Cash pooling is a powerful management tool that allows each division’s cash balance to be transferred to a single account managed by the corporate treasury. While the reported benefits of cash pooling are associated with the reduction of transaction and financing costs, the value of cash pooling is not clear from a perspective of improving operational efficiency. In this paper, we examine the benefit of cash pooling on inventory replenishment for a corporation with multiple divisions, each replenishing inventory to meet its local demand while receiving cash payments from customers in a finite-time horizon. The corporation can reserve part of the cash for purchasing inventory, and the rest for external investments that yield a positive return. There are holding costs for the on-hand inventory and unfilled demands incur backorder costs. The objective is to obtain the optimal joint cash retention and inventory replenishment policy that maximizes the expected net worth (equity). We show that the problem is equivalent to minimizing the expected total cost, consisting of the cash-related costs and the inventory-related costs. We formulate this problem into a dynamic program. Due to curse of dimensionality, we provide a simple and effective heuristic derived from the construction of a lower bound to the optimal value function. Our lower bound improves the so-called Lagrangian-relaxation bound and the induced-penalty bound in the literature. A numerical study suggests that the value of cash pooling is most significant when the demands are increasing and negatively correlated between the divisions. In addition, the benefit of cash pooling on reducing the inventory-related costs often outweighs that of reducing the cash-related costs. Our solution approach can be applied to the classic one-warehouse-multi-retailer inventory system with non-stationary demands. The resulting lower bound outperforms those in the multi-echelon literature.

Key words: multi-divisional firm, cash pooling, inventory management, working capital

History: December 4, 2018
1. Introduction
In 2002, the Swiss pharmaceutical company Roche, which was diversified both geographically and across business segments, had experienced a significant loss from its financial investments, whereas its core Pharmaceuticals and Diagnostics Divisions performed strongly and yielded positive cash flows. Because Roche lacked the flexibility to allocate resources between its operating units and divisions, it reported a net loss of CHF 4.02 billion for 2002. These events had triggered Roche to restructure its treasury and financing operations from a system with many local, division-based accounting departments to a centralized corporate treasury (Zhang et al. 2012). The corporate treasury serves not only as an in-house bank that provides financial services for the divisions, but also as a centralized payment factory to execute the necessary division payments with suppliers.

A similar scenario occurred to Tyco International, a multinational corporation, which produces a variety of products, including fire and safety, plastics, electronic, and health care products. At Tyco, despite having affluent cash in its balance sheet, the cash was locked up in subsidiaries scattered across the globe and was essentially inaccessible at the corporate level where it was critically needed (Zhang et al. 2011). Similarly, after acquiring the IBM personal computing division, Lenovo realized that cash was trapped in emerging markets because of lacking a financial system that allowed the company to manage capital and risk globally (Zhang et al. 2012). All these firms had created a centralized corporate treasury system that serves as an in-house bank whose function is to consolidate cash flows among divisions in order for better cash management. This is the so-called cash pooling strategy commonly implemented by many multinational corporations.

Under cash pooling, the cash balances on divisions’ accounts are pooled into a central master account collectively managed by the corporate treasury on a daily basis. While the benefits of consolidating cash have been reported in business cases and studied in the finance literature, e.g., reduction of interest charges, less exposure to exchange rate risks, better internal data control and transparency, as well as improving liquidity, they are centered around financial and managerial advantages. The value of cash pooling is not clear from an operations perspective. Specifically, cash consolidation can increase the liquidity so as to better support operational activities. For example, cash consolidation allows divisions to borrow from one another to cover temporary deficits, which eliminates interest payments on the short-term debt for purchasing inventory. Excess cash not being used for operations can be pooled into one single account for external investments, receiving the maximum cash gain due to a larger balance with a better interest rate. In this paper, we aim to build an analytical model to quantify the benefit of cash pooling.
We consider a firm with multiple divisions. Each division replenishes inventory from its supplier to meet the demand in each period. The demands between periods are independent but not necessarily identical, and demands between divisions may be correlated. The firm’s financial flows are centrally managed by the corporate treasury, which has a master account that nets the cash balances between divisions in each period. Specifically, a division receives cash payment after a customer order arrives and fulfills the order as much as possible. Backorders are accumulated; inventory holding and backorder costs are calculated in each period. The cash payment is transferred directly to the master account held by the corporate treasury, which in turn, pays the ordered inventory to the suppliers. In addition, the corporate treasury needs to decide how much cash to reserve for purchasing inventory so that the rest can be used for investing in assets (e.g., financial markets, facility expansion, R&D, etc.) with a positive return. The firm may liquidate the invested assets with transaction costs if it needs to increase the cash holding (e.g., purchasing additional inventory for the increasing demand). The objective is to obtain the optimal joint inventory replenishment and cash reservation policy such that the expected net worth (equity) is maximized at the end of the horizon.

We formulate this problem into a dynamic program and show that maximizing the expected net worth is equivalent to minimizing the total expected cost (= inventory related costs plus cash related costs). We first characterize the optimal cash retention policy – the corporate treasury monitors the system working capital (i.e., cash and monetary inventory value in the system) at the beginning of each period and maintains the cash holding within an interval determined by two thresholds. However, due to curse of dimensionality, it is not possible to fully characterize the joint optimal policy. Thus, from a perspective of implementation and revealing insights, we aim to derive a simple and effective heuristic policy. In order to evaluate the performance of heuristic policies, we construct a lower bound to the optimal cost. For the inventory model with a linear allocation constraint, two lower bounds have appeared in the literature, i.e., the induced-penalty bound for the i.i.d. demand model (Chen and Zheng 1994) and the Lagrangian-relaxation bound for the finite-horizon model (Goel and Gutierrez 2011). Using a novel idea of introducing a parameter that adjusts the cash holding amount in each period, we show that our lower bound dominates the above two known ones. We refer to the parameter as the cash-holding multiplier. In fact, our lower bound converges to the optimal value at the expense of computational efforts. More importantly, we are able to derive a heuristic policy based on the lower bound functions. The heuristic policy has a simple structure: the cash retention policy has exactly the same threshold structure as that of the optimal one; the inventory policy for the division is a modified base-stock policy (i.e., ordering
up to a level determined by cash available). The exact heuristic policy parameters require an input of effective cash-holding multipliers. We propose two methods: the static policy assumes that the cash-holding multipliers are constants across all time periods; the dynamic policy solves the best multipliers dynamically according to the system state in each period. A numerical study suggests that both heuristics are near-optimal. We note that our approach of constructing the lower bound and the heuristic can be applied to the classic one-warehouse-multi-retailer inventory system with non-stationary demands. As shown in Appendix A, the resulting lower bound outperforms those in the multi-echelon literature. Finally, one can view our model as a retailer who manages multiple products with a cash constraint. Thus, our results can comfortably be applied to such a setting.

To quantify the value of cash pooling, we compare the cash pooling system with the same system without cash pooling. The latter assumes that each division keeps a separate cash account for external investments and pays for the ordered inventory according to the demand. We find that value of cash pooling is most significant when the demands are negatively correlated between the divisions and increasing in time. This is because when demands are increasing, both divisions require cash for the increased inventory orders. This effect is most significant when the demands between divisions are negatively correlated. In addition, while cash pooling reduces both the inventory-related costs and the cash-related costs, the cost reduction of the former often exceeds that of the latter, suggesting that the cash-pooling practice is crucial to improve operational efficiency.

The rest of the paper is organized as follows. §2 reviews the relevant literature. We highlight the differences between our model and those in the literature. §3 describes the model in detail. §4 characterizes the properties of the optimal cash management policy. §5 constructs a novel lower bound on the optimal cost. §6 introduces a simple and effective heuristic policy based on the optimality analysis and the lower bound functions. §7 presents numerical results. §8 concludes our work and discusses some extensions. Appendix A demonstrates how our methodology can be applied to a multi-echelon distribution system and generates a new lower bound that outperforms those in the literature. Appendix B shows all proofs.

2. Literature Review

There are primarily four streams of research related to our work: cash management, capacitated inventory systems, multi-echelon distribution systems, and integrated cash and inventory models. From a modeling perspective, this paper is a generalization of a serial supply chain with multiple divisions studied in Luo and Shang (2015) to a distributed supply chain. Thus, to save space, we only review papers most relevant to our model and refer to the reader to Luo and Shang (2015)
for a complete review on the cash management and capacitated inventory literature, as well as the other related papers.

The considered problem and the classic distribution system share some similarities. In particular, the cash level held at the corporate treasury is similar to the inventory amount held at the warehouse, and both models require an allocation of resources to the downstream locations. For this reason, we first review multi-echelon distribution (or one-warehouse-multi-retailer) systems. In the seminal work of Clark and Scarf (1960), the authors point out that an optimal policy, if it exists, would be very difficult to compute and implement due to curse of dimensionality. Nevertheless, under the so-called balance assumption (i.e., inventory can be instantaneously transferred between the downstream locations), they show that an echelon base-stock policy is optimal. Federgruen and Zipkin (1984c) extend this result to an infinite-horizon problem. They show that the resulting echelon base-stock levels are stationary and provide a simple algorithm to compute the base-stock levels. Chen and Zheng (1994) consider the i.i.d. demand model with fixed order costs in each location. They streamline and simplify the optimality proof of Clark and Scarf, and construct two lower bounds based on innovative cost allocation schemes. Given that the optimal policy is difficult to obtain, researchers instead focus on easy and implementable policies. The research work in this category typically provides the optimization algorithms for given policies or develops heuristics to facilitate the implementation. For the systems without fixed order costs, base-stock policies are often considered, e.g., Sherbrooke (1968), Graves (1985), Axsäter (1990), and Gallego et al. (2007). For the system with decisions on order quantities, installation \((r, Q)\) policies and echelon \((r, Q)\) policies are often studied, e.g., Lee and Moinzadeh (1987), Chen and Zheng (1997), and Axsäter (1998, 2000). There is another stream of research that focuses on determining the reorder intervals, e.g., Graves (1996), Axsäter (1993), Cachon (1999), and Shang et al. (2015). We refer the reader to Simchi-Levi and Zhao (2012) and Shang (2011) for an extensive review.

