Joint Inventory and Cash Management for Multidivisional Supply Chains

Wei Luo
IESE Business School, University of Navarra, 08034 Barcelona, Spain, wluo@iese.edu
Kevin Shang
Fuqua School of Business, Duke University, Durham, North Carolina 27708, khshang@duke.edu

This paper develops a centralized supply chain model that integrates material flows with cash flows. The supply chain is owned by a single firm with two divisions. The downstream division (headquarters), facing random customer demand, replenishes materials from the upstream division. The firm installs a financial services platform that pools the divisions’ cash into a master account managed by the headquarters. In each period, cash is received from customers and paid to the outside vendor after materials are delivered. The headquarters determines how much cash to retain for inventory replenishment. The objective is to determine an optimal joint inventory replenishment and cash retention policy for the entire supply chain. We prove that the optimal policy has a surprisingly simple structure—both divisions implement a base-stock policy for inventory replenishment; the headquarters monitors the corporate working capital and implements a two-threshold policy for cash retention. This result is obtained by extending the well-known Clark-Scarf decomposition with newly derived cash-related penalty functions. The optimal policy enables us to investigate the interaction between cash and inventory decisions. We show that in the presence of transaction costs, a firm may stock more even if the inventory holding cost increases. To quantify the value of financial integration, we compare the cash pooling model with systems under different levels of financial integration. Our study suggests that the value of cash pooling can be significant when demand is increasing (respectively, stationary) and the internal transfer price is low (respectively, high). Nevertheless, a significant amount of cash pooling benefit may be recovered if the headquarters can optimize the internal transfer price.

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1. Introduction

The goal of supply chain management is to match demand with supply through effective coordination between material, information, and financial flows. When financial markets are efficient, i.e., external funding is plentiful and relatively inexpensive, firms have sufficient funds for daily operations. In this case, a downstream entity pays the inventory it orders from an upstream entity so financial flow becomes an output of logistics decisions. This perspective may explain why the supply chain literature has largely focused on the integration of material and information flows. Nonetheless, with the recent global financial crisis limiting the availability of external funding, many multinational, multidivisional corporations in their “hunt for cash” have witnessed a significant increase on their intragroup financial transactions (Rogers et al. 2009, Linebaugh 2013). The reason is simple: these multinationals realized that they can utilize the intragroup liquidity for centralized planning to achieve system efficiency. Cash pooling is a common practice for this purpose (Polak and Klusacek 2010).

Under cash pooling, the headquarters creates a corporate master account that aggregates divisions’ cash on a daily basis through a financial services platform (Jansen 2011). Although the value of cash pooling has been discussed in the finance literature, there is little research that assesses the value from a supply chain perspective. Indeed, the discussion of integrating financial flows into supply chain models is relatively sparse in the operations literature. In this paper, we make a small step in exploring these issues.

We consider a firm that owns a supply chain consisting of two divisions. The downstream division (division 1), facing stochastic customer demand, replenishes inventory from an upstream division (division 2), which further replenishes from an outside ample vendor. There is a positive delivery lead time for both divisions. The demands are independent between periods but not necessarily identical. The firm installs a financial services platform that creates a single cash account managed by the headquarters. To facilitate the discussion, we designate division 1 as the headquarters and refer to this system as the cash pooling model. In each period, division 1 receives cash payment from customers who order the material, and division 2 pays to the

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outside vendor for the material delivered. The headquarters has to decide how much cash the supply chain should hold for operations, i.e., inventory payments in our context. Typically, firms do not wish to hold excess cash for operations as it loses the potential benefit from external investments. We model this potential loss as the cash holding cost. On the other hand, liquidating the invested assets to assist operations incurs transaction costs, or may not be feasible in some cases (Baumol 1952, Miller and Orr 1966). Thus, the headquarters has to find a balance between the cash retained for internal operations and that invested for external assets. To focus on the cash flow generated by the system, we assume that there is no external financing. There are linear holding and backorder costs related to inventory. The objective is to determine a joint optimal inventory replenishment and cash retention policy within a finite horizon.

We first formulate a dynamic program for the cash pooling model that includes two inventory states and one cash state (i.e., the firm’s cash account). To be consistent with the inventory literature, we name division and stage interchangeably. The problem is difficult to solve as one cannot directly prove a structured joint optimal policy. Nevertheless, by redefining the state variables into echelon terms, we can transform the original two-stage system into a three-echelon system, under which the optimal joint policy can be characterized. The optimal policy is surprisingly simple. The inventory policy has the same structure as that of the traditional multi-echelon system (see Clark and Scarf 1960): Each stage reviews the echelon inventory position at the beginning of a period and orders up to a target echelon base-stock level. For the cash retention policy, stage 1 (or the headquarters) reviews the entire system working capital (= inventory on hand at both stages + inventory in transit – backorders at stage 1 + inventory-equivalent total system cash) at the beginning of each period and retains the cash holding within an interval determined by two threshold values. The above echelon policy can be converted into an equivalent local policy that can be implemented by each division with local information. A key technical contribution is that we simplify the computation by decoupling the original dynamic program with three states into three separate dynamic programs, each with one state variable. Thus, the optimal policy parameters can be easily calculated. The decoupling result is established by deriving a set of cash-related penalty cost functions, which appear to be new in the literature.

The cash pooling model reveals new insights in the inventory literature. First, although the optimal inventory policy has the same structure as that of the traditional multi-echelon inventory model, the inclusion of cash flows has a direct impact on the base-stock levels. In particular, when holding cash becomes more costly, the system should stock more in the presence of transaction costs. On the other hand, maintaining an optimal cash level depends on the system working capital that includes the total inventory in the supply chain. These two results suggest that a close inter-departmental collaboration between accounting, finance, and operations is crucial to improve the supply chain efficiency. Second, one important financial decision for a firm is to decide how much cash to hold in order to cope against the volatility of the external environment (Opler et al. 1999, Ramirez and Tadesse 2007, Baum et al. 2008). In a numerical study, we find that when the demand becomes more variable, the supply chain should simultaneously increase the optimal amount of cash and inventory holdings. Nevertheless, the change of cash holding is relatively smaller than that of inventory holding. Thus, our analysis is useful for firms to decide an optimal liquid-asset mix (i.e., cash and inventory).

We further investigate the value of financial integration under cash pooling. To assess this value, we compare the cash pooling model with a benchmark model in which the downstream division orders from the upstream division according to a fixed internal transfer price (i.e., the price a selling division charges for a product or service supplied to a buying division of the same corporation; see Abdallah 1989). We refer to this two-account system as the transfer pricing model. The optimal cost of the transfer pricing model is the minimum cost that a two-account system (including a decentralized model with two individual firms) can achieve. Thus, the cost difference between the cash pooling model and transfer pricing model is the actual value of financial integration. The transfer pricing model resembles some virtually integrated supply chains in which entities are individual firms but form a strategic alliance to minimize the supply chain cost (Porter 1985). It can also be viewed as a traditional supply chain in which cash flows are driven by logistics decisions.

Solving the transfer pricing is more involved. Unlike the traditional capacitated inventory systems (see Parker and Kapuscinski 2004), the on-hand cash level becomes an endogenous constraint determined by the order quantity. Although we are not able to characterize the optimal policy, we provide a lower bound to the optimal cost by connecting the transfer pricing model to an assembly system (see Rosling 1989). Our numerical result suggests that the value of cash pooling can be very significant when the internal transfer price is low and the demand is increasing or when the transfer price is high and the demand tends to be stationary. For the former case, the upstream division tends to have a cash shortage. Lacking cash restricts the order quantity, which affects the material supply to the downstream stage. For the latter case, excess cash will be accumulated in the upstream division, so cash pooling will help invest the excess cash to external assets.

The transfer pricing model assumes a fixed transfer price. This is commonly seen in the literature and in practice (Cohen and Lee 1989, Huh and Park 2013, Shunko et al. 2014). It is well known that transfer pricing is a powerful cash redistribution mechanism for multinationals. Thus,
theoretically, it would be interesting to see how much benefit can be recovered if the headquarters can dynamically optimize the transfer price. We refer to the system with an optimized transfer price as the *optimal pricing* model. Interestingly, solving the optimal pricing model is a simple extension of the cash pooling model. In a numerical study, we find that optimizing the transfer price can recover a significant amount of cash pooling benefit. We demonstrate this benefit of cash redistribution between divisions through a system with seasonal demand in §5.2: during the low demand season, the upstream division has sufficient fund for external inventory payment. Thus, the headquarters can reduce the transfer price so that the downstream division will pay less and dispose more cash to external investments. When preparing for the high demand season, the headquarters can increase the transfer price to ensure the upstream division has sufficient cash for a larger amount of inventory ordering. The cash redistribution may be also implemented through trade credit contracts between divisions. We refer the reader to §4.3 for such discussion.

## 2. Literature Review

Our work is related to four streams of research in the literature: cash management, multi-echelon inventory models, capacitated inventory models, and inventory models with financial issues.

For cash management in single firms, most papers treat cash as inventory and use inventory control tools to find the optimal cash balance for firms. Baumol (1952) studied the optimal cash level for a firm that uses cash either for paying transactions or for investment. We have a similar setup for the headquarters in our model. This line of research was further extended by Tobin (1956) and Miller and Orr (1966). For dynamic, periodic-review cash balance problems, Girgis (1968) modeled the selection of a cash level in anticipation of future net expenses as a single-product, multi-period inventory system. Heyman (1973) presented a model to minimize the average cash balance subject to a constraint on the probability of stock-out. The difference between these studies and ours is that we specifically model the cash and inventory dynamics as two interrelated flows.

For cash management in multidepartmental corporations, our model is related to resource allocation from a centralized planning perspective. This literature can be categorized into two groups. The first group is related to cash pooling. Most literature focuses on how cash pooling is implemented for multinational firms and discusses its potential benefits, e.g., Wündisch (1973), Polak and Klusacek (2010), and Jansen (2011). The second group concerns obtaining systemwide transfer prices to maximize the profit for a multidepartmental corporation, e.g., Merville and Petty (1978), Cohen and Lee (1989), Vidal and Goetschalckx (2001), Gjerdrum et al. (2002), Lakhal (2006), Villegas and Ouenniche (2008), and Perron et al. (2010). These two groups of research support our cash pooling model and transfer pricing model, respectively. Our model is different in that we aim to quantify the value of cash pooling from a supply chain perspective by characterizing the optimal joint inventory replenishment and cash retention policy.

