dc \preceq 0
\]
where the matrix \( D \) depending on \( \deg(p_1(t)) \) is one of the matrices \( D_{n,n} \), \( D_{n,n-1} \) or \( D_{n,n-2} \), then the polynomial \( \lambda p_0(t) + (1 - \lambda)p_1(t) \) is Hurwitz stable for all \( \lambda \in [0, 1] \).

3. Comparison with other methods

In this section we present the qualities of our approach comparing it with other known methods.

**3.1 Comparison with the Method in Aguirre et al.** There are segments of stable polynomials such that the stability can be verified using the approach introduced here, but it is not possible to check such stability with the test given in [1].

**Example (3.1.1).** Consider the Hurwitz stable polynomial \( p_0(t) = t^3 + 2t^2 + t + 1 \). The vector of coefficients \( c \) of the polynomial \( p_1(t) = t^3 + 8t^2 + 13t + 1 \) is a solution to the system of linear inequalities (1.2):
\[
E_{(3,3)}c = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 1 & -2 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 \\
8 \\
13 \\
1
\end{pmatrix} = \begin{pmatrix}
1 \\
2 \\
3 \\
1
\end{pmatrix}.
\]
Consequently the segment \([p_0, p_1] \) is stable. However, \( c \) is not a solution to (2.13) since
\[
D_{(3,3)}c = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 1 & -2 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix} \begin{pmatrix}
1 \\
8 \\
13 \\
1
\end{pmatrix} = \begin{pmatrix}
-6 \\
-18 \\
12
\end{pmatrix}.
\]

**3.2 Comparison with the Rantzer-type conditions.** Here we compare our approach with the known Rantzer-type conditions. Such conditions are explained in [13] and they are the following: Suppose that \( p_0 \) is a Hurwitz polynomial and \( p_1 \) is a semistable polynomial. Then the ray of polynomials \( p_0(t) + kp_1(t) \) consists of Hurwitz polynomials if one of the following four conditions holds:

1. The difference \( d = p_1 - p_0 \) satisfies
\[
\frac{\partial \arg(d(i\omega))}{\partial \omega} < 0, \quad \omega \in \{w > 0/d(iw) \neq 0\}.
\]

---

\(^1\)A polynomial is semistable if the real parts of its roots are not positive.
(ii) Each of the polynomials \( p_0, p_1 \) has at least one root in the open left half-plane and
\[
\frac{\partial \arg(d(i\omega))}{\partial \omega} < \left| \frac{\sin(2\arg(d(i\omega)))}{2\omega} \right|, \quad \omega \in \{w > 0/d(iw) \neq 0\}.
\]

(iii) Each of the polynomials \( p_0, p_1 \) has at least one root in the open left half-plane and
\[
\frac{\partial \arg(d(i\omega))}{\partial \omega} \leq 0, \quad \omega \in \{w > 0/d(iw) \neq 0\}.
\]

(iv) Each of the polynomials \( p_0, p_1 \) has at least two roots in the open left half-plane and
\[
\frac{\partial \arg(d(i\omega))}{\partial \omega} \leq \left| \frac{\sin(2\arg(d(i\omega)))}{2\omega} \right|, \quad \omega \in \{w > 0/d(iw) \neq 0\}.
\]

Although the Rantzer-type conditions offer four options to check the stability of segments of polynomials, they cannot cover all the possibilities, as is illustrated by the following example.

**Example (3.2.1).** Consider the Hurwitz stable polynomial \( p_0(t) = t^3 + 2t^2 + t + 1 \). The vector of coefficients \( c \) of the polynomial \( p_1(t) = t^3 + 7t^2 + 12t + 2 \) is a solution to the system of linear inequalities (1.2):
\[
E_{(3,3)c} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 1 & -2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
7 \\
12 \\
2
\end{pmatrix} = \begin{pmatrix}
1 \\
1 \\
1 \\
2
\end{pmatrix}.
\]

Therefore the segment \([p_0, p_1]\) is stable. Furthermore, we will see that this example does not satisfy the Rantzer-type conditions (see [13], [16]).

For this example \( p_0(t) \) and \( p_1(t) \) are Hurwitz polynomials, and \( d(t), d(i\omega) \) and \( \arg(d(i\omega)) \) are given by \( d(t) = (p_1 - p_0)(t) = 5t^2 + 11t + 1 \), \( d(i\omega) = 1 - 5\omega^2 + i11\omega \), \( \arg(d(i\omega)) = \arctan \left( \frac{11\omega}{1 - 5\omega^2} \right) \).