Our work differs from those in the literature in three aspects. First, most papers for the distribution system focus on the infinite-horizon model with given stationary policies. Our model, on the other hand, considers the finite horizon with nonstationary and correlated demands and we aim to obtain the optimal policy. Second, unlike the inventory level resumed by replenishment at the warehouse in the distribution system, the cash level in our model is resumed not only by the cash retention decision, but also by the random sales received from the customers. In addition, the cash level and the inventory value together define an important state variable, i.e., system working capital, which influences the optimal joint decision. Lastly, unlike the linear ordering costs in the distribution system, following Baumol (1952), we assume that linear transaction costs incur when
the cash is transferred to or from the corporate treasury account. This piece-wise linear transaction costs make the cash retention policy more complicated.

We next review papers that incorporate financial flows into inventory systems. Most of these papers are based on single-stage systems, see Buzacott and Zhang (2004), Chao et al. (2008), Babich (2010), Yang and Birge (2011), Tanrisever et al. (2012), Li et al. (2013), and references therein. A few papers study the joint operation and financial decisions in serial inventory systems. Hu and Sobel (2007) consider a two-stage model with financial constraints. The objective is to maximize the expected dividends in a finite horizon. They show that an echelon base-stock policy is no longer optimal. Song et al. (2014) introduce an accounting framework to study the impact of different payment times on the resulting system cost. Protopappa-Sieke and Seifert (2010) consider a two-stage supply chain and reveal qualitative insights on the allocation of working capital between the supply chain partners via a simulation study. The motivation of this work is most related to Luo and Shang (2015). They consider a serial system that integrates cash flows into material flows whereas we study a distribution system. Nevertheless, due to the cash allocation, our analysis is very different from that of Luo and Shang. The only paper to our knowledge that incorporates financial flows into a distribution system is Chou et al. (2013). They consider a distribution network in which trade credit contracts are employed between the supplier and retailers who face deterministic demands. They show that the supplier which receives a long trade credit term from its external vendor may not provide a long trade credit term to its retailers. Their research question as well as the model setup are quite different from ours.

3. Model and Problem Formulation

We consider a centralized-control supply chain in which a corporation manages the inventory and cash flows for its $N$ divisions over $T$ periods. Without loss of generality, for simplicity, we set $N = 2$ for the subsequent discussions. In each period, each division $i$ reviews its inventory level ($=$ on-hand inventory - backorders) and pipeline inventory, and places an order to an outside vendor with ample stock to satisfy the local demand $D_i$. Demands are stochastic and independent between periods but not necessarily identical. The demands between divisions may be correlated. There are no inventory transshipments between the divisions. (For example, the divisions may be far away from each other so transshipment is not economically feasible.) The replenishment lead time for division $i$, denoted by $L_i$, is a positive constant. Unsatisfied demands are fully backlogged.

To better manage cash, the corporation implements a cash pooling strategy; that is, the corporate treasury creates a master account that consolidates cash flows related to operational activities (i.e.,
inventory payments paid to the suppliers and collected from customers) of the entire supply chain. Specifically, at the beginning of each period, after receiving customers’ payments from the previous period, the treasury decides an amount of cash kept in the master account used for inventory replenishment for the current period. The remaining cash will be used for external investments, i.e., purchasing a portfolio of assets recorded in an investment account. We assume that the external investment yields a return rate $\eta$ in each period. On the other hand, the treasury may liquidate the invested assets to assist operational activities, i.e., inventory replenishment in our context, if necessary. We refer to the determination of cash amount for inventory replenishment as the 
\textit{cash retention decision}. The amount of available cash in the master account determines the total inventory amount that can be ordered by both divisions. We consider the so-called pay-on-order scheme, i.e., a payment is created when an order is placed or demand arrives. We assume that all cash transfers are instantaneous. Figure 1 shows the inventory and cash flows in the system. The circle represents the master account, and the oval represents the external investment account. The divisions are denoted by rectangles, and they order from outside vendors which are represented by triangles. Inventory and cash flows are denoted by solid and dash arrows, respectively.

The cash retention decision reflects the practice. A firm typically does not hold excess cash for operations as it loses the potential benefit from external investments. On the other hand, liquidating the invested assets to assist operations incurs transactions costs, or may not be feasible if the transaction costs are significant. Thus, the corporate treasury has to find a balance between the cash retained for inventory purchase and the cash invested for external assets. This modeling approach follows from Ba\textsuperscript{u}m\textsuperscript{ol} (1952) and Miller and Orr (1966). We denote by $\beta_I$ and $\beta_O$ the unit transaction cost charged on the cash transferred to and from the master account, respectively.

\textbf{Figure 1} Inventory and cash flows.
The former represents a cash deposit cost, whereas the latter represents a cash disposal cost. We assume $\beta_O < \eta < \beta_O + \alpha \beta_I$, where $\alpha$ is a discount rate. (As we shall see later, $\alpha = 1/(1 + \eta)$.) If the first inequality does not hold, the corporation will never invest in the external portfolio. Similarly, if the second inequality fails, the corporation will never hold any cash in the master account. Note that $\beta_O$ is often relatively small in practice as transferring cash for external investments within a firm causes minimal costs.

Define $c_i$ and $p_i$ as the unit ordering cost and selling price of the product for division $i$, respectively. We assume that $c_i(1 + \eta) < p_i$, for $i = 1, 2$, which means the unit profit is higher than the return from the external investment; otherwise, the corporation would never invest in inventory. For division $i$, there is a physical holding cost $h_i$ for each unit of inventory held in each period and a physical backorder cost $b_i$ for each unit of backorders in each period. Here, the physical holding cost rate refers to the costs related to inventory storage, insurances, shrinkage, etc., which does not include the financial opportunity cost due to holding inventory. The physical backorder cost rate should be viewed as the same way – it is the tangible, monetary penalty costs related to backlogging, e.g., expediting delivery costs. The objective of the corporation is to obtain the optimal joint cash retention and inventory replenishment policy such that the expected net worth (i.e., equity) at the end of the horizon is maximized. The corporation’s net worth is equal to the sum of the value of the investment assets, the cash balance in the master account, and the total inventory value. Notice that we do not consider the long-term assets and the liabilities. Thus, maximizing the expected net worth is the same as maximizing the expected working capital assumed in Luo and Shang (2018). As shown later, this objective is also equivalent to minimizing the total cash- and inventory-related costs assumed in Luo and Shang (2015).

The sequence of events in a period is summarized below (see Figure 2). At the beginning of the period, (1) division $i$ receives the order placed $L_i$ periods ago. After reviewing the inventory status of the two divisions and the cash balance in the master account, (2) the corporate treasury (or a central planner) makes the cash retention decision and places an order to the outside vendor for each division; the payments to the vendors are immediately deducted from the master account. During the period, demands are realized and sales revenue is transferred to the master account. At the end of the period, the physical inventory holding and backorder costs and cash transaction costs are calculated and deducted from the master account.

For periods $t = 1, 2, \ldots, T$, and divisions $i = 1, 2$, we define the following variables:

$\hat{x}_{i,t} =$ inventory level at division $i$ after event (1);
Figure 2  The events timeline in one period.

\( z_{i,t} \) = order quantity for division \( i \) in event (2);

\( q_{i,t} = (q_{1i,t}, q_{2i,t}, ..., q_{(L_i - 1)i,t}) \), the pipeline inventory for division \( i \) after event (1), where \( q_{i,t}^\tau \) is the units to be delivered in \( \tau \) periods, \( \tau = 1, 2, ..., L_i - 1 \);

\( \hat{w}_t \) = cash balance in the master account before event (2);

\( I_t \) = gross value of external investments before event (2);

\( W_t \) = net worth of the corporation before event (2);

\( v_t \) = cash amount transferred from the external investment to the master account in event (2).

Note that the decision variable \( v_t \) could be positive or negative. In particular, \( v_t^+ \) represents the cash transferred into the master account, whereas \( v_t^- \) is the cash transferred to the external investment account (we define \( x^+ = \max\{x, 0\} \) and \( x^- = \max\{-x, 0\} \)). The total cash transaction cost in period \( t \) can be calculated by \( F(v_t) = \beta_1 v_t^+ + \beta_2 v_t^- \). The net worth \( W_t \) is the sum of the gross value of the external investment assets, the cash balance in the master account, and the total inventory value, i.e., \( W_t = I_t + \hat{w}_t + \sum_{i=1}^{2} c_i (\hat{x}_{i,t} + 1^T q_{i,t}) \), where \( 1 \) represents a column vector with all elements equal to one.