Our research is also related to the multi-echelon literature. In particular, our model incorporates cash flows into the seminal supply chain model developed by Clark and Scarf (1960), who proved that an echelon base-stock policy is optimal. Furthermore, they showed that the problem can be decoupled into a series of one-dimensional dynamic programs by introducing the notion of echelon inventories. Federgruen and Zipkin (1984) and Chen and Zheng (1994) streamlined the analysis by considering an infinite horizon model. Gallego and Özer (2003) proved that the decomposition result can be extended to a serial system with extra demand information. Recently, Angelus (2011) considered a multi-echelon model that allows each stage to dispose excess inventory to a secondary market. He introduced a class of heuristic policies, called disposal saturation policies, which can be obtained by using the Clark-Scarf decomposition.

The capacitated inventory problem is related to our model since the cash constraint on inventory replenishment can be viewed as the supply capacity. For single-stage systems, Federgruen and Zipkin (1986) showed that the modified base-stock policy is optimal. Angelus and Porhous (2002) derived the optimal joint capacity adjustment and production plan with and without carryover of unsold inventory units. Their capacity adjustment decision is similar to our cash retention decision, but our cash holding amount is also affected by payment decisions and random sales. For serial systems, Parker and Kapuscinski (2004) demonstrated that a modified echelon base-stock policy is optimal in a two-stage system where there is a smaller capacity at the downstream facility. Huh et al. (2010) studied the stability issue of the system. The main difference between the serial capacitated models and ours is that the cash constraint is endogenously determined by the inventory and cash decisions.

Finally, there have been several recent studies to incorporate financial flows into inventory models. Most of these papers are based on single-stage systems. Buzzacott and Zhang (2004) incorporated asset-based financing into production decisions. They demonstrated the importance of joint consideration of production and financing decisions to capital constrained firms. Chao et al. (2008) considered a self-financing retailer who replenishes inventory under a cash budget constraint. They characterized the optimal inventory control policy. Babich (2010) studied a manufacturer’s joint inventory and financial subsidy decisions when facing a supplier whose financial state is governed by a firm-value model. He showed that an order-up-to policy and subsidize-up-to policy are optimal for the manufacturer. Yang and Birge (2011) modeled a Stackelberg game between a retailer and a supplier with the use of
a trade credit contract. They demonstrated that an effective trade credit contract can enhance supply chain efficiency. Bendavid et al. (2014) analyzed the material management practices of a self-financing firm under working capital requirement. Song et al. (2014) provided a new accounting framework to study inventory systems with different payment times. Tanrisever et al. (2012) built a two-period model to study a start-up firm’s trade-off between process investment and survival. Li et al. (2013) studied a dynamic model in which inventory and financial decisions are made simultaneously in the presence of uncertain demand. They characterized the policy that maximizes the expected present value of dividends. Luo and Shang (2013) modeled a firm with two-level trade credits and payment defaults. They proved the optimal and near-optimal policies and quantified the value of cash flow information. For multi-echelon models, Hu and Sobel (2007) studied a serial inventory model with the objective of optimizing the expected present value of dividends. They showed that an echelon base-stock policy is no longer optimal with financial constraints. Shang et al. (2009) provided a framework of supply chain finance that demonstrates how inventory decisions can be coordinated between supply chain partners through payment transfers in a serial system with fixed order costs. Protopappa-Sieke and Seifert (2010) conducted a simulation study on a two-stage supply chain to reveal qualitative insights on the allocation of working capital between the supply chain partners. Chou et al. (2013) studied a one-warehouse-multi-retailer system with trade credits. They showed that a longer trade credit term received from the external supplier may not lead to a longer trade credit term provided to the retailers.

The rest of this paper is organized as follows. Section 3 studies the cash pooling model and formulates the corresponding dynamic program. Section 4 focuses on the transfer pricing model. We develop a lower bound to the optimal cost. Section 5 discusses the qualitative insights through a numerical study. Section 6 concludes. Appendix provides proofs. Throughout this paper, we define $x^+ = \max(x, 0)$, $x^- = -\min(x, 0)$, $a \lor b = \max(a, b)$, and $a \land b = \min(a, b)$.

3. Cash Pooling (CP) System

We consider a periodic-review, two-stage serial supply chain where stage 1 orders from stage 2, which orders from an outside ample vendor. The supply chain is owned by a single corporation, with stage 1 being the headquarters and stage 2 the subsidiary. (We designate stage 1 as the headquarters to facilitate the discussion, but it does not have to be.) The logistics of this supply chain is fairly standard: Stage 1 faces a stochastic customer demand $D_i$ in period $i$. The demands are independent between periods, but the demand distributions may differ from period to period. We assume that the material lead time is one period for both stages (without loss of generality). In each period, each stage reviews its local inventory position (= inventory on-order + inventory on-hand – backorders) and places an order from its upstream stage. Unsatisfied demands are fully backlogged.

The headquarters creates a corporate master account that manages cash of the entire supply chain. In each period, after receiving the customer’s payment, the headquarters decides the amount of cash used for external investments, such as money and bond markets, facility expansion, or research and development (R&D), etc. The remaining cash will be used for operations, that is, paying inventory decisions can be coordinated between supply chain partners through payment transfers in a serial system with fixed order costs. Protopappa-Sieke and Seifert (2010) conducted a simulation study on a two-stage supply chain to reveal qualitative insights on the allocation of working capital between the supply chain partners. Chou et al. (2013) studied a one-warehouse-multi-retailer system with trade credits. They showed that a longer trade credit term received from the external supplier may not lead to a longer trade credit term provided to the retailers.

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amount of cash. The latter is generally true since inventory holding cost consists of both the financial opportunity cost and the physical shelf cost.

The inventory replenishment and cash retention decision is made centrally by the headquarters. The sequence of events in a period is as follows: at the beginning of the period, (1) shipments are received at both stages; (2) payment is made to the outside vendor; (3) cash retention decision is made; (4) orders are placed at both stages. During this period, demand is realized and sales revenue is collected. At the end of the period, all inventory and cash related costs are calculated. The planning horizon is $T$ periods, and the objective is to minimize the supply chain’s total expected discounted cost within the entire horizon.

We now define state and decision variables. For stage $i = 1, 2$ and period $t$, let

- $x'_{1,t} = \text{net inventory level at stage 1 after event (1)}$;
- $x'_{2,t} = \text{on hand inventory level at stage 2 after event (1)}$;
- $w'_t = \text{cash balance in the pooled account after event (2)}$;
- $v_t = \text{amount of cash transferred into the pooled account in event (3)}$;
- $z_{i,t} = \text{order quantity for stage $i$ made in event (4)}$;

Note that $w'^+_t$ is the cash amount that flows into the pooled account and $w'^-_t$ is the cash amount that flows out for investment. Clearly, $v_t$ cannot exceed $K'$. Let $p_1$ be the unit selling price to the end customer and $c$ be the unit procurement cost from the outside vendor. We assume $c < p_1$ to ensure profitability. The system dynamics are shown below:

\begin{align*}
x'_{1,t+1} &= x'_{1,t} + z_{1,t} - D_t, \tag{1} \\
x'_{2,t+1} &= x'_{2,t} + z_{2,t} - z_{1,t}, \tag{2} \\
w'_{t+1} &= w'_t + v_t - cz_{2,t} + p_1D_t. \tag{3}
\end{align*}

For the cash dynamics in (3), we assume that the actual payment transaction to the outside vendor occurs upon the receipt of shipments. That is, the vendor will not receive the payment determined in period $t$ until period $t+1$, when stage 2 receives the shipment (placed in period $t$). This payment practice is similar to a letter of credit. In other words, we can view that there is a one-period lead time for the cash payment. (Our analysis holds for an alternative assumption of payment on order by slightly changing the dynamics.) As for the payment received at stage 1, we assume that the customer will pay at the order epoch. This assumption is reasonable because all demands will be filled under the backorder model. It is also commonly seen in practice (such as iPhone; see Song et al. 2014 for different payment times adopted in practice) and in the dynamic pricing literature (e.g., Federgruen and Heching 1999). The payment-on-order assumption at stage 1 is for technical tractability. In §5.3, we show cash dynamics for an alternative assumption of payment on receipt and that the optimal policy obtained from the payment-on-order assumption remains effective for the payment-on-receipt system. We do not include inventory holding and backorder costs in (3) because inventory holding cost is usually not incurred in the periodic cash transactions, and backorder cost usually represents loss of goodwill, which is a nonmonetary cost.

Define $x'(t) = (x'_1(t), x'_2(t))$, and $z(t) = (z_1(t), z_2(t))$. The constraint set in each period is

\[ S(x'_2, w) = \{z, v \mid 0 \leq z_1 \leq x'_2, 0 \leq z_2 \leq (w_1 + v)/c, v \leq K'\}. \tag{4} \]

The first constraint states that stage 1’s order quantity cannot exceed stage 2’s on-hand inventory; the second constraint states that stage 2’s order quantity is constrained by the cash balance in the pooled account, which also implies that the investment amount in each period cannot exceed its on-hand cash level; i.e., $v' \geq w'$. Finally, the last constraint imposes a limit $K'$ on the amount of cash that can be injected into the pooled cash account.

The single-period expected cost function is

\[ G(x', w', z_2, v) = E_D[h'_1(x'_1 - D_t) + b(x'_1 - D_t)^-] + h'_2x'_2 + cz_2 + q'E_D(w' + v + p_1D_t) + \beta_1v + \beta_2v'^- \tag{5} \]

The first line in the cost function is the inventory-related cost, which includes inventory holding, backlogging, and procurement costs. By convention, we charge $h'_2$ to the pipeline inventory, so $h'_2x'_2$ is the cost for the inventories held at stage 2 plus those in the pipeline. The second line is the cash-related cost, which includes cash holding and transaction costs. As shown, we charge $q'$ for $w' + v + p_1D_t$ because the inventory payment to the outside vendor is held until the receipt of goods.

Let $\alpha$ be the single-period discount rate. Denote $J_i(x', w', z, v)$ as the expected cost over period $t$ to $T + 1$, given states and decisions $(x', w', z, v)$. Denote $\hat{V}_i(x', w')$ as the minimum expected cost over period $t$ to $T + 1$ over all feasible decisions. The dynamic program is

\[ J_i(x', w', z, v) = \hat{G}_i(x', w', z_2, v) + \alpha E_D[\hat{V}_{i+1}(x'_1 + z_1 - D_t, x'_2 + z_2 - z_1, w' + v - cz_2 + p_1D_t)] \tag{6} \]

\[ \hat{V}_i(x', w') = \min_{z, v \in S(x', w')} J_i(x', w', z, v), \tag{7} \]

with $\hat{V}_{T+1}(\cdot, \cdot) = 0$. Here we assume a zero terminating cost for simplicity. In the sequel, we omit the terminating cost from the dynamic program if it equals to zero.