It is not difficult to verify that \( i - iv \) are not satisfied:

1) Since \( \frac{\partial \arg(d(i\omega))}{\partial \omega} = \frac{11 + 55\omega^2}{(1 - 5\omega^2)^2 + 121\omega^2} > 0 \) for all \( \omega \in \{w > 0/d(i\omega) \neq 0\} \) = (0, \infty), \( i \) is not satisfied.

2) \( \sin(2\arg(d(i\omega))) = \frac{2\omega(11 - 55\omega^2)}{(1 - 5\omega^2)^2 + 121\omega^2} \), hence
\[
\frac{\partial \arg(d(i\omega))}{\partial \omega} < \left| \frac{\sin(2\arg(d(i\omega)))}{2\omega} \right|
\]
is satisfied if and only if
\[
\frac{11 + 55\omega^2}{(1 - 5\omega^2)^2 + 121\omega^2} < \frac{|11 - 55\omega^2|}{(1 - 5\omega^2)^2 + 121\omega^2}.
\]
If \( \omega = 1 \) we have that \( \frac{66}{137} < \frac{44}{157} \), which is a contradiction. Consequently \( ii \) is not satisfied.

3) From the above inequalities it is immediate that \( iii \) and \( iv \) are not satisfied either.

Consequently, although the segment \([p_0(t), p_1(t)]\) consists of Hurwitz polynomials, it is not possible to verify that using the Rantzer-type conditions obtained in [13].

Remark (3.2.2). With respect to the method in [1] and the Rantzer-type conditions we believe that the main contribution of our new approach is that it can be applied to cases where the others do not succeed. However this does not mean that our approach subsumes the other methods and in a given segment our method could fail and some of the other methods could work.

(3.3) Comparison with the Algorithm of Hwang-Yang. Now we compare our approach with the computational method given in [14].

Example (3.3.1). Consider the polynomials \( p_0(t) = t^5 + 6t^4 + 14t^3 + 16t^2 + 9t + 2 \) and \( p_1(t) = 2.165^2 + 6.47t^4 + 8.58t^3 + 6.57t^2 + 3.38t + 1.08 \). The polynomial \( p_0(t) \) is Hurwitz stable and the vector of coefficients \( c \) of the polynomial \( p_1(t) \) is a solution to the system of linear inequalities (1.2) since

\[
E_{(3,5)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-14 & 6 & -1 & 0 & 0 & 0 \\
9 & -16 & 14 & -6 & 1 & 0 \\
0 & 2 & -9 & 16 & -14 & 6 \\
0 & 0 & 0 & -2 & 9 & -16 \\
0 & 0 & 0 & 0 & 0 & 2 \\
\end{pmatrix} \begin{pmatrix}
2.16 \\
6.47 \\
8.58 \\
6.57 \\
3.38 \\
1.08 \\
\end{pmatrix} = \begin{pmatrix}
2.16 \\
0 \\
0 \\
0 \\
0 \\
2.16 \\
\end{pmatrix}.
\]

Then we can conclude that the segment with extremes \( p_0(t) \) and \( p_1(t) \) consists of Hurwitz polynomials.

On the other hand, if we apply the approach given in [14] we begin with the following calculations:

\[
p_0 + \lambda(p_1 - p_0) = [1 + 1.16\lambda]t^5 + [6 + 0.47\lambda]t^4 + [14 - 5.42\lambda]t^3 + [16 - 9.43\lambda]t^2 + [9 - 5.62\lambda]t + 2 - 0.92\lambda,
\]

\[
a_{0,0}(\lambda) = a_0(\lambda) = 2 - 0.92\lambda \\
a_{0,1}(\lambda) = a_2(\lambda) = 16 - 9.43\lambda \\
a_{0,2}(\lambda) = a_4(\lambda) = 6 + 0.47\lambda \\
a_{1,0}(\lambda) = a_1(\lambda) = 9 - 5.62\lambda \\
a_{1,1}(\lambda) = a_3(\lambda) = 14 - 5.42\lambda \\
a_{1,2}(\lambda) = a_5(\lambda) = 1 + 1.16\lambda \\
a_{2,0}(\lambda) = 116 - 151.07\lambda + 48.01\lambda^2 \\
a_{2,1}(\lambda) = 52 - 30.89\lambda - 1.5742\lambda^2 \\
a_{3,0}(\lambda) = 1156 - 2173.5\lambda + 1331.5\lambda^2 - 269.06\lambda^3 \\
a_{3,1}(\lambda) = 116 - 16.51\lambda - 127.23\lambda^2 + 55.692\lambda^3 \\
a_{4,0}(\lambda) = 5184 - 11128.5\lambda + 8745.6\lambda^2 - 3048.7\lambda^3 + 400.39\lambda^4.
\]