We define \( \hat{H}_i(x) = h_i x^+ + b_i x^- \), which represents the physical inventory holding and backorder costs at division \( i \) in period \( t \) given the end-of-period inventory level \( x \). The system dynamics can be characterized as follows: for \( i = 1, 2 \),

\[
\hat{x}_{i,t+1} = \hat{x}_{i,t} + q_{1i,t}^1 - D_{1,t},
\]

\[
q_{i,t+1} = (q_{2i,t}^2, ..., q_{(L_i - 1)i,t}^{(L_i - 1)}), z_{i,t}),
\]

\[
\hat{w}_{t+1} = \hat{w}_t + v_t - \sum_{i=1}^{2} (c_i z_{i,t}) + \sum_{i=1}^{2} (p_i D_{i,t}) - F(v_t) - \sum_{i=1}^{2} \hat{H}_i(\hat{x}_{i,t} - D_{i,t}),
\]

\[
I_{t+1} = (1 + \eta)(I_t - v_t).
\]
Equations (1) and (2) describe the inventory dynamics. As shown in (3), the transaction cost and physical inventory holding and backorder costs are deducted from the master account. Equation (4) describes that the value of external investments is increased by \((1 + \eta)\).

We first consider the constraints on the inventory and cash decisions. The total inventory value that can be ordered by both divisions should be less than the cash balance in the master account after the cash retention decision, i.e., \(\hat{w}_t + v_t - c_1 z_{1,t} - c_2 z_{2,t} \geq 0\). Thus, the feasible decision set \(S_t(\hat{w}_t)\) in period \(t\) can be expressed as

\[
S_t(\hat{w}_t) = \{v_t, z_{1,t}, z_{2,t} | z_{1,t} \geq 0, z_{2,t} \geq 0, (\hat{w}_t + v_t) \geq (c_1 z_{1,t} + c_2 z_{2,t})\}.
\]

We next turn to the objective function. From Equations (1) - (4), the net worth in period \(t + 1\) is

\[
W_{t+1} = I_{t+1} + \hat{w}_{t+1} + \sum_{i=1}^{2} c_i(x_{i,t+1} + 1^T q_{i,t+1})
\]

\[
= (1 + \eta)W_t + \sum_{i=1}^{2} [(p_i - c_i)D_{i,t}]
\]

\[
- \eta(\hat{w}_t + v_t + \sum_{i=1}^{2} c_i(x_{i,t} + 1^T q_{i,t})] - F(v_t) - \sum_{i=1}^{2} \hat{H}_i(x_{i,t} - D_{i,t}). \tag{5}
\]

Thus, by applying Equation (5) recursively, the corporation’s end-of-horizon net worth can be written as

\[
W_{T+1} = (1 + \eta)^T W_1 + \sum_{t=1}^{T} (1 + \eta)^{T-t} \left[ \sum_{i=1}^{2} [(p_i - c_i)D_{i,t}] \right.
\]

\[
- \eta(\hat{w}_t + v_t + \sum_{i=1}^{2} c_i(x_{i,t} + 1^T q_{i,t})] - F(v_t) - \sum_{i=1}^{2} \hat{H}_i(x_{i,t} - D_{i,t}) \bigg]. \tag{6}
\]

Note that the expected revenue \(\sum_{i=1}^{2} [(p_i - c_i)D_{i,t}]\) is a constant and \(W_1\) is the initial net worth. Thus, maximizing the expected total inventory and cash related costs over the whole horizon. Specifically, the original problem is equivalent to

\[
\min_{(v_{t}, z_{1,t}, z_{2,t}) \in \hat{S}_t(\hat{w}_t)} \mathbb{E} \left[ \sum_{t=1}^{T} (1 + \eta)^{T-t} \left[ \eta(\hat{w}_t + v_t + \sum_{i=1}^{2} c_i(x_{i,t} + 1^T q_{i,t})] + F(v_t) + \sum_{i=1}^{2} \hat{H}_i(x_{i,t} - D_{i,t}) \right] \right] \]

\[
= (1 + \eta)^{T-1} \min_{(v_{t}, z_{1,t}, z_{2,t}) \in \hat{S}_t(\hat{w}_t)} \mathbb{E} \left[ \sum_{t=1}^{T} \alpha^{T-t} \left[ \eta(\hat{w}_t + v_t + \sum_{i=1}^{2} c_i(x_{i,t} + 1^T q_{i,t})] + F(v_t) + \sum_{i=1}^{2} \hat{H}_i(x_{i,t} - D_{i,t}) \right] \right], \tag{7}
\]
where $\alpha = 1/(1+\eta)$, which can be interpreted as a discount rate.

The problem in (7) can be solved by the dynamic program shown below. Define $\hat{V}_t(\hat{w}_t, \hat{x}_t, q_{1,t}, q_{2,t})$ as the minimum expected total costs from period $t$ to $T$ over all feasible decisions, where $\hat{x}_t = (x_{1,t}, x_{2,t})$. The optimality recursion is

$$
\hat{V}_t(\hat{w}_t, \hat{x}_t, q_{1,t}, q_{2,t}) = \min_{(v_t,z_{1,t},z_{2,t})\in \hat{S}_t(\hat{w}_t)} \left\{ \hat{G}_t(\hat{w}_t, \hat{x}_t, q_{1,t}, q_{2,t}, v_t) + \alpha E\left[ \hat{V}_{t+1}(\hat{w}_{t+1}, \hat{x}_{t+1}, q_{1,t+1}, q_{2,t+1}) \right] \right\},
$$

where

$$
\hat{G}_t(\hat{w}_t, \hat{x}_t, q_{1,t}, q_{2,t}, v_t) = \eta (\hat{w}_t + v_t + \sum_{i=1}^{2} c_i(\hat{x}_{i,t} + 1^T q_{i,t})) + F(v_t) + \sum_{i=1}^{2} E[\hat{H}_i(\hat{x}_{i,t} - D_{i,t})],
$$

with $\hat{V}_{T+1}(\cdot) = 0$, and $\hat{x}_{t+1}, q_{1,t+1}, q_{2,t+1}$, and $\hat{w}_{t+1}$ following the dynamics in (1)-(3).

The minimum cost formulation provides a clear economic explanation. The first term on the right-hand side of (9) can be viewed as the opportunity cost of holding cash and inventory which incurs a potential profit loss from stable capital appreciation (see Allen and Hafer 1984; Luo and Shang 2015). The second term is the transaction cost for cash transfers between the master account and the investment account. The third term is the total expected physical inventory holding and backorder cost at both divisions in period $t$. Note that the single-period cost function in (9) does not include the inventory ordering cost $c^T z_t$. This is because the total working capital is not affected by inventory procurement: the increased inventory value is equal to the decreased cash amount in the master account.

**Simplified Model and Echelon Formulation**

It is difficult to obtain the optimal policy for the problem in (8) as it is a multi-dimensional dynamic program subject to curse of dimensionality. One immediate idea is to follow the approach of Clark and Scarf (1960) who defined echelon terms to reduce the dimension. However, our problem is more complicated as the cash transition in Equation (3) involves non-linear terms $F(v_t)$ and $\sum_{i=1}^{2} \hat{H}_i(\hat{x}_{i,t} - D_{i,t})$. To proceed, we propose a simplified model in which these two non-linear terms are omitted from the cash dynamics. (See Luo and Shang (2018) and Luo and Shang (2015) for the same treatment on a single-stage and serial model, respectively. They show that such omission does not affect quantitative properties of optimal policies in simulation studies.) Consequently, the cash dynamics in (3) become

$$
\hat{w}_{t+1} = \hat{w}_t + v_t - c^T z_t + p^T D_t.
$$
In this simplified model, the inventory order in period \( t \) does not affect the inventory level and cash flow until it arrives at the division. It allows us to use the idea of echelon transformation to simplify the formulation. Define \( x_{i,t} \) as the inventory position at division \( i \), and \( w_t \) as the cash balance in the master account plus the total inventory value. We refer to \( w_t \) as the system working capital.

For period \( t = 1, 2, \ldots, T + 1 \), and division \( i = 1, 2 \), we have

\[
x_{i,t} = \hat{x}_{i,t} + 1^T q_{i,t}, \quad \text{and} \quad w_t = \hat{w}_t + \sum_{i=1}^2 c_i (\hat{x}_{i,t} + 1^T q_{i,t}).
\]

Accordingly, we also define the following echelon variables:

\[
y_{i,t} = x_{i,t} + z_{i,t}, \quad i = 1, 2, \quad \text{and} \quad r_t = w_t + v_t,
\]

where \( y_{i,t} \) represents the order-up-to inventory position at division \( i \), and \( r_t \) is the corporation’s working capital after the cash retention decision.

Under these echelon variables, the system dynamics become

\[
x_{i,t+1} = y_{i,t} - D_{i,t}, \quad i = 1, 2, \quad \text{and} \quad w_{t+1} = r_t + (p - c)^T D_t,
\]

and the feasible decision set becomes

\[
S_t(x_t) = \{r_t, y_t | y_t \geq x_t, \ r_t \geq c^T y_t\}, \quad (11)
\]

where \( r_t \geq c^T y_t \) is the new cash allocation constraint that requires the cash used for inventory replenishment cannot exceed the cash balance in the master account.