The local formulation in (6) and (7) is difficult to solve. Specifically, one can show the joint convexity of $J_i(\cdot)$ and
derive a state-dependent global minimum solution. However, computing the solution is quite hard because of the curse of dimensionality. In the next section, we transform the original problem into a new system, from which the exact optimal joint policy can be shown to have a surprisingly simple structure.

### 3.1. Echelon Formulation

We transform the original two-stage system into a three-stage serial model by introducing new system variables. First, define the following echelon variables:

\[ x_1 = x'_1, \quad x_2 = x'_1 + x'_2, \quad w = x'_1 + x'_2 + w'/c. \]

Let \( x = (x_1, x_2) \). We refer to \( x \) as the echelon net inventory level and \( w \) as the net working capital level measured in inventory unit, which is obtained by converting cash to inventory at the value of \( c \). This state transformation explicitly treats cash as inventory. More specifically, the financial inventory at the value of \( c \) is obtained by converting cash to inventory unit, which is obtained by converting cash to inventory.

Similar to the multi-echelon inventory model, we derive the echelon holding cost rate as follows:

\[ h_i(x_i) = E[D_i((h_i + \eta + b)(D_i - x_i) + h_1(x_1 - D_i))], \]
\[ H_2(x_2) = E[D_2h_2(x_2 - D_i)], \]
\[ H_3(r) = E[D_3\eta(r + \theta D_i)]. \]

Then we can rewrite the dynamic program in (6) and (7) as follows:

\[ J(x, w, y, r) = G(x, w, y, r) + \alpha E[D_iV_{i+1}(y_i - D_i, y_{i+1} - D_i, r + \theta D_i)], \]
\[ V_i(x, w) = \min_{y, r \in S(x, w)} J_i(x, w, y, r), \]

where the single-period cost function can be shown as

\[ G(x, w, y, r) = H_i(x_i) + H_2(x_2) + H_3(r) + c(y_2 - x_2) + \beta_x(r - w)^{+} + \beta_y(r - w)^{+}. \]

We refer to (9) and (10) as the echelon formulation of the CP model.

### 3.2. The Optimal Policy

We first state the optimal joint policy for the CP model, which includes two types of decisions made through four control parameters \((y^*_i, y^*_2, l^*, u^*)\) in each period. For the inventory ordering decisions, each stage implements an echelon base-stock policy. That is, stage \( i \) reviews its \( x_i \) at the beginning of each period. If \( x_i < y^*_i \), it orders up to \( y^*_i \) or as close as possible if its upstream does not have sufficient stock; otherwise, it does not order. For the cash retention decision, stage 1 reviews \( w \); if \( w > u^* \), it dispenses cash down to the maximum of \( u^* \) and \( x_2 \); if \( w < l^* \), it retrieves cash up to \( l^* \) or as close as possible (because of the upper bound \( K \)); otherwise, it does not transfer cash.

For the traditional multi-echelon inventory model, there exists an equivalence result between echelon and local base-stock policies. Namely, each stage will generate exactly the same inventory orders based on the local and echelon policies; see Chapter 8 of Zipkin (2000). This result can dramatically simplify the implementation of the optimal policy because each stage can monitor its local information to execute the optimal policy. We have a similar result here: the optimal echelon policy \((y^*_i, y^*_2, l^*, u^*)\) can be converted back to the local term \((y^*_i, y^*_2, l^*, u^*)\), where \( y^*_i = y^*_i, y^*_2 = y^*_2 - y^*_i, l^* = l^* - y^*_2, \) and \( u^* = u^* - y^*_2 \). In this way, stage \( i \) can implement a local base-stock policy based on its local inventory level \( x_i \); the headquarters (or the accounting department at stage 1) can implement a local two-threshold policy based on the cash position \( w^* \).

We next explain how the optimal policy is derived and how to calculate these policy parameters. This is done by transforming a three-state dynamic program into three single-dimensional dynamic programs. We summarize the main result in the following proposition.
Proposition 1. For all \( t \) and \((x, w)\), \( V_t(x, w) = f_{1,t}(x_1) + f_{2,t}(x_2) + f_{3,t}(w) \), where \( f_{3,t}(\cdot) \) is convex.

We define \( f_{1,t}(\cdot) \) as the expected optimal cost for echelon \( i \) in period \( t \). Starting from echelon 1, we have

\[
f_{1,t}(x_1) = H_{1,t}(x_1) + \min_{y_1 \leq y_1^*} \{ \alpha E_{D_t} f_{1,t+1}(y_1 - D_t) \}.
\]

(11)

Let \( g_{1,t}(y_1) = \alpha E_{D_t} f_{1,t+1}(y_1 - D_t) \); then the optimal control parameter \( y_1^* \) can be obtained by solving the minimization problem:

\[
y_1^* = \arg \min_{y_1} \{ g_{1,t}(y_1) \}.
\]

Now, we express the expected optimal cost functions of echelon 2 as follows:

\[
f_{2,t}(x_2) = H_{2,t}(x_2) + \Gamma_{2,t}(x_2) + \Lambda_{2,t}(x_2) + \min_{y_2 \leq y_2^*} \{ c(y_2 - x_2) + \alpha E_{D_t} f_{2,t+1}(y_2 - D_t) \}.
\]

(12)

Here, \( \Gamma_{2,t}(\cdot) \) is the so-called induced penalty cost functions defined in Clark and Scarf (1960); i.e.,

\[
\Gamma_{2,t}(x_2) = \begin{cases} \alpha E_{D_t} f_{1,t+1}(x_2 - D_t - f_{1,t+1}(y_1^* - D_t)), & x_2 \leq y_1^*, \\ 0, & \text{otherwise} \end{cases}
\]

(13)

It represents the penalty cost charged to echelon 2 if stage 2 cannot ship up to stage 1’s target base-stock level \( y_1^* \). Similar to echelon 1, let \( g_{2,t}(y_2) = c(y_2 + \alpha E_{D_t} f_{2,t+1}(y_2 - D_t)) \) and \( y_2^* = \arg \min_{y_2} \{ g_{2,t}(y_2) \} \).

For echelon 3,

\[
f_{3,t}(w) = \Lambda_{3,t}(w) + \begin{cases} L_t(w), & w \leq l_t^*, \\ H_{3,t}(w) + \Gamma_{3,t}(w) + \alpha E_{D_t} f_{3,t+1}(w + \theta D_t), & l_t^* < w \leq u_t^*, \\ U_t(w), & w > u_t^* \end{cases}
\]

(14)

where

\[
\Gamma_{3,t}(r) = \begin{cases} c(r - y_2^*), & r \leq y_2^*, \\ -f_{2,t+1}(y_2^* - D_t), & r > y_2^*, \\ 0, & \text{otherwise} \end{cases}
\]

(15)

Although bearing the same mathematical structure as \( \Gamma_{2,t}(\cdot) \), \( \Gamma_{3,t}(\cdot) \) has a different economic meaning: it represents the penalty cost charged to the headquarters’ accounting department if it fails to hold sufficient cash to pay for the inventory procurement up to the target echelon base-stock level \( y_2^* \).

There are new penalty cost functions \( \Lambda_{2,t}(\cdot) \) and \( \Lambda_{3,t}(\cdot) \) in (12) and (14). To illustrate their meanings, we define

\[
g_{3,t}(w) = H_{3,t}(w) + \Gamma_{3,t}(w) + \alpha E_{D_t} f_{3,t+1}(w + \theta D_t),
\]

(16)

\[
L_t(w) = -\beta_t(w - l_t^*) + g_{3,t}(l_t^*),
\]

(17)

\[
U_t(w) = \beta_u(w - u_t^*) + g_{3,t}(u_t^*).
\]

(18)

One can view \( g_{3,t}(w) \) as the optimal cost for echelon 3 when the system working capital \( w \) is in \([l_t^*, u_t^*]\). Under the optimal policy, when \( w < l_t^* \), stage 1 should retrieve cash until \( w \) reaches \( l_t^* \). Thus, \( L_t(w) \) can be viewed as the optimal cost when \( w < l_t^* \). Similarly, \( U_t(w) \) can be viewed as the optimal cost when \( w > u_t^* \) because in this case stage 1 should dispose cash down to \( u_t^* \). With these explanations, the two new penalty cost functions can be defined as follows:

\[
\Lambda_{2,t}(x_2) = \begin{cases} 0, & x_2 \leq u_t^*, \\ g_{3,t}(x_2 - U_t(x_2)), & w \leq l_t^* - K,
\end{cases}
\]

(19)

\[
\Lambda_{3,t}(w) = \begin{cases} g_{3,t}(w + K) + \beta_t K - L_t(w), & w \leq l_t^* - K,
\end{cases}
\]

(20)

Let us first consider \( \Lambda_{2,t}(x_2) \) in (19). This is a penalty cost charged to echelon 2 if the system carries too much inventory. Intuitively, if echelon inventory \( x_2 \) is less than or equal to \( u_t^* \), echelon 3 (the headquarters accounting department) can always maintain a system of working capital between \( l_t^* \) and \( u_t^* \). However, if \( x_2 > u_t^* \), the best that stage 1 can do is to dispose all cash on hand, making \( w = x_2 \). In such a case, the extra cost \( g_{3,t}(x_2 - U_t(x_2)) \) incurred at echelon 3 should be charged to echelon 2 because of its excess inventory. For this reason, we call \( \Lambda_{2,t}(x_2) \) the excess inventory penalty. (Recall that \( \Gamma_{2,t}(x_2) \) is the penalty cost charged to echelon 2 due to insufficient inventory holding.) The cash retention control thresholds can be obtained from the following equations:

\[
l_t^* = \sup \{ w : \frac{\partial}{\partial w} g_{3,t}(w) \leq -\beta_t \}.
\]

\[
u_t^* = \sup \{ w : \frac{\partial}{\partial w} g_{3,t}(w) \leq \beta_u \}.
\]

With a similar logic, \( \Lambda_{3,t}(w) \) in (20) can be explained: this is a self-induced penalty cost charged to echelon 3 if the system working capital \( w \) is less than \( l_t^* - K \) due to too much cash disposal in the previous period. In this case, stage 1 is penalized with the extra cost \( g_{3,t}(w + K) + \beta_t K - L_t(w) \) for over-disposing cash. Figure 3 illustrates the relationship between three echelons and four penalty cost functions in our problem. The direction of the arrow indicates to which echelon the penalty cost is charged. It also shows how the inventory and cash decisions interact with each other via the \( \Lambda_{2,t}(\cdot) \) and \( \Gamma_{2,t}(\cdot) \) functions. Figure B.1 in Appendix B depicts the induced penalty functions at echelon 2 and 3 as well as the optimal control thresholds \( l^* \) and \( u^* \).