To finish the algorithm one must check that \( a_{4,0}(\lambda) > 0 \) for every \( \lambda \in [0, 1] \).
**Remark (3.3.2).** In general, to apply the algorithm of Hwang-Yang one must calculate the all \( a_{j,k} \)'s. Next one must check whether \( a_{n-1,0}(\lambda) > 0 \) for \( \lambda \in [0,1] \). That is, this algorithm reduces the problem of determining the stability of a segment of polynomials to checking the positivity of a polynomial, which is usually verified by using Sturm sequences. But note that the number of calculations increases with the degree of the extremes of the segment of polynomials since the degree of \( a_{n-1,0}(\lambda) \) is \( n - 1 \).

**(3.4) Comparison with the Algorithm of Bouguerra et al.** Now we compare the calculations of our approach with the algorithm given in [8].

**Example (3.4.1).** Consider the Hurwitz polynomial \( p_0(t) = t^6 + 7t^5 + 20t^4 + 30t^3 + 25t^2 + 11t + 2 \) and let \( p_1(t) = 15t^6 + 58t^5 + 100t^4 + 100t^3 + 65t^2 + 29t + 7.5 \) be a polynomial. The vector of coefficients \( c \) of the polynomial \( p_1(t) \) is a solution to the system of linear inequalities (1.2) since

\[
E_{[6,6]}c = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-20 & -7 & -1 & 0 & 0 & 0 \\
25 & -30 & -20 & -7 & 1 & 0 \\
-2 & 11 & -25 & 30 & -20 & 7 & 1 \\
0 & 0 & 0 & 0 & 21 & 11 & 25 \\
0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix} \begin{pmatrix}
15 \\
58 \\
100 \\
65 \\
29 \\
7.5
\end{pmatrix} = \begin{pmatrix}
15 \\
6 \\
0 \\
3.5 \\
5.0 \\
1.5 \\
15.0
\end{pmatrix}.
\]

Consequently the segment \([p_0, p_1]\) is stable.

On the other hand, if we apply the algorithm posed in [8] first we have to determine the polynomials

\[
\hat{a}(x) = 2 - 25x + 20x^2 - x^3 \\
\hat{b}(x) = 11 - 30x + 7x^2 \\
\hat{c}(x) = 7.5 - 65x + 100x^2 - 15x^3 \\
\hat{d}(x) = 29 - 100x + 58x^2
\]

Next we must make the following calculations:

1) Find the positive real roots of \( \hat{a}(x), \hat{b}(x), \hat{c}(x) \) and \( \hat{d}(x) \).

2) From these positive roots, one looks for intervals where both \( \hat{a}(x)\hat{c}(x) \) and \( \hat{b}(x)\hat{d}(x) \) are negative.

3) If such intervals exist, one needs to check for the existence of positive real roots of \( \hat{a}(x)\hat{d}(x) - \hat{b}(x)\hat{c}(x) = 0 \) inside these intervals. If \( \hat{a}(x)\hat{d}(x) - \hat{b}(x)\hat{c}(x) = 0 \) admits roots inside these intervals, then the segment is unstable, and stable otherwise.

**Remark (3.4.2).** In this algorithm one uses Sturm sequences as well. Observe that if the degrees of \( p_0 \) and \( p_1 \) are increased, the degrees of \( \hat{a}(x), \hat{b}(x), \hat{c}(x), \hat{d}(x), \hat{a}(x)\hat{c}(x), \hat{b}(x)\hat{d}(x) \) and \( \hat{a}(x)\hat{d}(x) - \hat{b}(x)\hat{c}(x) \) are also increased.

Hence if the degrees of \( p_0 \) and \( p_1 \) are large, the application of 1), 2) and 3) requires costly effort.

**Remark (3.4.3).** It is natural that the algorithms of Hwang-Yang and Bouguerra require more work that our condition since they are based on necessary and sufficient conditions and consequently can check both situations: stable or unstable segments.
4. Stability of a segment when only a extreme is given

Now we study a different problem: a Hurwitz polynomial $p_0(t)$ is given and we ask whether there exist polynomials $p_1(t)$ such that $[p_0(t), p_1(t)]$ is a segment of Hurwitz polynomials.