The simplified problem can be reformulated as

\[
V_t(w_t, x_t) = \min_{(r_t, y_t) \in S_t(x_t)} \left\{ G_t(w_t, r_t, y_t) + \alpha \mathbb{E}\left[ V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)\right]\right\}, \quad (12)
\]

where

\[
G_t(w_t, r_t, y_t) = \eta r_t + F(r_t - w_t) + \sum_{i=1}^2 H_i(y_{i,t}), \quad (13)
\]

\[
H_i(y_{i,t}) = \alpha L_i \mathbb{E}[\hat{H}_i(y_{i,t} - \sum_{j=0}^{L_i} D_{i,t+j})], \quad (14)
\]

The corresponding terminal function is \( V_{T+1}(w_{T+1}, x_{T+1}) = 0 \). We refer to (12) as the echelon formulation of the simplified model. Our subsequent analysis will be imposed on the problem (12).
4. Optimality Analysis

Unlike Luo and Shang (2015), the simplified problem still suffers from the issue of curse of dimensionality as there is a working capital allocation problem for the corporate treasury. This is the same issue as the inventory allocation problem in the multi-echelon distribution system. Thus, our objective here is to derive a simple and effective heuristic. To that end, we first explore some properties for the optimal joint policy in this section. These optimality analysis will motivate us to develop a novel lower bound on the optimal cost as well as the heuristic policy.

The following lemma shows that the optimal value function is jointly convex. All proofs can be found in Appendix B.

**Lemma 1.** The function $V_t(w_t, x_t)$ is jointly convex in $(w_t, x_t)$ for all $t$.

With Lemma 1, we form the following KKT conditions for problem (12) to investigate the optimal solution.

\[
\begin{align*}
\eta + \beta_I + \alpha \frac{\partial}{\partial r_t} \mathbb{E}[V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)] - \lambda_t &= 0, \text{ if } r_t > w_t; \\
\eta - \beta_O + \alpha \frac{\partial}{\partial r_t} \mathbb{E}[V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)] - \lambda_t &= 0, \text{ if } r_t < w_t; \\
\eta - \beta_O + \alpha \frac{\partial}{\partial r_t} \mathbb{E}[V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)] < \lambda_t \\
&< \eta + \beta_I + \alpha \frac{\partial}{\partial r_t} \mathbb{E}[V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)], \text{ if } r_t = w_t; \\
\frac{d}{dy_{i,t}} H_{i,t}(y_{i,t}) + \alpha \frac{\partial}{\partial y_{i,t}} \mathbb{E}[V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)] - \mu_{i,t} + c_i \lambda_t &= 0, \text{ for } i = 1, 2; \\
\lambda_t (r_t - c^T y_t) &= 0; \\
\mu_{i,t}(y_{i,t} - x_{i,t}) &= 0, \text{ for } i = 1, 2.
\end{align*}
\]

The first four equations represent the first-order conditions for decision variables $r_t$ and $y_t$. The last two equations are the complementary slackness conditions, where $\lambda_t$ and $\mu_{i,t}$ are the nonnegative Lagrange multipliers associated with constraints $r_t \geq c^T y_t$ and $y_{i,t} \geq x_{i,t}$, respectively. We next explore the following bounds on the first-order partial derivatives of the objective functions $V_t(w_t, x_t)$, which will be useful in deriving further results.

**Lemma 2.** For any period $t$, the first-order partial derivative of the optimal value function $V_t$ to the working capital $w_t$ is bounded by $-\beta_I \leq \frac{\partial}{\partial w_t} V_t(w_t, x_t) \leq \beta_O$, for all $(w_t, x_t)$.

Lemma 2 indicates that the marginal optimal cost of the working capital is bounded below by $-\beta_I$ and above by $\beta_O$. To see why this marginal optimal cost is bounded above by $\beta_O$, imagine that we have achieved the system optimal cost with given $(w_t, x_t)$. For $w_t$ increased by one unit,
this implies that cash increases by one unit as \( x_t \) is fixed. With this additional cash, one strategy, which may not be optimal, is to invest externally, resulting in a transaction cost \( \beta_O \). Thus, the marginal optimal cost to this additional cash must be smaller than \( \beta_O \), as one can always perform no worse than simply investing externally. On the other hand, the maximum optimal system cost that can be saved from this additional cash is that it had been kept in the master account, instead of transferring from the investment account. Thus, the maximum cost saving is \( \beta_I \).

Let \( r^*_t \) represent the optimal working capital level after the cash retention decision and \( y^*_i,t \) be the optimal order-up-to inventory position at division \( i \). Let \( (\lambda^*_i, \mu^*_i) \) be the optimal Lagrangian multipliers. As a consequence of conditions (15)-(20) and Lemma 2, we can obtain the structural properties of \( (\lambda^*_i, \mu^*_i) \) summarized in the following theorem.

**Theorem 1.** For any period \( t \), the optimal working capital \( r^*_t \) and the optimal values of Lagrangian multipliers \( (\lambda^*_i, \mu^*_i) \) are related as follows:

(i) If \( r^*_t > w_t \), then \( \eta + \beta_I - \alpha \beta_I \leq \lambda^*_i \leq \eta + \beta_I + \alpha \beta_O \) and \( \prod_{i=1}^{2} \mu^*_i,t = 0 \).

(ii) If \( r^*_t = w_t \), then \( 0 \leq \lambda^*_i \leq \eta + \beta_I + \alpha \beta_O \) and \( \prod_{i=1}^{2} \mu^*_i,t \geq 0 \).

(iii) If \( r^*_t < w_t \), then \( 0 \leq \lambda^*_i \leq \eta - \beta_O + \alpha \beta_O \) and \( \prod_{i=1}^{2} \mu^*_i,t \geq 0 \).

Theorem 1 shows three possible optimal solutions that depend on the system working capital level. Part (i) describes a scenario in which cash is not sufficient and additional cash is transferred from the investment account to the master account for inventory replenishment. From Lemma 2, it is clear that \( \lambda^*_i \geq \eta + \beta_I - \alpha \beta_I > 0 \). This implies that the working capital constraint is binding, i.e., \( r^*_t = c_1 y^*_1,t + c_2 y^*_2,t \). In other words, if cash is transferred into the master account, all of the transferred cash has to be used in purchasing inventory, making \( \prod_{i=1}^{2} \mu^*_i,t = 0 \) (at least one division places an order). Note that \( \lambda^*_i \) is the shadow price, which means how much cost can be reduced if the system has one unit of “free” cash to order. In such case, the system can avoid the cash holding cost \( \eta \), the transaction cost \( \beta_I \), and the potential cost of disposing it for investment in the next period, i.e., \( \alpha \beta_O \). This explains the right-hand side of the bound for \( \lambda^*_i \). The left-hand side can be explained similarly: \( (\beta_I - \alpha \beta_I) \) represents the actual transferring cost reduction if the cash is transferred this period instead of the following period.

Part (ii) describes a scenario in which the optimal system working capital level after the cash retention decision is the same as the initial working capital level in period \( t \). This implies that no cash is transferred. In such case, both divisions may not necessarily order in period \( t \) so \( \prod_{i=1}^{2} \mu^*_i,t \geq 0 \). The right-hand size of the bound for \( \lambda^*_i \) has the same economic meaning as that of Part (i). As
for the left-hand side, if the system has a right amount of working capital, it is possible that the additional free cash does not bring any benefit so the left-hand side bound is zero. Lastly, Part (iii) describes a scenario in which there is sufficient cash, and the excess cash is transferred from the master account to the outside investment account. In this case, similar to Part (ii), both divisions may not necessarily order in period $t$. As for the right-hand side of the bound for $\lambda^*_t$, consider the cost of keeping one additional unit of cash. The system incurs one unit of the cash holding cost but transfers one unit less of cash to the investment account. In addition, this unit potentially has to be disposed in the following period. Thus, the net cost of this additional unit is $(\eta - \beta_O + \alpha\beta_O)$. In other words, if the firm has one free unit of cash, the net benefit would be the net cost shown above.

In summary, Theorem 1 suggests that the optimal cash retention policy is a two-threshold policy. If the working capital is too low, cash should be transferred into the master account up to a lower-threshold level. In this case, all of the transferred cash should be used for inventory ordering. The benefit of transferring cash in this case is to avoid a significant backorder cost (at the expense of incurring the transaction cost $\beta_I$). On the other hand, if the working capital is too high, cash should be disposed to the external investment account until an upper-threshold level. The benefit of transferring cash externally is to avoid the cash holding cost (at the expense of the transaction cost $\beta_O$). These properties are useful to develop our heuristic in the subsequent section.

While it is difficult to obtain the optimal joint policy, we are able to characterize the exact one for the system with i.i.d. demands. Consider the case in which the divisions independently manage the inventory and cash flows and the cash generated from sales is always sufficient for inventory replenishment (the divisions therefore transfer all excess cash to external investments). As such, division $i$ faces the following dynamic program:

$$V_{i,t}(x_t) = \min_{y_t \geq x_t} \left\{ (\eta - \beta_O + \alpha\beta_O)c_i y_t + H_{i,t}(y_t) + \alpha\mathbb{E}[V_{i,t+1}(y_t - D_{i,t})] \right\}, \quad (21)$$

where $V_{i,T+1}(x_{T+1}) = 0$.

The unit cost $(\eta - \beta_O + \alpha\beta_O)$ reflects the opportunity cost of inventory replenishment. On the one hand, ordering one more unit of inventory incurs a potential profit loss from stable capital appreciation (i.e., $\eta$). On the other hand, as the division transfers all excess cash to external investments, ordering one more unit of inventory reduces the cash disposal cost in the current period (i.e., $-\beta_O$). However, this additional inventory unit will be sold and bring in additional cash in the next period, it will make the division dispose more cash from the master account in the next period (i.e., $\alpha\beta_O$).
In the stationary setting, it is well known that the optimal policy for problem (21) is a myopic base-stock policy with reorder points \( S_i \) defined by

\[
S_i = \arg \min \left\{ (\eta - \beta O + \alpha \beta O)c_i y_t + H_{i,t}(y_t) \right\}.
\]

The following proposition demonstrates the optimal inventory and cash policy in the stationary setting under some mild conditions.