The optimal policy enables us to investigate the impact of cash flows on the inventory stocking decision. Proposition 2 summarizes our findings.
Figure 3. Induced penalty functions of the cash pooling model.

**Proposition 2.** When demand is independent and identically distributed (i.i.d.) and the initial (first period) echelon inventory level \(x_{1,1} \leq u^*_1\),

1. the optimal echelon base-stock levels \(y^*_1, t\) and \(y^*_2, t\) increase in \(\eta\);
2. the upper threshold of cash retention \(u^*_t\) increases in \(\beta_o\) and \(\beta_i\), and the lower threshold \(l^*_t\) decreases in \(\beta_o\) and \(\beta_i\).

Note that Proposition 2 holds for the system with stochastically increasing demand, provided that \(u^*_t\) is non-decreasing in \(t\). Proposition 2(1) suggests that the firm should increase inventory stocking levels when holding cash is more costly. In particular, the optimal total system stock \(y^*_2\) should increase in the cash holding cost rate \(\eta\). A higher \(\eta\) implies a better external investment return. Intuitively, the firm should dispose more cash to external investment, which leads to a low cash level. Thus, it is important for the divisions to stock more to prevent inventory shortage because it might be costly to transfer cash for purchasing inventory when the transaction costs are sufficiently large. This effect is more prominent when the demand is increasing because more cash is needed.

It is interesting to compare the solution obtained from our cash pooling model with that of the Clark-Scarf model when \(\eta\) increases. To facilitate the subsequent discussion, let us define the optimal echelon base-stock levels obtained from the Clark-Scarf model as \((y^*_1, y^*_2)\). When \(\eta\) increases, the local holding cost rates \(h^*_2 (= h_2 + \eta)\) and \(h^*_1 (= h_1 + h_2 + \eta)\) increase. Under such condition, it is known that the optimal base-stock level for stage 1, \(y^*_1\), will increase, whereas the optimal system stock level, \(y^*_2\), will decrease (Shang and Song 2003). This is intuitive—since it is more costly to hold inventory for the entire system, the system tends to reduce the total system stock; however, since the “marginal” holding cost at stage 1 (i.e., \(h_1 = h^*_1 - h^*_2\)) remains the same, the system tends to push more stock to stage 1 in order to maintain the same service level.

The different conclusion on how \(\eta\) affects the optimal system stock between the cash pooling model and the Clark-Scarf model is due to the financial restrictions. The Clark-Scarf model can be viewed as a special cash pooling system in which \(\beta_i = \beta_o = 0\) and \(K = \infty\). Under these parameters, any cash amount can be obtained instantaneously without incurring any costs. Thus, the system has no incentive to stock more when the inventory becomes more expensive. On the other hand, when the transaction costs are sufficiently high, it becomes costly to transfer cash to purchase inventory. To maintain the same service level, the system has an incentive to increase the total system stock. Our finding provides a refinement to a traditional wisdom: A firm should stock less when the inventory holding cost increases. Our finding suggests that this statement is true only when there are no costs or restrictions for cash transfers.

The impact of \(\beta_o\) and \(\beta_i\) in Proposition 2(2) is more intuitive. When \(\beta_o\) and \(\beta_i\) increase, it will be more costly to transfer cash between the system and the external assets. Thus, \(u^*_t\) should increase and \(l^*_t\) should decrease, resulting in a fewer number of cash transfers.

The optimal policy also allows us to investigate the impact of operational requirement, such as service level, on the cash retention decision. Proposition 3 summarizes our finding.

**Proposition 3.** The optimal cash retention thresholds \(u^*_t\) and \(l^*_t\) and the optimal echelon base-stock levels \(y^*_1\) and \(y^*_2\) increase in the backorder cost \(b\) for all \(t\).

When the backorder cost rate \(b\) increases, the service level becomes larger. It is a known result that the echelon base-stock levels increase in \(b\) in the traditional model (Shang and Song 2003). We have the same effect here in our cash pooling model. That is, if the system requires a higher service level, the total system working capital will increase, and the cash level tends to increase as well.

### 4. Transfer Pricing (TP) System

Let us now consider the transfer pricing system. In this setting, stage \(i\) holds its own, separate cash account \(w^*_i\), and stage 1 pays stage 2 for the ordered material according to a fixed transfer price \(p_2\). The investment function is held by the headquarters (stage 1). Thus, the cash retention decision directly affects the dynamics of stage 1’s cash balance \(w^*_1\). Similar to the cash pooling system, we denote the cash holding cost rate \(\eta^*_1\) for stage 1, which represents the opportunity cost of holding cash at the division \(i\). Note that \(\eta^*_1\) and \(\eta^*_2\) may be different. For example, the divisions may have different investment opportunities in their respective geographical region or different administrative costs that make the return different. Here, we make no ante assumption on the order of \(\eta^*_1\) and \(\eta^*_2\). The rest of the notation remains the same as that in §3. Figure 4(a) shows the material and financial flows of the TP model.

The inventory dynamics of the TP model are identical to those of the CP model, as in Equation (1) and (2). Due to separate accounts, the cash dynamics of the TP model become

\[
\begin{align*}
w^*_2,t+1 &= w^*_2,t + p_2 z^*_1,t - c z^*_2,t, \\
w^*_1,t+1 &= w^*_1,t + v_t - p_2 z^*_1,t + p_1 D_t.
\end{align*}
\]
As shown in the first inequality, the receipt of shipment. Define \( \mathbf{w} = (w_1', w_2') \). The constraint set for the TP model is

\[
\tilde{S}(x_2', w) = \left\{ z, v \mid 0 \leq z_1 \leq \min\left(w_1' + v/p_2, x_2'\right), 0 \leq z_2 \leq w_2'/c, v \leq K' \right\}.
\]  

As shown in the first inequality, \( p_2z_1 \) cannot exceed the available cash \( (w_1' + v) \).

The single-period expected cost function is

\[
\tilde{G}_t(x', w', z_2, v) = E_{D_t}\left[ h_1'(x_1' - D_t) + b(x_1' - D_t) - z_2 x_2' + \frac{c z_2 + \gamma_i w_i' + \gamma_i' E_{D_t}(w_1' + v + p_1 D_t) + \beta_i' v + \beta_i' v'}{2} \right].
\]

The dynamic program of the TP model can be expressed as follows:

\[
\hat{J}_t(x', w', z, v) = \tilde{G}_t(x', w', z_2, v) + \alpha E_{D_t}\left[ \tilde{V}_{t+1}(x_1' + z_1 - D_t, x_2' + z_2 - z_1, w_2' + p_2 z_1 - c z_2, w_1' + v - p_2 z_1 + p_1 D_t) \right],
\]

\[
\tilde{V}_t(x', w') = \min_{z, v \in \tilde{S}(x', w')} \hat{J}_t(x', w', z, v).
\]

The TP model is essentially a serial inventory problem with capacities (in the form of cash constraints) at both stages. However, these constraints are random and endogenous and are different from those assumed in the traditional capacitated inventory model (e.g., \textit{Parker and Kapuscinski 2004}).

We are not able to obtain the exact optimal joint policy for the TP model because the endogenous capacity constraints make the analysis complicated. Nonetheless, we can obtain a lower bound to the optimal cost of the TP model. In the subsequent sections, we shall introduce a different echelon notion from that of the CP model. From this new echelon formulation, we can connect the TP problem to an assembly system from which the lower bound cost is derived.

4.1. Echelon Formulation

We shall create a different echelon transformation scheme for the TP model. Define

\[
x_1 = x_1', \quad y_1 = x_1' + z_1, \quad x_2 = x_2', \quad y_2 = x_1' + x_2' + z_2,
\]

\[
w_1 = x_1' + w_1'/p_2, \quad r_1 = x_1' + (w_1' + v)/p_2,
\]

\[
w_2 = x_1' + x_2' + w_2'/c.
\]

Here, \( x \) and \( y \) are the same as in the CP model; \( w_1 \) is defined to be stage 1’s working capital (in inventory units); \( w_2 \) is defined as stage 2’s echelon working capital, which includes inventory at both stages and stage 2’s cash balance (in inventory units). With these state transformations, we redefine the echelon holding cost parameters for the TP model: \( h_1 = h_1 c, h_2 = h_2 - \gamma_2 c, \gamma_i = \gamma_i p_2, \gamma_i' = \gamma_i p_2, \) and \( h_1 = h_1 - h_2 - \gamma_1 c, h_1 = h_1 - h_2 - \gamma_1 c, \gamma_i = \gamma_i p_2, \gamma_i' = \gamma_i p_2. \) Also redefine \( \beta_i = p_2 \beta_i', \beta_o = p_2 \beta_o', \theta = p_1/p_2 - 1 > 0, K = K'/p_2; \) and, finally, \( p = p_2/c. \) With the new echelon terms, the feasible set becomes

\[
S(x, w) = \left\{ y, r_1 \mid x_1 \leq y_1 < r_1 \leq w_1 + K, x_1 \leq y_1 < x_2 \leq y_2 \leq w_2 \right\}.
\]

As shown in Figure 4(b), the transformed TP system is similar to an assembly system. We further redefine the holding and backorder cost associated with each echelon as

\[
H_{1,1}(x_1) = E_{D_t}\left[ (h_1 + h_2 + \gamma_2 + \eta_1 + b)(D_t - x_1') + h_1(x_1 - D_t) \right],
\]

\[
H_{2,1}(x_2) = E_{D_t} h_2(x_2 - D_t), \quad H_{1,1}(w_2) = E_{D_t} \eta_2(w_2 - D_t),
\]

\[
H_{4,1}(r_1) = E_{D_t} \eta_1(r_1 + \theta D_t).
\]

The echelon formulation of the TP model becomes

\[
J_t(x, w, y, r_1) = G_t(x, w, y_2, r_1) + E_{D_t} V_{t+1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t, r_1 + \theta D_t),
\]

\[
V_t(x, w) = \min_{y, r_1 \in S(x, w)} J_t(x, w, y, r_1),
\]
where the single-period cost function can be shown as
\[
G_t(x, w, y, r) = H_{i,t}(x_1) + H_{2,t}(x_2) + H_{3,t}(w_2) \\
+ H_{4,t}(r_1) + c(y_2 - x_2) + \beta_t(r_1 - w_1)^+ \\
+ \beta_t(r_1 - w_1)^-.
\] (29)

After the new transformation, some of the complexities caused by the endogenous constraints disappear. More specifically, the dynamics of the new echelon variable \( w_1 \) is no longer depend on \( z_t \). However, the dynamics of echelon \( w_2 \) still depend on the decision \( y_1 - x_1 \) associated with echelon 1, as shown in (27). This unique property underlines the decomposition structure in the CP model and differentiates the TP model from the traditional assembly system (Rosling 1989). Below, we derive lower bounds to the optimal cost of the TP model.