Remark (4.1). Let $\mathcal{H}_n$ denote the set of Hurwitz stable polynomials of degree $n$. If the vector of coefficients of the polynomial $p_1(t) = c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n$ is a solution to the system of linear inequalities $E_n c \geq 0$, then it can be proved that the segment of polynomials $[p_0(t), p_1(t)]$ is Hurwitz stable. Observe that $p_1 \notin \mathcal{H}_n$ since $\deg(p_1) = n - 1$. However it is clear that $p_1(t)$ is on the boundary of $\mathcal{H}_n$.

Remark (4.2). Consider the Hurwitz polynomial $p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n$. Let $E_{(n,n)}$ be the corresponding matrix defined by (1.1). If $p_1(t)$ is given by $p_1(t) = \sum_{i=1}^{n+1} c_i t^{n+1-i}$ and the vector $c = (c_1, c_2, \ldots, c_{n+1})^T \geq 0$ is a solution to the system of linear inequalities (1.2), then following a idea similar to that of Theorem (2.3) it can showed that $[p_0(t), p_1(t)]$ is a segment of Hurwitz polynomials. Contrary to Remark (4.1) we have in this case that $p_1 \in \mathcal{H}_n$ since $\deg(p_1) = n$. But the question is whether there exists any polynomial $p_1(t)$ that fulfills this property. In the following subsection we work on this problem. First we describe an example.

In the following example we present a segment of Hurwitz stable polynomials $[p_0(t), p_1(t)]$ such that the vector of coefficients of $p_1(t)$ does not satisfy the linear inequalities (1.2) and (2.13). This example proves that conditions (1.2) and (2.13) are only sufficient.

Example (4.3). Consider the Hurwitz stable polynomials $p_0(t) = t^3 + 2t^2 + t + 1$ and $p_1(t) = t^3 + \frac{5}{2} t^2 + 2t + \frac{19}{4}$. First, observe that the linear matrix inequality (1.2) is not satisfied:

$$E_{(3,3)} c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{5}{2} \\ 2 \\ \frac{19}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -10 \\ \frac{19}{4} \end{pmatrix}. $$

On the other hand, the segment $[p_0(t), p_1(t)]$ is a segment of Hurwitz stable polynomials since the Routh-Hurwitz conditions $(\frac{5}{2} - \frac{3}{2} \lambda), (2 - \lambda), \frac{19}{4} - \frac{15}{4} \lambda > 0$ and $(\frac{5}{2} - 1 - \lambda)(2 - \lambda)t - \frac{19}{4} + \frac{15}{4} \lambda = \frac{1}{2} \lambda^2 + \frac{1}{4} \lambda + \frac{1}{4} > 0$ associated with the polynomial $(1 - \lambda)p_0(t) + (1 - \lambda)p_1(t) = t^3 + (\frac{5}{2} - \frac{1}{2} \lambda)t^2 + (2 - \lambda)t + \frac{19}{4} - \frac{15}{4} \lambda$ are satisfied for all $\lambda \in (0, 1)$.

Furthermore, this example does not satisfy the condition (2.13) either, since

$$D_{(3,3)} c = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{5}{2} \\ 2 \\ \frac{19}{4} \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{9}{4} \\ -\frac{11}{4} \end{pmatrix}. $$

This example and the Remark (4.2) illustrate the relevance of studying the problem of characterizing the solution set of (1.2), which will be addressed in the next subsection.
(4.1) **Characterization of the set of solutions of (1.2).** In this subsection, given the Hurwitz stable polynomial \( p_0(t) \), we will find the polynomials \( p_1(t) \) whose vector of coefficients satisfies the linear inequalities (1.2).

As was proved in [1] for the matrix \( D_{n,n-1} \), it can be seen that the matrix \( E_{n,n} \) is of monotone kind (i.e., \( E_{n,n}z \geq 0 \) implies \( z \geq 0 \)), which implies that it is invertible and \( E_{n,n}^{-1} \geq 0 \), where \( E_{n,n}^{-1} \geq 0 \) means that all its entries are nonnegative (see [10]). Denote by \( V = \{ z \in \mathbb{R}^{n+1}/\{0\} \mid z_i \geq 0, \forall i = 1, 2, \ldots, n+1 \} \). The following result characterizes the solution set of (1.2).