**Proposition 1.** For the system with i.i.d. demands, if the initial working capital satisfies \( w_1 \geq (c_1 S_1 + c_2 S_2) \) and the terminal function is \( V_{T+1}(w_{T+1}, x_{T+1}) = \beta_O w_{T+1} \), the optimal inventory policy is a base-stock policy with the reorder point \( S_i \) whereas the corporate treasury transfers excess cash to the investment account after inventory payment.

This result is quite intuitive. Recall that \( p_i > c_i \). Thus, whenever a customer orders, the corporation will receive sufficient funds for the inventory replenishment in the following period. Thus, the cash allocation is no longer a concern as the corporation always has sufficient cash to fulfill the inventory order.

5. **Lower Bound**

As the optimal policy is difficult to characterize, we aim to propose a simple and effective heuristic. This section develops a tight lower bound to the optimal cost in order to evaluate the effectiveness of a heuristic policy. It turns out that this lower bound will lead to an effective heuristic presented in Section 6.

The main idea is to introduce a linear function \( a_t (r_t - c^T y_t) \), where \( (r_t - c^T y_t) \) is the cash amount in the master account after inventory replenishment, and \( a_t \) is nonnegative and can be considered as an incentive for the corporate treasury to hold one more unit of cash in the master account. For this reason, we shall refer to \( a_t \) as the *cash-holding multiplier*. By incorporating this additional term to our problem (12), we can construct an auxiliary system as follows:

\[
V_t(w_t, x_t | a_t) = \min_{r_t \geq c^T y_t} \left\{ G_t(w_t, r_t, y_t | a_t) + \alpha \mathbb{E} \left[ V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t | a_{t+1}) \right] \right\}, \tag{22}
\]

where

\[
G_t(w_t, r_t, y_t | a_t) = \eta r_t + F(r_t - w_t) + \sum_{i=1}^{2} H_{i,t}(y_{i,t}) \underbrace{-a_t(r_t - c^T y_t)}_{\text{savings due to holding cash}}.
\]

Here, \( a_t \) is a \((T - t + 1)\)-dimensional vector \((a_t, a_{t+1}, ..., a_T)\). Clearly, compared with the original system, the auxiliary system has exactly the same cost function except the additional savings term.
Given a nonnegative $a_t$, the problem in (22) is clearly a lower bound on the optimal one, i.e.,
\[ V_{t-1}(w_t, x_t | a_t) \leq V_t(w_t, x_t) \]. Moreover, the original system is a special case of the auxiliary system with $a_t = 0$.

We next derive a lower bound on the optimal cost of the auxiliary system. We first sketch the idea, which is similar to that of Chen and Zheng (1994)'s induced-penalty bound. Imagine that each inventory unit at the division $i$ is composed of a cash-equivalent component 0 and a division-specific component $i$. The cash-equivalent component 0 is distributed through the corporate treasury to each division. The component $i$ is replenished directly from division $i$'s supplier. For a given $a_t$, let $J_{i,t}(y_t | a_t)$ denote total costs of division $i$ in period $t$ when its inventory position after ordering is $y_t$ and the optimal inventory decisions are employed for period $t + 1, ..., T$, i.e.,
\[ J_{i,t}(y_t | a_t) = H_{i,t}(y_t) + a_t c_i y_t + \alpha E[V_{i,t+1}(y_t - D_{i,t}, a_{t+1})]. \] (23)

The function $J_{i,t}(y_t | a_t)$ is convex in $y_t$. Let
\[ S_{i,t}(a_t) = \arg \min_{y_t \in \mathbb{R}} J_{i,t}(y_t | a_t). \] (24)

The function $J_{i,t}(y_t | a_t)$ can be separated into two parts. The first part, $\Gamma_{i,t}(y_t | a_t)$, is the cost resulted from an insufficient amount of cash-equivalent component 0 (so that division $i$ cannot produce up to the desired level $S_{i,t}$). The second part is the remaining of $J_{i,t}(y_t | a_t)$. Now, we assign the first part of the cost to the corporate treasury, as it is the cost caused by insufficient cash. Let the optimal cost for the corporate treasury in period $t$ under such a cost allocation scheme be $V_{H,t}(|a_t|)$. These cost functions are shown below:
\[ V_{H,t}(w_t | a_t) = \min_{r_t \geq c^T x_t} \left\{ (\eta - a_t) r_t + F(r_t - w_t) + \sum_{i=1}^2 \Gamma_{i,t}(y_t | a_t) + \alpha E[V_{H,t+1}(r_t + (p - c)^T D_t | a_{t+1})] \right\}, \]
\[ V_{i,t}(x_{i,t} | a_t) = \min_{y_t \geq x_{i,t}} \left\{ H_{i,t}(y_t) + a_t c_i y_t - \Gamma_{i,t}(y_t | a_t) + \alpha E[V_{i,t+1}(y_t - D_{i,t}, a_{t+1})] \right\}, \text{ for } i = 1, 2, \] (25)

where
\[ \Gamma_{i,t}(y_t | a_t) = \begin{cases} J_{i,t}(y_t | a_t) - J_{i,t}(S_{i,t}(a_t) | a_t), & \text{if } y_t < S_{i,t}(a_t); \\ 0, & \text{otherwise}. \end{cases} \] (26)

With this cost allocation scheme, we have decoupled the total optimal cost in period $t$ into three separate ones; $V_{i,t}(|a_t|)$ is managed by division $i$ whereas $V_{H,t}(|a_t|)$ is managed by the corporate treasury. As the three subsystems do not need to be coordinated due to the decoupling, the resulting sum of the optimal costs is a lower bound to that of the auxiliary system, which, in turn, a lower bound to that of the original system. Theorem 2 summarizes the result.
Theorem 2. For all \(t\) and \((w_t, x_t)\), \(V_t(w_t, x_t) \geq V_t(w_t, x_t|a_t) \geq V_{H,t}(w_t|a_t) + \sum_{i=1}^{2} V_{i,t}(x_i, t|a_t)\).

It is worth noting that when \(a_t = 0\), the additional savings term \(a_t(r_t - c^T y_t)\) becomes zero, and the above cost decomposition scheme is degenerated to that of Chen and Zheng’s induced-penalty bound. While the difference is subtle, it turns out that this generalization is very useful for the non-stationary, finite-horizon model. More specifically, we can optimize the value of \(r_t\) to obtain an improved lower bound than the one obtained directly by using Chen and Zheng’s approach. As shown in a numerical study in Section 7, under the i.i.d. demand case, \(a_t\) turns out to be close to zero for almost all \(t\) and instances. This suggests that Chen and Zheng’s induced-penalty bound works well for the i.i.d. demand case. However, when the demands are non-stationary, the choice of the cash-holding multipliers \(a_t\) becomes an important factor that affects the performance of the lower bound. We shall report the connection between \(a_t\) and the system parameters in a numerical study.

Our lower bound formulation also is different from the well-known Lagrangian relaxation for the stochastic optimization problem (e.g., Goel and Gutierrez 2011). The subtle difference lies in the constraint set: Unlike the corresponding Lagrangian-relaxation problem, we keep the cash allocation constraint in the constraint set (see Equation (22)). Clearly, the constraint set is smaller than that of the Lagrangian-relaxation problem, which suggests that the optimal cost obtained from the auxiliary system is an upper bound of the cost of the Lagrangian-relaxation problem. In fact, let \(\lambda^*_t\) be the optimal Lagrangian multipliers obtained from the Lagrangian-relaxation problem. We can show that when \(a_t = \lambda^*_t\), i.e., when the cash-holding multiplier is exactly the same as the reduced cost of the system with one unit of “free” cash, our lower bound cost is the optimal cost of the original system.

Theorem 3. For all \(t\) and \((w_t, x_t)\), \(V_t(w_t, x_t) = V_{H,t}(w_t|\lambda^*_t) + \sum_{i=1}^{2} V_{i,t}(x_i, t|\lambda^*_t)\).

With these lower bound functions, one can construct an effective heuristic by obtaining an effective \(a_t\) that accommodates non-stationary demands. This will be presented in the next section.

6. Heuristic

We now analyze the optimal policy of the lower bound system, which is composed of three separable subsystems defined by (25). Let \(u_t(a_t)\) and \(l_t(a_t)\) denote the thresholds of optimal working capital levels, where

\[ u_t(a_t) = \sup \left\{ r_t : \frac{dJ_{H,t}(r_t|a_t)}{dr_t} \leq \beta_0 \right\}, \tag{27} \]
\[ l_t(a_t) = \sup \left\{ r_t : \frac{dJ_{H,t}(r_t|a_t)}{dr_t} \leq -\beta_t \right\}, \text{ and} \]

\[ J_{H,t}(r_t|a_t) = (\eta - a_t)r_t + \min_{c^T y_t \leq r_t} \sum_{i=1}^{c} \Gamma_{i,t}(y_{i,t}|a_t) + \alpha E[V_{H,t+1}(r_t + (p - c)^T D_t|a_{t+1})]. \]

We shall use these two thresholds to control the cash flow for the corporate treasury. Theorem 4 summarizes the optimal policy for each subsystem.