### 4.2. Lower Bounds

This subsection establishes two lower bounds to the optimal cost for the TP model. Recall that the TP model is similar to an assembly system. The main idea of constructing these lower bounds is to decompose this assembly system. Specifically, the expression of \( S(x, w) \) indicates that stage 1’s decision \( y_1 \) is subject to two constraints: one is \( y_1 \leq r_1 \leq w_1 + K \), which represents the cash constraint on the order quantity; the other is \( y_1 \leq x_2 \leq y_2 \leq w_2 \), which can be viewed as a material order constraint in a two-stage system with an endogenous, random capacity \( w_2 \) at the upstream stage 2. Figure 5(a) shows these two sets of constraints.

Now, imagine that the final product sold at stage 1 consists of two components: a physical component (depicted by triangles) supplied from stage 2’s stock, and a “cash” component (depicted by circles) supplied from stage 1’s operating account. The constraint \( 0 \leq z_t \leq \min\{w_1, w_2\} \) in (23) (or, equivalently, \( x_1 \leq y_1 \leq \min\{r_1, x_2\} \)) implies a similar structure to an assembly system: the same amount of inventory and cash equivalent are matched through replenishment at stage 1.

To derive a lower bound to the optimal cost, we relax the above matching constraint by assuming that the components can be ordered and sold separately. As a result, the original system is decoupled into two independent subsystems as shown in Figure 5(b)—subsystem 1 represents the cash flows; subsystem 2 represents the material flow. The sum of the minimum costs of subsystems is a lower bound on the minimum cost of the original system.

We specify the total cost function for each of the subsystems. Let \( h_1 \) and \( h_2 \) be the inventory holding cost for subsystem 1 and 2, respectively, where \( h_1 + h_2 = h \). Let \( b_1 \) and \( b_2 \) be the backorder cost for subsystem 1 and 2, respectively, where \( b_1 + b_2 = b \).

\[
H_{i,t}(x_1) = E_D [h_1 + \eta_1 + b_1(D_r - x_1)^+] + h_1(x_1 - D_r),
\] (30)

\[
H_{1,t}(x_1) = E_D [(h_1 + \eta_x + b_x)(D_r - x_1)^+ + h_1(x_1 - D_r)].
\] (31)

Now, let us define

\[
G_1(x_1, w_1, r_1) = H_{1,1}(x_1) + H_{4,1}(r_1) + \beta_t(r_1 - w_1)^+
\]

\[
+ \beta_t(r_1 - w_1)^-.
\] (32)

\[
G_2(x_1, x_2, w_1, w_2) = H_{2,2}(x_2) + H_{4,2}(w_2)
\]

\[
+ c(y_2 - x_2).
\] (33)

Note that \( H_{1,1}(x_1) = H_{4,1}(r_1) + H_{1,1}(x_1) \), hence \( G_1(x_1, w_1, r_1) \) + \( G_2(x_1, x_2, w_1, w_2) = G(x_1, w_1, w_2) \). With this cost allocation, the dynamic program for subsystem 1 can be expressed as

\[
V_1(x_1, w_1) = \min_{x_1 \leq y_1 \leq r_1 \leq w_1 + K} \{G_1(x_1, w_1, r_1) + \alpha E_D V_{r+1}(y_1 - D_r, r_1 + \theta D_r)\}.
\] (34)

And the dynamic program for subsystem 2 is

\[
V_2(x_1, x_2, w_2) = \min_{x_1 \leq y_1 \leq x_2 \leq y_2 \leq w_2} \{G_2(x_1, x_2, w_1, w_2) + \alpha E_M V_{y+1}(y_2 - D_r, y_2 - \rho(y_1 - x_1) - D_r)\}.
\] (35)

**Proposition 4.** \( V_1(x, w) \geq V_1(x_1, w_1) + V_2(x_1, x_2, w_2) \) for all \( (x, w) \) and \( t \).
Proposition 4 shows that for any combination of \((h_1^c, h_2^c)\) and \((b_1, b_2)\), the sum of the two subsystems forms a cost lower bound to the original system. Maximizing expected cost over all combinations of \(h_1\) and \(b\) yields the best lower bound.

The remaining question is how to find the optimal cost of these subsystems. A careful examination of subsystem 1 described in (34) reveals that it is the echelon transformation for a single-stage system with a joint inventory and cash retention decision. Thus, we can characterize the optimal joint policy, i.e., using the base-stock policy to control the inventory replenishment and the two-threshold policy to manage the working capital level.

Solving subsystem 2 is more difficult. The dynamic problem described in (35) is the echelon expression of a two-stage inventory model with random, endogenous capacity \(w_2\) at the upstream stage. There exists no known optimal policy for this model. Thus, we provide two approaches to further develop a lower bound to the optimal cost for subsystem 2. The first idea is to remove \(w_2\) from the constraint set. We term the resulting lower bound as constraint relaxation bound. One can show that the resulting problem is a two-stage serial system that can be solved by the Clark-Scarf algorithm. The second idea is a simulation-based bound. For each demand sample path, we are able to obtain the exact optimal solution for the corresponding deterministic problem. Averaging the optimal costs over all demand sample paths yields a lower bound (due to Jensen’s inequality). We term this bound as sample path bound. We refer the reader to Appendix A for a detailed derivation. We will compare the optimal cost of the CP model with the maximum of these two bounds to quantify the value of financial integration under cash pooling.

### 4.3. Optimal Transfer Pricing (OP) Model

For some multidivisional corporations, it is possible that the headquarters can determine transfer price to efficiently distribute liquidity. This section extends the TP model to optimize the transfer price between the divisions. Notice that the optimal transfer price can be obtained by the optimal order quantity and the optimal cash payment in each period. Thus, we modify the transfer pricing model to incorporate the inter-divisional cash payment decision. For period \(t\), define

\[
m_t = \text{amount of cash payment paid from stage 1 to stage 2 before the demand occurs.}
\]

The optimal transfer pricing model can be obtained by replacing \(p_2z_{1,t}\) with \(m_t\) in the TP model, as shown in Figure 6(a).

Interestingly, solving the OP model is no harder than solving the CP model. More specifically, we can follow the same logic of solving the CP model by defining a set of new echelon variables and cost parameters, transforming the original two-stage model into a four-stage serial system. Figure 6(b) shows the transformed system with division 2’s and division 1’s cash account being stage 3 and stage 4, respectively. Similarly, we can decompose the resulting four-state dynamic program into four separable, single-state dynamic programs. We refer the reader to Luo and Shang (2012) for the detailed analysis. In summary, we can obtain the exact optimal joint inventory, cash payment, and retention policy for the OP model. The optimal transfer price is equal to the optimal cash payment divided by the optimal order quantity.

**Proposition 5.** The optimal policy for the OP model can be described as follows. For inventory replenishment, both stages implement an echelon base-stock policy; for cash payment, stage 2 monitors its echelon working capital \((x_1 + x_2 + w_2/c)\) and receives payment up to a target level; for cash retention, stage 1 monitors the system working capital and maintains it within an interval.

Note that the cash redistribution by optimizing the transfer price may be implemented through trade credit contracts if fixing the transfer price \(p_2\) is necessary under certain circumstances. Trade credit is a loan offered by a supplier to allow its customer to delay inventory payments for certain periods of time. In our model, when division 2 (the supplier) has more cash than needed, it can offer trade credit to division 1 that specifies the delayed payment of \((p_2z_1^* - m^*)^+\). Division 1 can disburse these delayed payments in a later period when division 2 experiences cash shortage. We illustrate this idea in a system with seasonal demand in §5.2.

**Figure 6.** (Color online) The transformed optimal transfer pricing model.
5. Numerical Study

5.1. Value of Cash Pooling

We assess the value of cash pooling by comparing the optimal cost of the CP model, $C^*$, with the lower bound cost of the TP model, $C_L = \max\{C_R, C_S\}$, where $C_R$ and $C_S$ represent the cost of the constraint relaxation bound and the sample path bound, respectively. We define the value of cash pooling as

$$% value = \frac{C_L - C^*}{C_L} \times 100\%.$$ 

This represents the percentage of cost reduction due to financial integration under cash pooling.

We conduct a numerical study by starting with a test bed that has the time horizon of 10 periods. We fix parameters $\alpha = 0.95$, $c = 1$, $\eta'_1 = 0.05$, and $h'_1 = 1$ and vary the other parameters, with each taking two values: $p_2 = (1.2, 2)$, $p_1 = (2.5, 4)$, $b = (5, 10)$, $\eta_2 = (0.05, 0.2)$, $h_1 = (0.25, 0.75)$, $\beta'_o = (0.05, 0.15)$, and $\beta'_t = (0.05, 0.2)$. In addition, two demand forms are considered. For the i.i.d. demand case, $D_t$ is Poisson distributed with mean $\mu = 10$ for all $t$; for the increasing demand case, $D_t$ is Poisson distributed with the first period mean $\mu_t = 10$ and $\mu_t$ increasing at a rate of 1.2 per period. In both demand cases, we fix the liquidity level $K'_t = \mu$. For each demand form, we generate 128 instances.

The total number of instances in the test bed is 256. For all cases we assume the initial on-hand inventory and cash level $(x'_{1,1}, x'_{2,1}, w_{1,1}', w_{1,1}') = (16, 10, 10, 10)$, roughly equal to the steady-state inventory/cash level under the i.i.d. demand. When computing $C_S$, we run a simulation of 1,000 iterations for each instance. When computing $C^*$, we assume $\eta = \min\{\eta_1, \eta_2\}$, and set the initial balance of the cash pool as $w'_t = w'_{1,1} + w'_{2,1}$.