**Theorem (4.1.1).** The set \( H \) of solutions of the system of linear inequalities (1.2) can be written as \( H = E_{n,n}^{-1} V \).

**Proof.** First we prove that \( H \subseteq E_{n,n}^{-1} V \). Let \( u \in H \), then \( u \geq 0 \) and \( E_{n,n}u \geq 0 \). Consequently, \( u = E_{n,n}^{-1}E_{n,n}u = E_{n,n}^{-1}V \). That is, \( u \in E_{n,n}^{-1} V \).

Now, we prove \( H \supseteq E_{n,n}^{-1} V \). Let \( u \in E_{n,n}^{-1} V \), then \( u = E_{n,n}^{-1}E_{n,n}u \) with \( v \geq 0 \) and \( v \neq 0 \). Hence, \( E_{n,n}u = v \geq 0 \) and \( E_{n,n}^i u > 0 \) for some row \( E_{n,n}^i \) with \( 1 \leq i \leq n \), that is, \( u \in H \).

**Corollary (4.1.2).** Let \( p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n \) be a Hurwitz stable polynomial. Let \( E_{n,n} \) be the corresponding matrix defined by (1.1). If the vector \( c = (c_1, c_2, \ldots, c_n + 1)^T \in E_{n,n}^{-1} V \), then \( [p_0(t), p_1(t)] \) is a segment of Hurwitz polynomials, where the polynomial \( p_1(t) \) is given by \( p_1(t) = \sum_{i=1}^{n+1} c_i t^{n+1-i} \).

**Remark (4.1.3).** Observe that the set of vectors that satisfies (1.2) is given by the polyhedral cone \( C \) generated by \( w_1 = E_{n,n}^{-1}e_1, w_2 = E_{n,n}^{-1}e_2, \ldots, w_{n+1} = E_{n,n}^{-1}e_{n+1} \), where \( e_1, e_2, \ldots, e_{n+1} \) are the canonical vectors in \( \mathbb{R}^{n+1} \). Given the vector of coefficients \( w_0 = (1, a_1, \ldots, a_n) \) of the Hurwitz stable polynomial \( p_0(t) \), the vectors \( w \in C \) are vectors of coefficients of polynomials \( p_1(t) \) such that \( [p_0(t), p_1(t)] \) is a segment of Hurwitz polynomials.

**Example (4.1.4).** Consider the Hurwitz stable polynomial \( p_0(t) = t^3 + 2t^2 + t + 1 \). The matrices \( E_{3,3} \) and \( E_{3,3}^{-1} \) are given by

\[
E_{3,3} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad E_{3,3}^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} - \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{pmatrix}.
\]

From Theorem (4.1.1), the set of vectors that satisfy \( E_{3,3} c \geq 0 \) can be seen as the polyhedral cone \( C \) generated by

\[
\{ (1, 1, 1, 0)^T, (0, 1, 1, 0)^T, (0, 1, 2, 0)^T, (0, 2, 4, 1)^T \}.
\]

5. **The minimum left extreme**

In this section, given the Hurwitz stable polynomials \( p_0(t) \) and \( p_1(t) \), we are concerned with the problem of estimating the minimum \( h_{min} < 0 \) such that \( p_0(t) + kp_1(t) \) is a Hurwitz stable polynomial \( \forall k > h_{min} \) (see [4]). Using the results presented in the above sections, we will find a number \( k_0 < 0 \) such that \( p_0(t) + kp_1(t) \) is Hurwitz stable for every \( k > k_0 \), if the vector of coefficients of \( p_1 \) satisfies (1.2) or (1.4). Here \( k_0 \) is an estimate of \( h_{min} \) (\( k_0 \geq h_{min} \)) because we do not know if \( k_0 \) is the smallest number with this property. The problem
of calculating the minimum left extreme \( k_{\min} \) was solved by Bialas [4]. In our approach \( k_0 \) is obtained by an algebraic calculation.