**Theorem 4.**

(i) The optimal inventory policy for division \( i \) is a base-stock policy with the base-stock level \( S_{i,t}(a_t) \), which is nonincreasing in the value of \( a_t \) for all time periods \( \tau \geq t \).

(ii) The cash retention policy for the corporate treasury is a two-threshold policy with the lower threshold \( l_t(a_t) \) and the upper threshold \( u_t(a_t) \).

Theorem 4 suggests that the optimal policy for the lower bound system has a simple structure. The corporate treasury maintains the working capital between \( l_t(a_t) \) and \( u_t(a_t) \). If the working capital is lower than the lower threshold, cash is retrieved up to \( l_t(a_t) \); if the working capital is higher than the upper threshold, cash is disposed down to \( u_t(a_t) \). This two-threshold structure is consistent with the optimal cash retention policy in Theorem 1. For the inventory policy, each division simply implements a base-stock policy.

The value of the cash-holding multiplier \( a_t \) can be viewed as a lever to coordinate cash holding and inventory stocking in our heuristic policy. To see this, consider a scenario in which the demand is increasing in some periods. Both divisions would order up to the optimal base-stock level, which is increasing within these periods. In such case, the cash received from the previous period may not be sufficient to pay for the ordered inventory for the following period, so borrowing externally is necessary. However, borrowing incurs a transaction cost in our model. Thus, \( a_t \) should become larger because it will lead to more cash holding in the current period to avoid the future transaction costs. In addition, a larger \( a_t \) will motivate the division to order less in the current period by reducing the base-stock level (compared to the no-cash constraint case) so that cash can be saved for the future surging demand. We shall illustrate this point in a numerical study.

We propose the aforementioned optimal policy of the lower bound system as our heuristic. We need to refine the resulting policy for some rare situations during implementation. First, the cash upper threshold \( u_t(a_t) \) may be lower than the initial inventory value \( c^T x_t \). As inventory disposal is not allowed, adjusting the working capital down to \( u_t(a_t) \) is not feasible. In our heuristic, if the situation occurs, we just do nothing and keep \( r_t^* = c^T x_t \). Second, it is possible that the division
cannot order up to $S_{i,t}(a_i)$ due to the cash limitation in the master account. In such case, the division just orders up to the level that minimizes the lower-bound cost determined by the cash allocation constraint.

We now discuss how to choose $a_t$ in our heuristic. We suggest two methods. For the static policy, the vector $a_1$ is predetermined at the beginning of the first period and stays unchanged over the entire horizon. For the dynamic policy, we dynamically adjust $a_t$ according to the real-time system states.

The Static Policy

The vector $a_1$ is predetermined at the beginning of the first period and stays unchanged over the entire horizon. To reduce computational complexity, we restrict our attention to vectors $a_1$ such that $a_1 = a_2 = ... = a_T$. This heuristic can be applied to a scenario where the corporation has a forecasted demand for divisions. With the demand data as an input, the corporation can solve the following problem:

$$\arg \max_{0 \leq a_1 \leq \eta + \beta I + \alpha O} \left\{ V_{H,1}(w_1|a_1) + \sum_{i=1}^{2} V_{i,1}(x_{i,1}|a_1) \right\},$$

where $a_1 = (a_1, a_1, ..., a_1)$. The right-hand side of the constraint for $a_1$ is due to Theorem 1. We obtain the best $a_1$ through a numerical search in this range.

The Dynamic Policy

For the dynamic heuristic policy, we update the subvector $a_t$ at the beginning of each period $t$ based on real-time system states. Specifically, in each period $t$, the corporation solves the following problem:

$$\arg \max_{0 \leq a_t \leq \eta + \beta I + \alpha O} \left\{ V_{H,t}(w_t|a_t) + \sum_{i=1}^{2} V_{i,t}(x_{i,t}|a_t) \right\},$$

where $a_t = (a_t, a_t, ..., a_t)$.

After the selection of $a_t$, we can generate the corresponding inventory and cash control thresholds and then implement the policy stated by Theorem 4. Clearly, these thresholds depend on the real-time system states $(w_t, x_t)$.

7. Numerical Study

In this section, we present a set of comprehensive numerical experiments. The goal of this numerical study is threefold. First, we compare our lower bound with the induced-penalty bound developed by Chen and Zheng (1994) for our cash pooling system. (We also compare these two lower bounds for the classic one-warehouse-multi-retailer system in Appendix A). Second, we test the effectiveness of
the static and dynamic heuristic policies in Section 7.2. Third, we assess the value of cash pooling by comparing our system with the one in which the divisions independently manage cash under different demand patterns. We also investigate how the demand correlation between the divisions affects the value of cash pooling.

### 7.1. Comparison of Lower Bounds

As stated, our lower bound is exactly the optimal cost if the cash-holding multipliers $a_i$ are equal to the optimal Lagrangian multipliers $\lambda^*_t$. However, obtaining $\lambda^*_t$ is computationally infeasible. Following the logic of the static heuristic, we calculate $a_i$ according to (30) and report the resulting lower bound cost in our study.

#### System Primitives

We assume that the divisions are identical and share the same parameters unless specified below. We fix the planning horizon $T = 10$, the replenishment lead time $L_i = 1$, the transaction cost $\beta_O = 0.01$, the physical holding cost $h_i = 0.25$, and the ordering cost $c_i = 1$. We test different values of the other seven parameters: the number of divisions $N$, the transaction cost $\beta_I$, the return rate $\eta$, the physical backorder cost $b_i$ and the selling price $p_i$. These parameters are given in the following sets: $N \in \{2, 4, 8\}$, $\beta_I \in \{0.2, 0.4, 0.6\}$, $\eta \in \{0.15, 0.2, 0.3\}$, $b_i \in \{1, 2, 3.25\}$, and $p_i \in \{1.4, 1.7, 2\}$.

#### Demand Distributions

The demands between divisions are independent, and the demand at division $i$ in period $t$ follows Poisson distribution with mean $\mu_{i,t}$. For each division, we test three demand types including stationary, increasing and seasonal demands with respect to the time period. We set $\mu_{i,t} = 5$ for the stationary demand, $\mu_{i,t} = 5 \times 1.2^{t-1}$ for the increasing demand, and $\mu_i = (5, 7, 10, 7, 5, 3, 1, 3, 5, 7, 10)$ for the seasonal demand.

#### Initial States

We set the initial cash level in the master account to be zero. To verify the effect of inventory imbalance among divisions, we classify the divisions into two groups and set different initial inventory positions specified in Table 1. The combination of the described parameters generates a total of $3^7 = 2187$ instances.

We define the percentage improvement of our lower bound over the induced-penalty bound as

$$\frac{C - C_{IP}}{C_{IP}} \times 100\%,$$

where $C$ and $C_{IP}$ represent our lower bound and the induced-penalty bound, respectively.

Table 1 summarizes the improvement of our lower bound over the induced-penalty bound. There are three observations. First, the improvement is sensitive to the demand type. For the systems with stationary demand, the improvement of our lower bound is marginal. To explain this observation,
Table 1  The performance improvement of the lower bound against the induced-penalty bound.

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<th>(w_1, x_{i,1}, x_{j,1})</th>
<th>Stationary</th>
<th>Seasonal</th>
<th>Increasing</th>
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<tr>
<td>i = 1, ..., N/2; j = N/2 + 1, ..., N</td>
<td>Avg. (Max., Std.) %</td>
<td>Avg. (Max., Std.) %</td>
<td>Avg. (Max., Std.) %</td>
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<td>3.06 (6.54 , 0.64)</td>
<td>3.67 (6.76 , 0.95)</td>
</tr>
<tr>
<td>8</td>
<td>(56, 7, 7)</td>
<td>0.00 (0.00 , 0.00)</td>
<td>1.30 (3.27 , 0.32)</td>
<td>1.51 (3.23 , 0.40)</td>
</tr>
<tr>
<td></td>
<td>(56, 4, 10)</td>
<td>0.00 (0.02 , 0.00)</td>
<td>2.30 (4.14 , 0.70)</td>
<td>2.62 (5.80 , 0.84)</td>
</tr>
<tr>
<td></td>
<td>(56, 1, 13)</td>
<td>0.01 (0.05 , 0.01)</td>
<td>3.71 (6.97 , 0.82)</td>
<td>4.42 (7.91 , 1.01)</td>
</tr>
</tbody>
</table>

recall that the two lower bounds coincide when the incentive factor a_1 = 0. Figure 3 illustrates that the optimal a_1 of problem (30) is close to zero for the stationary systems. It is consistent with Proposition 1 which states that under stationary demands, the cash received in a period is sufficient to pay the ordered inventory so a_1 is close to zero. On the contrary, the improvement is significant for the non-stationary demand cases. For example, for the systems with N = 8, the average improvement achieves 4.42% with a maximum of 7.91% for the increasing demand when the initial states are (56,1,13). Figure 3 demonstrates that the optimal value of a_1 appears to be much larger than zero under non-stationary demands. Second, the improvement becomes rather significant when the initial inventory levels between the divisions are imbalanced. For example, for the systems with N = 2 and increasing demand, the average improvement is about 1.25% when the initial state is (14,7,7) (balanced inventory between divisions), whereas the average improvement increases to 3.21% for the initial state (14,1,13) (imbalanced inventory between divisions). This observation confirms that reserving more cash in the master account will help coordinate the inventory imbalance between divisions. Third, the improvement becomes larger as the number of divisions increases. This is because when the number of divisions increases, there is a higher probability that the inventory imbalance between the divisions will occur.