In this test bed of 256 cases, the average cost reduction of adopting cash pooling is 29.29%. More specifically, the cost reduction is 13.62% when demand is i.i.d and 44.96% when demand is increasing. Table 1 (left) summarizes the value of cash pooling under the i.i.d. demand (128 cases). The results are further aggregated, each displaying the average value of 32 cases with the same $p_2$ and $\eta_2$. As shown in Table 1 (left), cash pooling does not add much value if the transfer price is low (e.g., $p_2 = 1.2$). This is because under the i.i.d. demand, division 2 has to purchase inventory in every period to cover the stationary order received from division 1. With a lower transfer price, division 2’s average inventory procurement cost per period will be close to the average payment received per period. Thus, division 2 will not accumulate too much cash that leads to system inefficiency (hence the value of cash pooling is small). On the other hand, with a high transfer price (e.g., $p_2 = 2$), cash pooling will then play a significant role—it will be better off to allocate more cash to division 1 so less cash will be accumulated at division 2. The value of cash pooling is more significant when division 2 cash holding cost $\eta'_2$ increases as the excess cash will be charged with a higher rate.

Under the increasing demand, the TP system will perform poorly when the transfer price is low. More specifically, as division 1’s order size increases with the demand, ideally division 2 should in turn increase its inventory stocking to prepare for the future increasing order sizes. However, under fixed transfer pricing, division 2 will not have sufficient cash to do so due to its low markup ($p_2 = 1.2$). Table 1 (right) demonstrates this inefficiency. As shown, when the transfer price is low and demand is increasing, the value of cash pooling can be very significant. The value of cash pooling is clearly higher when backorder cost is larger. Figure 7(a) summarizes the conditions under which cash pooling has a significant value.4

Figure 7(b) illustrates the impact of downstream liquidity level $K$ on the value of cash pooling when demand is increasing under different selling prices $p_1$. Here, we set the transfer price $p_2 = 1.05$, $b = 5$, $\eta'_2 = 0.2$, $h'_2 = 0.25$, and $\beta'_o = \beta'_t = 0.05$, and the other parameters are the same as in the test bed. For a fixed selling price, the value of cash pooling is increasing in $K$, but the marginal benefit of cash pooling decreases in $K$. Thus, cash pooling makes a more significant cost improvement when $K$ is small. In addition, we find that $p_1$ and $K$ complement each other’s role as a liquidity source. For example, $(K, p_1) = (8, 1.2)$ and $(4, 1.5)$ yield a similar CP value.

We next examine the impact of demand volatility on the optimal mix between cash and inventory in a simulation study for the cash pooling model. The instances have the following parameters: $p_1 = 2.5$, $b = 10$, $h'_2 = 0.25$, $\eta' = 0.05$, $\beta'_o = \beta'_t = 0.05$, and $K'_t = 0$. All the other parameters are same as those in the test bed. We assume that the demand is negative binomial with mean equal to 10 and variance increasing from 15 to 55 with an increment of 10. We calculate the ratio of cash holding amount to the total system inventory level for each simulated instance and then take an average over these ratios. While both cash and inventory holding amount increase in the demand variability, the ratio of cash to inventory decreases. More specifically, when the variances are 15, 25, 35, 45, and 55, the corresponding ratios are 0.32, 0.30, 0.28, 0.26, and 0.25, respectively. This result suggests that cash holding is less sensitive to the change of demand variability than the inventory holding.

Table 1. Value of cash pooling—i.i.d. demand (left) and increasing demand (right).

<table>
<thead>
<tr>
<th>Cash holding cost $\eta'_2$</th>
<th>Transfer price $p_2$ (%)</th>
<th>Backorder cost $b$</th>
<th>Transfer price $p_2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>1.2 2</td>
<td>5</td>
<td>66.14 16.29</td>
</tr>
<tr>
<td>0.2</td>
<td>12.87 27.38</td>
<td>10</td>
<td>80.25 17.16</td>
</tr>
</tbody>
</table>

[1] Joint Inventory and Cash Management for Multidivisional Supply Chains
5.2. Optimal Transfer Pricing System

Although we have demonstrated significant value of cash pooling, one interesting question to investigate is how much benefit can be recovered by optimizing the transfer price. To answer this question, we compare the optimal cost from the CP model, $C^*$, with the optimal cost from the OP model, $C_o$. We define the percentage cost reduction as $(C_o - C^*)/C_o \times 100\%$. We test the same 256 instances. The average (maximum, minimum) percentage cost reduction of adopting cash pooling is 6.38\% (9.98\%, 3.34\%) for the i.i.d. demand and 9.33\% (14.75\%, 4.65\%) for the increasing demand case. The cost reduction is more significant when the cash holding cost $\eta_i$ or the backorder cost $b$ is high. For the former, cash pooling can consolidate the entire system cash to a single account that has a smaller cash holding cost rate, so the cost saving is more significant when $\eta_i$ is larger. For the latter, cash pooling allows cash to move bidirectionally upstream or downstream instantaneously, making the supply chain more responsive and leading to a smaller number of backorders. Thus, the benefit of cash pooling is more prominent when the backorder cost is high. Comparing the cost reduction between the TP model and the OP model, we find that optimizing the transfer price can retain a big portion of the benefit achieved by cash pooling.

It is interesting to see how the optimized transfer price helps to redistribute cash between these two divisions for the supply chain facing a demand with seasonality. We consider a time horizon of 11 periods with Poisson demand in each period. The demand mean starts from 16 in the high demand period (period 1), declines gradually to 5 in the low demand period (period 6), and then increases back to 16 (period 11). See Figure 8(a). To illustrate the change of transfer prices, we consider a instance with $p_1 = 1.5$, $b = 55$, $\eta_1 = 0.15$, $h_2 = 0.02$, $\beta_o = \beta_i = 0.05$, $K_i = 50$, initial states $(x_{1,1}', x_{2,1}', w_{2,1}', w_{i,1}') = (28, 16, 16, 16)$, and the other fixed parameters in the test bed. We obtain the optimal transfer price $p^*_1$ by dividing the optimal payment $m^*$ with optimal order quantity, $z^*_i$ and plot the optimal transfer price in Figure 8(b).

In this example, we fix $p_2 = 1.1$. One can view that division 1 delays payment to division 2 if $p^*_2$ is lower than $p_2$ and disburses the previously owed payment or otherwise advances payment. Figure 8(b) illustrates this idea. Given

Figure 7. (Color online) Value of cash pooling.

(a) Demand form, transfer price, and CP value

(b) Downstream liquidity and CP value

Figure 8. (Color online) (a) (Top) the mean of seasonal demand; (b) (middle) optimal transfer price; (c) (bottom) trade credit and aggregated account payable.
that the demand declines gradually in the first four periods, division 2 does not need to hold too much cash for purchasing inventory. Thus, division 1 can delay the payment and dispose more cash for external investments. However, notice that there is an increasing trend for the demand after period 6. Consequently, the optimal transfer price increases gradually and is larger than \( p_2 \) after period 5, which represents the need to hold more cash at division 2. Therefore, starting from period 5 until period 8, division 1 can disburse the previously owed inventory payments by setting \( p_2 > p_2 \). From an implementation perspective, trade credit can be applied to facilitate the above cash redistribution. When division 2 has sufficient cash in periods 1 to 4, it can offer trade credit to division 1 that specifies the delayed payment of \( (p_2 z_i^t - m^*)^+ \). Division 1 then disburse these delayed payments in periods 5 to 8. This is illustrated in Figure 8(c), which shows the trade credit amount and the accumulated account payable at division 1. Given that there is a strong demand in period 9, some advance payment from division 1 to division 2 in periods 8 and 9 is necessary to keep the system efficient.

5.3. Payment on Receipt

In the cash pooling model, we assume that customers pay upon ordering rather than at the receipt of an order. In this section, we consider an alternative assumption that customers pay on receipt. We define the following state variables: for period \( t \),

\[
M_t = \text{shipment from stage 1 to the customers;}
\]

\[
B_t = \text{backorder at stage 1 after shipment;}
\]

Under the base-stock policy, it can be shown that (see Chen et al. 2014)

\[
M_t = B_{t-1} - B_t + D_t
\]

\[
= (D_{t-1} - x_{1,t-1})^+ - (D_t - x_{1,t})^+ + D_t. \tag{36}
\]

The new dynamic program can be obtained by replacing \( p_t D_t \) with \( p_t M_t \) in (3). It turns out that the shipment expression in (36) makes the resulting dynamic program encounter the curse-of-dimension issue and the exact optimal policy is hard to obtain.

Nevertheless, we shall demonstrate that the optimal solution obtained from the cash pooling model still is an effective heuristic policy for the payment-on-receipt model. Note that a cash pooling system with payment on order yields a higher discounted cash flow. Thus, the optimal cost of the payment-on-order model serves as a cost lower bound to that of the payment-on-receipt model. Below we conduct a simulation study to examine the performance of the heuristic policy by comparing the resulting cost with the lower bound cost.

We conduct a numerical study using the same test bed as that in §5.1. The average cost increase is 0.51% for the i.i.d. demand case and 2.28% for the increasing demand case. When demand is increasing, the system needs more cash to order inventory in preparation for the higher future demand. In such case, the payment-on-order scheme is more attractive due to its fast cash collection. This explains why the percentage cost increase is higher under increasing demand. As we can see, the overall difference between the heuristic cost and the lower bound cost is minimal even when the service levels are not extremely high. This suggests that the optimal policy developed in §3.2 is robust under the payment-on-receipt model.

6. Concluding Remarks

This paper studies a joint inventory replenishment and cash retention policy for a centralized supply chain under cash pooling. We characterize the optimal policy by transforming the system into a serial system with newly defined echelon variables. We show that the optimal policy can be obtained by solving a series of single-state dynamic programs. From the optimal policy, we demonstrate how inventory decisions and cash retention decisions interact with each other. Finally, we assess the value of financial integration by comparing the cash pooling model with the transfer pricing model. We characterize the conditions under which the value of cash pooling is significant. Our paper contributes to different business disciplines. For general management, we show that an inter-departmental collaboration between accounting, finance, and operations is crucial to ensure supply chain efficiency. For operations, we show that the inventory decision under cash considerations can be determined in a similar fashion as that of the traditional inventory system. Finally, for accounting and finance, we show that it is necessary to take inventory into account when making the cash retention decision.

We end this paper with two remarks. First, we emphasize that the simple structure of the optimal policy is derived from a supply chain in which the headquarters can fully pool their cash for centralized planning and the customers pay on order. While such a model has practical representations (such as vertically integrated supply chains), it remains an interesting and open question to study a decentralized system in which the entities are individual firms, each with its own interests. We hope this work will motivate future research in this area. Second, the value of cash pooling obtained from our analysis may be overestimated because we assume that there are no external financing options in our model. If such external options exist, the value of cash pooling will be smaller.