Consider the polynomial

\[
(5.1) \quad p(t, k) = p_0(t) + kp_1(t)
\]

where \( p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n \) is the nominal polynomial. Assume \( p_0(t) \) is a Hurwitz stable polynomial, and let \( E_{(n,n)} \) be the corresponding matrix defined as in (1.1). If the vector of coefficients \( c = (c_1, c_2, \ldots, c_{n+1})^T \geq 0 \) of the polynomial \( p_1(t) = \sum_{i=1}^{n+1} c_i t^{n+1-i} \) is a solution to the system of linear inequalities (1.2), then \( p_0(t) + kp_1(t) \) is a Hurwitz stable polynomial \( \forall k \geq 0 \). In [4] it was proved that

\[
(5.2) \quad k_{\min} = \frac{1}{\lambda_{\min}[-H^{-1}(p_0)H(p_1)]}
\]

where \( H(p_0), H(p_1) \) are the Hurwitz matrices of \( p_0 \) and \( p_1 \) respectively and \( \lambda_{\min}[-H^{-1}(p_0)H(p_1)] \) is the minimum negative eigenvalue of the matrix \(-H^{-1}(p_0)H(p_1)\). Observe that numerically (5.2) is not easy to calculate because the calculation implies solving an \( n \)th-order eigenvalue problem. In what follows we give an algebraic procedure to obtain an estimate of \( k_{\min} \).

Define the matrix

\[
(5.3) \quad Z_{(n,n)} = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & a_1 & -2 & 0 & \cdots & 0 & 0 \\
0 & 0 & a_2 & -2a_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{n-1} & -2a_{n-2} \\
0 & 0 & 0 & 0 & \cdots & 0 & a_n
\end{pmatrix}
\]

and denote by \( Z_{i(n,n)}^i \) the \( i \)-th row of the matrix \( Z_{(n,n)} \) and let \( a = (1, a_1, \ldots, a_n)^T \).

**Theorem (5.4).** Let \( p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n \) be a Hurwitz stable polynomial. Let \( E_{(n,n)} \) be the corresponding matrix defined by (1.1). If the vector \( c = (c_1, c_2, \ldots, c_{n+1})^T \geq 0 \) is a solution to the system of linear inequalities (1.2) and each component of \( E_{(n,n)}c \) is positive and the polynomial \( p_1(t) \) is given by \( p_1(t) = \sum_{i=1}^{n+1} c_i t^{n+1-i} \) then, \( p_0(t) + kp_1(t) \) is a Hurwitz stable polynomial for all \( k > k_0 \), where

\[
k_0 = \max_{i=1,\ldots,n+1} \left( \frac{Z_{i(n,n)}^i a}{E_{i(n,n)}^i c} \right).
\]

**Proof.** In a similar way to the proof of Theorem (2.3) we get

\[
p_0(-i\omega)p_0 + kp_1(i\omega) = \left[ P^2(\omega^2) + \omega^2 Q^2(\omega^2) \right] + k \left[ P(\omega^2)p(\omega^2) + \omega^2 Q(\omega^2)q(\omega^2) \right] + i\omega k \left[ P(\omega^2)q(\omega^2) - Q(\omega^2)p(\omega^2) \right].
\]

Note that the expression \( P^2(\omega^2) + \omega^2 Q^2(\omega^2) + k \left[ P(\omega^2)p(\omega^2) + \omega^2 Q(\omega^2)q(\omega^2) \right] \) can be rewritten as \( \omega^{2(n+1)} + \sum_{i=1}^{n+1} (Z_{i(n,n)}^i a + kE_{i(n,n)}^i c) \omega^{2(n+1-i)} \). If \( k > k_0 \) then...
$k > -\frac{Z_{i,n,n}^i}{E_{i,n,n}^i} \quad \forall i = 1, \ldots, n+1$. Since $E_{i,n,n}^i > 0 \quad \forall i = 1, \ldots, n+1$ it follows that $kE_{i,n,n}^i > -Z_{i,n,n}^i$ and then $Z_{i,n,n}^i + kE_{i,n,n}^i > 0 \quad \forall i = 1, \ldots, n+1$. Consequently, for all $\omega > 0$, $p_0(-\imath \omega)\|p_0 + kp_1(\imath \omega)$ does not intersect the imaginary axis, from which we have that $p_0(t)$ and $p_0(t) + kp_1(t)$ satisfy the hypotheses of Lemma (2.1). This implies that the polynomial $p_0(t) + kp_1(t)$ is Hurwitz stable for all $k > k_0$, and the theorem is proved.