We next investigate how the selling price p_i and the physical backorder cost b_i affect the optimal value of a_1. We fix parameters T = 10, N = 2, η = 0.15, β_I = 1.0, β_O = 0.01, h_i = 0.25, c_i = 1, and the initial states (w_1, x_{i,1}, x_{2,1}) = (28,7,7). To study the effect of p_i, we fix b_i = 1 and vary p_i ∈ {1.2,1.6,2.0,2.4,2.8,3.2}. Figure 3(a) illustrates the optimal element a_1 with respect to p_i.
under different demand patterns. It is shown that the cash-holding multiplier is decreasing in the selling price for all demand patterns. This observation can be explained as follows: the cash holding in the master account of the next period comes from two sources – the remaining cash \((r_t - c^T y_t)\) after inventory ordering and the product sales \(p^T D_t\) from customers. When the selling price \(p_i\) becomes larger, the corporation can gain more profit from product sales, and thus reserving cash in the master account becomes less valuable, leading to a smaller \(a_1\). To study the effect of \(b_i\), we fix \(p_i = 1.4\) and vary \(b_i \in \{1, 1.5, 2, 2.5, 3, 3.5\}\). Figure 3(b) shows the corresponding \(a_1\) values under different demand patterns. As shown, the optimal value of \(a_1\) is increasing in the backorder cost \(b_i\), because the corporate treasury needs to reserve more cash to make up higher shortage costs.

### 7.2. Heuristic Performance

We test effectiveness of the static and dynamic heuristics by comparing them with our lower bound. The effectiveness of the heuristic is defined as
\( N_i = 1, \ldots, N/2; j = N/2 + 1, \ldots, N \) Stationary Seasonal Increasing

<table>
<thead>
<tr>
<th>( N )</th>
<th>Stationary</th>
<th>Seasonal</th>
<th>Increasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>((14, 7, 7))</td>
<td>0.07 (0.15, 0.02)</td>
<td>0.79 (1.56, 0.25)</td>
</tr>
<tr>
<td></td>
<td>((14, 4, 10))</td>
<td>0.12 (0.21, 0.06)</td>
<td>1.59 (2.36, 0.52)</td>
</tr>
<tr>
<td></td>
<td>((14, 1, 13))</td>
<td>0.16 (0.32, 0.04)</td>
<td>2.60 (4.55, 0.76)</td>
</tr>
<tr>
<td>4</td>
<td>((14, 7, 7))</td>
<td>0.09 (0.19, 0.02)</td>
<td>1.36 (2.73, 0.43)</td>
</tr>
<tr>
<td></td>
<td>((14, 4, 10))</td>
<td>0.14 (0.31, 0.09)</td>
<td>2.32 (4.21, 0.62)</td>
</tr>
<tr>
<td></td>
<td>((14, 1, 13))</td>
<td>0.21 (0.63, 0.16)</td>
<td>3.48 (6.45, 1.12)</td>
</tr>
<tr>
<td>8</td>
<td>((14, 7, 7))</td>
<td>0.15 (0.31, 0.10)</td>
<td>1.72 (3.11, 0.70)</td>
</tr>
<tr>
<td></td>
<td>((14, 4, 10))</td>
<td>0.27 (0.51, 0.12)</td>
<td>3.24 (6.20, 1.30)</td>
</tr>
<tr>
<td></td>
<td>((14, 1, 13))</td>
<td>0.33 (0.72, 0.20)</td>
<td>4.57 (7.70, 1.23)</td>
</tr>
</tbody>
</table>

Table 2  The performance of the static policy.

\[
\frac{\overline{C} - C}{C} \times 100\% ,
\]

where \( \overline{C} \) is the system-wide costs under a certain heuristic and \( C \) is our lower bound.

We test the parameter combinations in Section 7.1 with a total \( 3^7 = 2187 \) instances. For each instance, we run a simulation of 1000 iterations to calculate the expected heuristic cost. Tables 2 and 3 present the overall performance of the static heuristic and the dynamic heuristic, respectively. Both heuristic policies perform surprisingly well for the stationary systems (the maximum gap is below 0.7%). For the non-stationary systems, it is conceivable that the heuristics would perform less effectively, and the dynamic heuristic should outperform the static one. The result confirms this conjecture: The average percentage gap is 2.13% for the seasonal demand cases, and 2.85% for the increasing demand cases. Interestingly, the dynamic policy does not significantly improve the performance over the static policy, which suggests that an effective initial cash-holding parameter is crucial. Note that the heuristic is compared with the lower bound cost. Thus, the actual performance would be better than we reported in the table if compared with the optimal cost.

7.3. Value of Cash Pooling

We assess the value of cash pooling by comparing the cash-pooling system with the cash-separating system in which each division manages its inventory and cash independently. The optimal joint cash and inventory policy for division \( i \), a single-stage system, has been established in Luo and Shang (2015). The optimal joint policy for each division \( i \) has a similar structure as that of the
Table 3  The performance of the dynamic policy.

<table>
<thead>
<tr>
<th>N</th>
<th>((w_1, x_{i,1}, x_{j,1}))</th>
<th>Stationary Avg. (Max., Std.) %</th>
<th>Seasonal Avg. (Max., Std.) %</th>
<th>Increasing Avg. (Max., Std.) %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>((14, 7, 7))</td>
<td>0.07 (0.13 , 0.02)</td>
<td>0.42 (1.14 , 0.15)</td>
<td>1.21 (2.10 , 0.23)</td>
</tr>
<tr>
<td></td>
<td>((14, 4, 10))</td>
<td>0.10 (0.17 , 0.03)</td>
<td>1.01 (2.14 , 0.27)</td>
<td>1.75 (3.20 , 0.25)</td>
</tr>
<tr>
<td></td>
<td>((14, 1, 13))</td>
<td>0.15 (0.26 , 0.05)</td>
<td>1.89 (3.85 , 0.46)</td>
<td>2.02 (4.29 , 0.41)</td>
</tr>
<tr>
<td>4</td>
<td>((14, 7, 7))</td>
<td>0.08 (0.15 , 0.04)</td>
<td>0.91 (1.96 , 0.25)</td>
<td>1.15 (2.71 , 0.53)</td>
</tr>
<tr>
<td></td>
<td>((14, 4, 10))</td>
<td>0.12 (0.26 , 0.09)</td>
<td>1.78 (3.86 , 0.42)</td>
<td>2.19 (4.01 , 0.52)</td>
</tr>
<tr>
<td></td>
<td>((14, 1, 13))</td>
<td>0.16 (0.35 , 0.10)</td>
<td>2.91 (5.85 , 0.61)</td>
<td>3.10 (5.71 , 0.56)</td>
</tr>
<tr>
<td>8</td>
<td>((14, 7, 7))</td>
<td>0.10 (0.22 , 0.07)</td>
<td>1.61 (2.96 , 0.40)</td>
<td>1.22 (2.19 , 0.22)</td>
</tr>
<tr>
<td></td>
<td>((14, 4, 10))</td>
<td>0.18 (0.32 , 0.08)</td>
<td>2.69 (5.66 , 0.42)</td>
<td>2.90 (6.01 , 0.50)</td>
</tr>
<tr>
<td></td>
<td>((14, 1, 13))</td>
<td>0.25 (0.47 , 0.14)</td>
<td>3.81 (6.75 , 0.66)</td>
<td>4.03 (7.71 , 0.82)</td>
</tr>
</tbody>
</table>

cash-pooling model: the working capital is managed by a two-threshold policy, and the inventory is controlled according to a base-stock policy.

To evaluate the benefit of cash pooling, we compare the total cost of the cash-separating system with that of the cash-pooling system under the static heuristic policy. Define \(C_S\) is the total cost of the cash-separating system (i.e., the sum of the optimal costs of the divisions). We define the value of cash pooling as

\[
\frac{C_S - \overline{C}}{C_S} \times 100\%,
\]

which represents the cost reduction due to cash pooling. Here \(\overline{C}\) is the total cost under the static heuristic.