Acknowledgments

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Appendix A: Lower Bounds for the Transfer Pricing Model

Constraint Relaxation (CR) Bound

We form the lower bound by relaxing the constraint \( y_2 \leq w_2 \) at stage 2. Once \( w_2 \) is removed from the constraint set, it only appears in the single-period cost function in the dynamic program. The following lemma characterizes the expected value of \( w_2 \) through the flow conservation.

**Lemma 1.** Given the initial states \( w_{2,1} \) and \( x_{1,1} \), for any policy we have

\[
E_{D_1,\ldots,D_{n-1}} w_{2,t} = \rho \cdot E_{D_1,\ldots,D_{n-1}} x_{1,t} + B_t,
\]

where \( B_t = (\rho - 1) \sum_{s=1}^{t-1} \mu_s + w_{2,1} - \rho x_{1,1} \).

Recall that in the single-period cost function (33), the function \( H_{3,1}(w_2) \) is a linear function of \( w_2 \). Therefore, by using Lemma 1, we can replace \( H_{3,1}(w_2) \) with \( H_{3,1}(\rho x_1 + B_t) \) without affecting the optimal decision in period \( t \). With this construction, \( w_2 \) can be replaced by \( x_1 \) and subsystem 2 becomes a classic two-stage serial system in which Clark and Scarf’s algorithm can be applied to find the optimal echelon base-stock levels for both stages. The CR bound generally works well when the constraint \( y_2 \leq w_2 \) is not binding, i.e., when stage 2 holds sufficient cash. This occurs if the stage 2’s markup \((p_2/c_2 - 1)\) is high and demand tends to be stationary. However, under increasing demand, it is optimal for stage 2 to order more in anticipation of a future rise in demand. In this case, stage 2’s cash constraint could become binding, especially if its markup is low. Thus, we need another lower bound to complement the performance of the CR bound.

Sample Path (SP) Bound

The difficulty of solving subsystem 2 comes from keeping track of the state \( w_{2,t} \), because the current period’s \( w_{2,t} \), depends on the previous period’s demand and order quantity. However, if we consider a specific demand sample path, \( w_{2,t} \), can be fully characterized by flow conservation.

**Lemma 2.** Let \( d_t(\omega) \) represent the demand realization in period \( t \) given a demand sample path \( \omega \). With initial states \( w_{2,1} \) and \( x_{1,1} \), we have \( w_{2,t} = \rho x_{1,t} + B_t(\omega) \), where

\[
B_t(\omega) = (\rho - 1) \sum_{s=1}^{t-1} d_s(\omega) + w_{2,1} - \rho x_{1,1}.
\]

The proof of Lemma 2 is similar to that of Lemma 1 and thus omitted. Given the initial states and a demand sample path, \( B_t(\omega) \) is a constant. If we replace \( w_{2,t} \) (according to Lemma 2) in both the constraint set and the periodic cost function, subsystem 2 can be reduced to a two-stage serial system with deterministic demand subject to the following constraint (at time \( t \)):

\[
S^L_t(x_1, x_2 | \omega) = \{ y_1, y_2 | x_1 \leq y_1 \leq y_2 \leq c_2 + B_t(\omega) \}.
\]

The constraints state that stage 1’s order decision \( y_1 \) is affected by stage 2’s echelon inventory level \( x_2 \); stage 2’s order decision \( y_2 \) is affected by a linear function of stage 1’s inventory level \( x_1 \). The optimal \( y_1^* \) and \( y_2^* \) can be obtained by solving a two-dimensional convex program in each period. To facilitate the computation, we prove that this problem can be decoupled into two one-dimensional convex programs. Let \( V^d_t(x_1, x_2 | \omega) \) represent the optimal cost for subsystem 2 for any demand sample path \( \omega \) after \( w_{2,t} \) is substituted with \( \rho x_{1,t} + B_t(\omega) \). The following proposition shows the decoupling result.

**Proposition 6.** \( V^d_t(x_1, x_2 | \omega) = v^d_1(x_1 | \omega) + v^d_2(x_2 | \omega) \), where \( v^d_1(x_1 | \omega) \) is a convex function.

We refer the reader to the proof for the detailed formulation of \( v^d_1 \) and \( v^d_2 \) functions. A lower bound to the optimal cost of the subsystem 2 under the SP approach can be found by averaging total costs over all demand sample paths. In summary, we are able to generate two lower bounds—the sum of the optimal cost obtained from subsystem 1 and the optimal cost obtained from either the CR approach or the SP approach.

Appendix B: Induced Penalty Functions

Figure B.1 depicts functions \( f_1(\cdot) \), \( U(\cdot) \), \( g_3(\cdot) \), \( f_3(\cdot) \), as well as induced penalty functions \( A_2(\cdot) \) and \( A_3(\cdot) \) created while decoupling echelon 2 and 3 (with time subscripts suppressed). The optimal control threshold \( l^* (u^*) \) derived as the tangent point of curve \( g_3(\cdot) \) and a line with slope \(-\beta_i (\beta_i)\). Function \( f_3(\cdot) \) is shown as the bold convex curve connected by four different functions, which are, from the right to the left, the linear function \( U(\cdot) \), the convex function \( g_3(\cdot) \), the linear function \( L(\cdot) \), and the convex function \( g_i \) shifted from point \((l^* - K, L(l^* - K))\) to point \((l^* - K, L(l^* - K))\). The induced penalty function \( A_{3i}(w) \) is the difference between \( f_3(w) \) and \( L(w) \) to the left of \( l^* - K \); the induced penalty function \( A_2(x_2) \) is the difference between \( g_3(x_2) \) and \( U(x_2) \) to the right of \( u^* \).

Appendix C: Proofs

Lemma 3 (Karush 1959) shows the additive separation of a function value.

**Lemma 3.** If a function \( f(y) \) is convex on \(( -\infty, \infty) \) and attains its minimum at \( y^* \), then

\[
\min_{a \leq y < b} f(y) = f_L(a) + f_U(b),
\]

where \( f_L(a) = \min_{a \leq y} f(y) \) is convex nondecreasing in \( a \), and \( f_U(b) = f(b) - f(b \lor y^*) \) is convex nonincreasing in \( b \).

Figure B.1. (Color online) Induced penalty functions at echelon 2 and 3.
Proposition 1

Proof. We prove by induction. The claim trivially holds for \( t = T + 1 \). Assume \( V_{i+1}(x, w) = f_{1,i+1}(x_1) + f_{2,i+1}(x_2) + f_{3,i+1}(w) \); then

\[
V_i(x, w) = \min_{y, r \leq \min\{x, w\}} \left[ H_{1,i}(x_1) + \alpha E_{D_{1,i}} f_{1,i+1}(y_1 - D_t) \\
+ H_{2,i}(x_2) + c(y_2 - x_2) \\
+ \alpha E_{D_{2,i}} f_{2,i+1}(y_2 - D_t) \\
+ H_{3,i}(r) + \beta(r - w)^+ + \beta_n(r - w)^- \right]. \tag{C1}
\]

Let \( g_{1,i}(y_1) = \alpha E_{D_{1,i}} f_{1,i+1}(y_1 - D_t) \), and \( g_{2,i}(y_2) = cy_2 + \alpha E_{D_{2,i}} f_{2,i+1}(y_2 - D_t) \). Since \( f_{1,i+1}(\cdot) \) is convex (from induction), by Lemma 3 we can decompose the cost function of echelon 1 and 2:

\[
\min_{y_1 \leq y_2 \leq x_2} g_{1,i}(y_1) = \min_{y_1 \leq y_2} g_{1,i}(y_1) + \Gamma_1, \quad \Gamma_1 \geq 0,
\]

\[
\min_{y_2 \leq x_2} g_{2,i}(y_2) = \min_{y_2 \leq x_2} g_{2,i}(y_2) + \Gamma_2, \quad \Gamma_2 \geq 0.
\]

Let \( \Gamma_1 \) and \( \Gamma_2 \) be the induced penalty functions \( \Gamma_1, \Gamma_2 \) and \( g_3,i(r) = H_3,r + \Gamma_3,i + \alpha E_{D_{2,i}} f_{3,i+1}(r + \theta D_t) \). Then, we define

\[
V_i(x, w) = f_{1,i}(x_1) + H_{2,i}(x_2) + \Gamma_{1} + \min_{y_2 \leq x_2} g_2,i(y_2) + g_3,i(r) + \alpha E_{D_{2,i}} f_{3,i+1}(r + \theta D_t) \tag{C2}
\]

Let \( \hat{r}_i = \min \{g_3,i(r) + \alpha E_{D_{2,i}} f_{3,i+1}(y_2 - D_t)\} \). Therefore, we verify that (C6) holds in all cases. Substituting (C6) into (C2) and defining \( f_{2,i}(\cdot) \) in (12), we complete the induction \( V_i(x, w) = f_{1,i}(x_1) + f_{2,i}(x_2) + f_{3,i}(w) \). Using Lemma 3, all induced penalty functions are convex; thus, \( f_{2,i}(\cdot) \) is convex (i = 1, 2, 3).

Proposition 2

Proof. (1) We prove by induction. Clearly \( f_{1,i+1} \) is submodular in \( (x_1, \eta) \). Suppose \( f_{1,i+1} \) is submodular in \( (x_1, \eta) \). Proposition 1 shows that \( f_{1,i}(x_1, \eta) \) is convex in \( x_1 \) for all \( \eta \). By applying Theorem 3.10.2 in Topkis (1998), we have \( V_{1,i+1}(x_1, \eta) = \min_{x_2 \leq \min\{x, w\}} \{\alpha E_{D_{1,i}} f_{1,i+1}(y_1 - D_t)\} \) is submodular in \( (x_1, \eta) \); i.e.,

\[
\frac{\partial V_{1,i+1}(x_1, \eta)}{\partial x_1} \eta = 0.
\]

Therefore,

\[
\frac{\partial f_{1,i}(x_1, \eta)}{\partial x_1} \eta = -\hat{F}_i(x_1) + \frac{\partial V_{1,i+1}(x_1, \eta)}{\partial x_1} \eta = 0,
\]

where \( \hat{F}_i(\cdot) \) is the cdf of \( D_{i+1} \). This shows that \( f_{1,i} \), and hence \( V_{i-1} \), is submodular in \( (x_1, \eta) \). Therefore, \( y_{1,i}^* \) increases in \( \eta \).