**Remark (5.5).** The extension of Theorem (5.4) to the case where $\deg p_1(t) = n - 1$ turns out to be as follows: $p_0(t) + kp_1(t)$ is Hurwitz stable for all $k > k_0$, if $k_0 = \max_{i=1,\ldots,n,n} \left( -\frac{Z_{i,n,n}^i}{E_{i,n,n}^i} \right)$ and

\begin{equation}
Z_{i,n,n}^i = \begin{pmatrix}
a_1 & -2 & 0 & 0 & \ldots & 0 & 0 \\
0 & a_2 & -2a_1 & 2 & \ldots & 0 & 0 \\
0 & 0 & a_3 & -2a_2 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & a_{n-1} & -2a_{n-2} \\
0 & 0 & 0 & 0 & \ldots & 0 & a_n \\
\end{pmatrix}.
\end{equation}

**Remark (5.7).** For the segment of polynomials $p(t, q) = p_0(t) + q[p_1(t) - p_0(t)]$ for $q \in [0, 1]$, it follows from Theorem (5.4) that $p(t, q)$ is Hurwitz stable for all $q \geq q_0$, where $q_0 = \frac{k_0}{1+k_0}$. If $k_0 \leq -1$, it results that $p(t, q)$ is Hurwitz stable for all $k \in (-\infty, 1)$.

Next we present an example where $k_0 = k_{\min}$.

**Example (5.8).** Let $p_0(t) = t^3 + 7t^2 + 14t + 8$, $p_1(t) = t^2 + 4t + 6$. To calculate $k_0$, we first have that

\begin{equation}
Z_{i,3,2}a = \begin{pmatrix}
7 & -2 & 0 \\
0 & 14 & -14 \\
0 & 0 & 8 \\
\end{pmatrix}
\begin{pmatrix}
7 \\
14 \\
8 \\
\end{pmatrix}
= \begin{pmatrix}
21 \\
84 \\
64 \\
\end{pmatrix},
\end{equation}

\begin{equation}
E_{i,3,2}c = \begin{pmatrix}
7 & -1 & 0 \\
-8 & 14 & -7 \\
0 & 0 & 8 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
4 \\
6 \\
\end{pmatrix}
= \begin{pmatrix}
3 \\
6 \\
48 \\
\end{pmatrix}.
\end{equation}

Then $k_0 = \max(\frac{-21}{3}, \frac{-84}{6}, \frac{-64}{48}) = -\frac{4}{3}$.

To calculate $k_{\min}$, we find that

\begin{equation}
H(p_0) = \begin{pmatrix}
1 & 0 \\
1 & 4 \\
0 & 1 \\
\end{pmatrix},
H(p_1) = \begin{pmatrix}
1 & 6 & 0 \\
0 & 4 & 0 \\
0 & 1 & 6 \\
\end{pmatrix}
\end{equation}

and

\begin{equation}
H^{-1}(p_0)H(p_1) = \begin{pmatrix}
\frac{7}{4} & \frac{26}{45} & 0 \\
\frac{1}{720} & \frac{2}{9} & \frac{1}{4} \\
\frac{1}{720} & \frac{2}{9} & \frac{1}{4} \\
\end{pmatrix},
\sigma(-H^{-1}(p_0)H(p_1)) = \left\{ -\frac{3}{4}, -\frac{1}{5} \pm \frac{1}{15}i \right\}.
\end{equation}

Therefore $\lambda_{\min} = -\frac{3}{4}$, and thus $k_{\min} = -\frac{4}{3} = k_0$. 

Example (5.9). For the polynomials \( p_0(t) = t^3 + 7t^2 + 14t + 8 \), \( p_1(t) = 26t^2 + 137t + 90 \) we obtain \( k_0 > k_{\text{min}} \). Defining the matrices \( Z_{(3,2)}, E_{(3,2)} \) as in (5.6) and (1.3) respectively, we have

\[
Z_{(3,2)}a = \begin{pmatrix} 21 \\ 84 \\ 64 \end{pmatrix}, \quad E_{(3,2)}c = \begin{pmatrix} 45 \\ 1080 \\ 720 \end{pmatrix}.
\]

From which \( k_0 = \max \left\{ \frac{21}{45}, \frac{84}{1080}, \frac{64}{720} \right\} = \frac{7}{90} = -0.07778 \). On the other hand, given the Hurwitz matrices \( H(p_0) \) and \( H(p_1) \), \( \sigma \left( -H^{-1}(p_0)H(p_1) \right) = \{ -11.25, -4.1399, -9.5601 \} \), and \( \lambda_{\min} = -11.25 \). Finally, \( k_{\text{min}} = -0.088889 < k_0 = -0.07778 \).

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