Next, we show that \( y_{1,i}^* \) increases with \( \eta \) using the same induction logic. Note that when demand is i.i.d. and \( x_{2,i} \leq \eta^* \), under
steady state we have \( x_{2,t+1} = y_{2,t} - D_t \leq u^*_t \). Hence by following the optimal policy, \( \Lambda_{2,t}(x_2) = 0 \) for all \( t \). According to (12),
\[
f_{2,t}(x_2, \eta) = h_2(x_2 - \mu_t) + \alpha E_{D_t}[f_{2,t+1}(x_2 - D_t, \eta)] - f_{2,t+1}(y^*_t(\eta), \eta)] I_{1_t < y^*_t(\eta)}(0) + \min_{y_2 \leq y_2} \{c(y_2 - x_2) + \alpha E_{D_t, f_{2,t+1}}(y_2 - D_t, \eta)\},
\]
where \( \mu_t = E(D_t) \) and \( I_{1_t} \) is the indicator function. By induction, \( f_{2,t+1}(y_2 - D_t, \eta) \) is submodular in \( (y_2, \eta) \). Similar to the above proof, we have
\[
V_{2,t+1}(x_2, \eta) = \min_{y_2 \leq y_2} \{c(y_2 - x_2) + \alpha E_{D_t, f_{2,t+1}}(y_2 - D_t, \eta)\}
\]
is submodular in \( (x_2, \eta) \); i.e., \( \partial^2 V_{2,t+1}(x_2, \eta)/\partial x_2 \partial \eta \leq 0 \). Since \( \partial^2 f_{2,t}(x_2, \eta)/\partial x_2 \partial \eta \leq 0 \) for all \( t \) (as shown above), we have \( \partial^2 f_{2,t}(x_2, \eta)/\partial x_2 \partial \eta \leq 0 \). Hence \( f_{2,t} \) and \( V_{2,t} \) are submodular in \( (x_2, \eta) \). Therefore, \( y^*_2 \) is increasing in \( \eta \).

(2) We show by induction that \( u^*_t \) increases in \( \beta_t \) and \( l^*_t \) decreases in \( \beta_t \). The proof regarding \( \beta_t \) is similar and hence omitted. Clearly \( 0 \leq \partial^2 f_{1,t+1}(w, \beta_t)/\partial \omega \partial \beta_t \leq 1 \). Suppose \( 0 \leq \partial^2 f_{1,t+1}(w, \beta_t)/\partial \omega \partial \beta_t \leq 1 \), according to (14),
\[
\frac{\partial^2 f_{1,t}(w, \beta_t)}{\partial \omega \partial \beta_t} = \begin{cases} 
0, \text{ if } w \leq l^*_t; \\
\alpha E_{D_t} \left[ \frac{\partial^2 f_{1,t+1}(w + \theta D_t, \beta_t)}{\partial \omega \partial \beta_t} \right], \text{ if } l^*_t < w \leq u^*_t; \\
1, \text{ if } u^*_t < w
\end{cases}
\]
where the middle line in (C7) is due to \( \partial^2 \Gamma_{1,t}(w, \beta_t)/\partial \omega \partial \beta_t = 0 \), given \( \Lambda_{2,t}(x_2) = 0 \) along the optimal decision path. Therefore we have \( 0 \leq \partial^2 f_{1,t}(w, \beta_t)/\partial \omega \partial \beta_t \leq 1 \), hence \( 0 \leq \partial^2 \Gamma_{1,t}(w, \beta_t)/\partial \omega \partial \beta_t \leq 1 \). The monotonicity results can be easily shown from the expressions of \( l^*_t \) and \( u^*_t \). □

**Proposition 3**
Proof. From the expression for \( H_{1,t} \), we have \( \partial^2 H_{1,t}(x_1, b)/\partial x_1 \partial b = -f(x_1) \leq 0 \). Applying the same induction logic as in the proof of Proposition 2 to (11), we have \( \partial^2 f_{1,t+1}(x_1, b)/\partial x \partial b \leq 0 \), which leads to \( \partial^2 \Gamma_{1,t}(x_1, b)/\partial \omega \partial \beta_t \leq 0 \), which in turn leads to \( \partial^2 f_{1,t}(x_1, b)/\partial \omega \partial b \leq 0 \), which in turn leads to \( \partial^2 \Gamma_{1,t}(x_1, b)/\partial \omega \partial b \leq 0 \), which finally leads to \( \partial^2 \Gamma_{1,t}(x_1, b)/\partial \omega \partial b \leq 0 \). The monotonicity results can be easily shown from the expressions \( y^*_1, z^*_1, l^*_t \), and \( u^*_t \). □

**Proposition 4**
Proof. We prove by induction. \( V_{1,t+1}(x, w) = 0 = V_{1,t+1}(x_1, w_1) + V_{1,t+1}(x_1, x_2, w_2) \). Suppose \( V_{1,t+1}(x, w) \geq V_{1,t+1}(x_1, w_1) + V_{1,t+1}(x_1, x_2, w_2) \) for all \( (x, w) \), then
\[
V_t(x, w) = \min_{y_1, y_2 \in S^{(1,2)}} \{G_t(x, w, y_2, y_1) + \alpha \mathbb{E}_{D_t}[V_{1,t+1}(y_1 - D_t, y_2 - D_t, w_2) + \rho(y_1 - x_1) - D_t, r_t + \theta D_t)] \geq \min_{y_1, y_2 \in S^{(1,2)}} \{G_t(x, w, y_2, y_1) + \alpha \mathbb{E}_{D_t}[V_{1,t+1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)] + \alpha \mathbb{E}_{D_t}[V_{1,t+1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)] \geq \min_{y_1, y_2 \in S^{(1,2)}} \{G_t(x, w, y_2, y_1) + \alpha \mathbb{E}_{D_t}[V_{1,t+1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)] + \min_{y_1, y_2 \in S^{(1,2)}} \{G_t(x, w, y_2, y_1) + \alpha \mathbb{E}_{D_t}[V_{1,t+1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)] = V_1(x_1, w_1) + V_1(x_1, x_2, w_2).
\]
The inequality in (C9) and (C11) are due to induction and constraint relaxation, respectively. The above relationship holds for all \( (x, w) \) in period \( t \), completing the induction. □

**Lemma 1**
Proof. We write out the flow conservation of \( w_2 \) and \( x_1 \) from \( s = 1 \) to \( s = t - 1 \).
\[
E_{D_1, \ldots, D_{t-1}}[w_{2,t} - w_{2,1}] = \sum_{s=1}^{t-1} \rho z_{1,s} - \sum_{s=1}^{t-1} \mu_s,
\]
\[
E_{D_1, \ldots, D_{t-1}}[x_{1,t} - x_{1,1}] = \sum_{s=1}^{t-1} z_{2,s} - \sum_{s=1}^{t-1} \mu_s.
\]
The result is shown by subtracting \( \rho \times (C14) \) from (C13). □

**Proposition 6**
Proof. We first specify the cost functions when demand is deterministic. Define
\[
H_{1,t}^D(x_1) = (h_1^D + y_1 + D_t)(d_1^D - x_1) + h_1^D(x_1 - d_1^D),
\]
\[
H_{2,t}^D(x_2) = h_2(x_2 - d_2^D), \quad H_{3,t}^D(\alpha) = \alpha(a - d_3^D).
\]
We then prove by induction. The claim trivially holds for \( t = T + 1 \). Now, we assume \( V_{1,t}^D(x_1, x_2 | \omega) = v_{1,t}^D(x_1 | \omega) + v_{2,t}^D(x_2 | \omega) \). Let \( g_d^1 = \alpha v_{1,t}^D(\omega - d_1^D) \) and \( g_d^2 = \alpha v_{2,t}^D(\omega - d_2^D) \). From the convexity of \( v_{1,t}^D(\cdot) \) and Lemma 3, we can decompose the cost functions of echelon 1 and 2 as follows:
\[
\min_{x_1 \in S^{x_1}} g_d^1(\omega) = \min_{x_1 \in S^{x_1}} [\alpha v_{1,t}^D(\omega - d_1^D)] + \Gamma_{1,t}(x_1),
\]
\[
\min_{x_2 \in S^{x_2}} g_d^2(\omega) = \min_{x_2 \in S^{x_2}} [\alpha v_{2,t}^D(\omega - d_2^D)] + \Gamma_{2,t}(x_1),
\]
where \( a = \rho x_1 + B_t(x_1) \). Let \( y_{1,t}^D \) minimize \( g_d^1(\omega) \); the induced penalty functions are
\[
\Gamma_{1,t}(x_1) = \begin{cases} 
\alpha [v_{1,t}^D(x_1 - d_1^D | \omega)] - v_{1,t}^D(y_{1,t}^D - d_1^D | \omega), & x_1 \leq y_{1,t}^D; \\
0, & \text{otherwise}.
\end{cases}
\]
\[
\Gamma_{2,t}(x_1) = \begin{cases} 
\alpha [v_{2,t}^D(x_1 - d_2^D | \omega)] - v_{2,t}^D(y_{1,t}^D - d_2^D | \omega), & a \leq y_{1,t}^D; \\
0, & \text{otherwise}.
\end{cases}
\]
From Lemma 3, the functions above are convex. Therefore, the following functions are convex:

\[ v^1_t(x_1 | \omega) = H^d_{x_2}(x_1) + H^c_{x_1}(x_1) + \sum_{i=1}^{T} \nu^c_i(a) \]
\[ + \min_{x_1 \leq y_1} \{a v^1_{i+1}(y_1 - d_i(\omega) | \omega)\} \]
\[ v^2_t(x_2 | \omega) = H^d_{x_2}(x_2) + \sum_{i=1}^{T} \nu^c_i(x_2) \]
\[ + \min_{x_2 \leq y_2} \{c(y_2 - x_2) + a v^2_{i+1}(y_2 - d_i(\omega) | \omega)\} . \]

Furthermore, \( V^d_t(x_1, x_2 | \omega) = v^1_t(x_1 | \omega) + v^2_t(x_2 | \omega) \), completing the proof.

**Endnotes**

1. Allen and Hafer (1984) conducted an empirical study that shows the cash holding cost rate is positively correlated to a company’s interest return on short-term money markets and long-term on bond markets.
2. All the cash related parameters can be nonstationary or follow an exogenous Markov process that reflects the states of the world.
3. In our extensive numerical experiments, this condition always holds, although we are unable to prove it.
4. Figure 7(a) also shows that CR (SP) bound performs better in the upper left (lower right) quadrant.

**References**


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**Wei Luo** is an assistant professor in the Department of Production, Technology and Operations Management at IESE, University of Navarra. His research interests include supply chain management, inventory planning and control, and interface of operations and finance.

**Kevin Shang** is an associate professor at the Fuqua School of Business, Duke University. His research mainly focuses on developing simple and effective inventory policies for supply chains. He is also interested in issues regarding interface of operations and finance and corporate sustainability.