We set \(N = 2\), and keep the values of the other parameters the same as those in the System Primitives paragraph of Section 7.1. Let \(\eta_i = \eta\), \(\beta_{i,t} = \beta_I\) and \(\beta_{i,O} = \beta_O\) for \(i = 1,2\). We again consider three types of demand forms with the mean specified in Section 7.1 and set the demand correlation between the two divisions in each period as \(\rho = \{-0.5, 0, 0.5\}\). The initial states are fixed as \((w_1, x_1, x_2) = (14, 7, 7)\). The average value of cash pooling is summarized in Table 4. We find that cash pooling does not add much value for the system with stationary demand. This is because under i.i.d. demand, the systems are more likely to have sufficient cash (see Proposition 1) and hence holding cash in the master account does not add much value. On the other hand, the value of cash pooling is significant for the non-stationary systems especially when the backorder cost rate is large. For instance, under the increasing demand form with the demand correlation \(\rho = -0.5\),
the cost reduction due to cash pooling increases from 13.19% to 19.45% when the backorder cost increases from $b_i = 2$ to 3.25. This observation implies that the cash pooling can effectively reduce the inventory shortage. Finally, the value of cash pooling is related to the demand correlation and tends to be more significant when the demands of the divisions are negatively correlated. This benefit comes from two sources: On the one hand, cash pooling can reduce the volatility of the sale revenue, and this benefit is amplified when the demands at divisions are negatively correlated. On the other hand, negatively correlated demands lead to a more stable total demand which increases inventory management efficiency.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$b_i$</th>
<th>$b_i$</th>
<th>$b_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3.25</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3.25</td>
</tr>
<tr>
<td>0.5</td>
<td>0.72</td>
<td>1.11</td>
<td>1.77</td>
</tr>
<tr>
<td>0</td>
<td>0.53</td>
<td>0.69</td>
<td>1.14</td>
</tr>
<tr>
<td>−0.5</td>
<td>0.27</td>
<td>0.36</td>
<td>0.60</td>
</tr>
</tbody>
</table>

To understand the impact of cash pooling on the cost reduction in the cash and inventory related costs, we divide the total system cost into two parts: the cash-related cost that includes the cash holding and transaction costs, and the inventory-related cost that includes the inventory holding and backorder costs. Specifically, we define

Cash-related Cost Reduction $= \frac{C_{\text{cash}}^S - \overline{C}_{\text{cash}}}{C_{\text{cash}}^S} \times 100\%,$

Inventory-related Cost Reduction $= \frac{C_{\text{inv}}^S - \overline{C}_{\text{inv}}}{C_{\text{inv}}^S} \times 100\%,$

where $C_{\text{cash}}^S$ and $\overline{C}_{\text{cash}}$ are the cash-related costs of cash-separating and cash-pooling systems, respectively, and $C_{\text{inv}}^S$ and $\overline{C}_{\text{inv}}$ are the inventory-related costs of cash-separating and cash-pooling systems, respectively. For the cash separating system, we use the optimal cost, whereas for the cash pooling system, we use the cost under the static heuristic.

Table 5 illustrates the cash pooling effect on the cost reduction of the inventory-related cost and cash-related cost. It is interesting to observe that the cost reduction on the inventory-related cost is more significant than that on the cash-related cost when the demands are non-stationary. For example, under the increasing demand with $\rho = −0.5$, the inventory-related cost reduction is about
Table 5 The percentage of cost reduction on the cash-related cost and the inventory-related cost (in solid rectangle) under cash pooling.

<table>
<thead>
<tr>
<th>ρ</th>
<th>Stationary</th>
<th>Seasonal</th>
<th>Increasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5</td>
<td>2.02</td>
<td>0.32</td>
<td>7.20</td>
</tr>
<tr>
<td>0</td>
<td>1.34</td>
<td>0.11</td>
<td>4.89</td>
</tr>
<tr>
<td>0.5</td>
<td>0.77</td>
<td>0.07</td>
<td>2.96</td>
</tr>
</tbody>
</table>

16.51%, while the cash-related cost reduction is 11.01%. This result suggests that cash pooling reduces not only the cost resulted from transaction costs and financing costs, as suggested in the finance literature, but also the cost of managing inventory by matching supply with demand more efficiently. The latter benefit often outweighs the former.

8. Conclusion

This paper studies a joint inventory and cash management problem for a corporation with multiple divisions. We formulate the problem into a dynamic program and partially characterize the optimal policy. Because of curse of dimensionality, we develop a novel lower bound which is a generalization of the two known lower bounds in the literature. We provide two efficient and simple heuristic policies based on the lower bound functions. We numerically show that the proposed heuristic policies perform near optimally and also examine how system parameters affect the value of cash pooling. We conclude that the value of cash pooling is most significant when the demands are increasing and negatively correlated between the divisions. We also show that cash pooling can effectively alleviate inventory shortage and reduce mismatches of demand and supply. Our model and analysis are comfortably applied to a setting where a cash-constrained retailer replenishes inventory for multiple products. They can also be applied to the classic multi-echelon distribution system under a finite-time horizon with non-stationary demands.

There are two possible extensions of the current work. First, our model and the analysis are based on a centralized control scheme. It is of interest to study a coordination mechanism under which the decisions are decentralized made by the divisions and the corporate treasury. Second, we do not consider a warehouse that can store inventory for both divisions. It is interesting to investigate the benefits of cash pooling and inventory pooling and their relationships (i.e., substitute or complementary) under such model.
References


Online Appendix

Managing Inventory for a Multi-divisional Corporation with Cash Pooling

A. Two-Echelon Distribution Systems

Our lower bound can be extended to the traditional distribution inventory system consisting of one warehouse (stage 0) and multiple retailers (stage $i$, $i = 1, ..., N$). The retailers replenish their stock from the warehouse, which in turn places orders at an outside supplier with unlimited supply. We define $c_0$ as the unit ordering cost of the warehouse and set without loss of generality the unit ordering costs of the retailers to be zero. Inventory replenishment at each echelon takes a constant lead time. We assume the material lead time from the outside supplier to warehouse is one period and the lead time from warehouse to retailer $i$ is denoted by $L_i$. In each period, the warehouse first places an order to the outside supplier if necessary and then distributes the on-hand inventory to the retailers. Left inventory will be carried over to the next period and unsatisfied demands at retailer $i$ are fully backlogged with a unit backorder cost $b_i$. We denote by $h_0$ and $h_i$ the unit echelon holding costs at the warehouse and retailer $i$, respectively. The corporation aims to minimize its total expected costs over $T$ periods.

Let $x_{i,t}$ be the echelon inventory position at stage $i$ in the beginning of period $t$ after receiving shipment and $y_{i,t}$ be order-up-to echelon inventory position. We define the echelon holding/backorder cost associated each echelon as follows:

\[
H_{0,t}(y_{0,t}) = \alpha E[h_0(y_{0,t} - \sum_{j=1}^{N} D_{i,t+j})],
\]
\[
H_{i,t}(y_{i,t}) = \alpha L_i E[h_i(y_{i,t} - \sum_{j=0}^{L_i} D_{i,t+j}) + (b_i + h_0 + h_i)(y_{i,t} - \sum_{j=0}^{L_i} D_{i,t+j})^-], \text{ for } i = 1, ..., N.
\]

Define $x_t = (x_{1,t}, x_{2,t}), y_t = (y_{1,t}, y_{2,t})$ and $D_t = (D_{1,t}, D_{2,t})$. The dynamic program is written as

\[
V_t(x_{0,t}, x_t) = \min_{y_{0,t} \geq x_{0,t}} \min_{y_t \geq x_t} \left\{ c_0(y_{0,t} - x_{0,t}) + H_{0,t}(y_{0,t}) + \sum_{i=1}^{N} H_{i,t}(y_{i,t}) + \alpha E[V_{t+1}(y_{0,t} - 1^T D_t, y_t - D_t)] \right\}.
\]

Incorporating the linear savings due to holding inventory at warehouse $a_t(x_{0,t} - c^T y_t)$, the auxiliary system is

\[
\underline{V}_t(x_{0,t}, x_t | a_t) = \min_{y_{0,t} \geq x_{0,t} \geq \Gamma^T} \min_{y_t \geq x_t} \left\{ c_0(y_{0,t} - x_{0,t}) + H_{0,t}(y_{0,t}) + \sum_{i=1}^{N} H_{i,t}(y_{i,t}) - a_t(x_{0,t} - c^T y_t) + \alpha E[\underline{V}_{t+1}(y_{0,t} - 1^T D_t, y_t - D_t | a_t)] \right\}.
\]

Decoupling this system by the cost allocation scheme in Chen and Zheng (1994), our lower bound can be expressed as the sum of following subsystems

\[
\underline{V}_{0,t}(x_{0,t} | a_t) = -(c_0 + a_t) x_{0,t} + \min_{y_{0,t} \geq x_{0,t} \geq \Gamma^T} \left\{ c_0 y_{0,t} + H_{0,t}(y_{0,t}) + \sum_{i=1}^{2} \Gamma_{i,t}(y_{i,t} | a_t) + \alpha E[\underline{V}_{0,t}(y_{0,t} - 1^T D_t | a_t+1)] \right\},
\]
\[
\underline{V}_{i,t}(x_{i,t} | a_t) = \min_{y_t \geq x_{i,t}} \left\{ H_{i,t}(y_t) + a_t c_i y_t - \Gamma_{i,t}(y_{i,t} | a_t) + \alpha E[\underline{V}_{i,t}(y_{i,t} - D_{i,t} | a_t+1)] \right\}, \text{ for } i = 1, ..., N.
\]
The induced-penalty cost is
\[
\Gamma_{i,t}(y_t|a_t) = \begin{cases} 
J_{i,t}(y_t|a_t) - J_{i,t}(S_{i,t}(a_t)|a_t), & \text{if } y_t < S_{i,t}(a_t); \\
0, & \text{otherwise};
\end{cases}
\]

where
\[
J_{i,t}(y_t|a_t) = H_{i,t}(y_t) + a_t c_i y_t + \alpha \mathbb{E}[\mathcal{V}_{i,t}(y_t - D_{i,t}|a_{t+1})],
\]
\[
S_{i,t}(a_t) = \arg\min_{y_t \in \mathbb{R}} J_{i,t}(y_t|a_t).
\]

Next we conduct a comprehensive numerical study to test the performance of our lower bound for the traditional distribution model. We obtain the value of \(a_1\) through a numerical search
\[
\max_{0 \leq a_1 \leq \max(\alpha L_i b_i)} \left\{ V_{0,1}(x_{0,1}|a_1) + \sum_{i=1}^{N} V_{i,1}(x_{i,1}|a_1) \right\},
\]
where \(a_1 = (a_1, a_1, ..., a_1). \max\{\alpha L_i b_i\} \text{ for } i = 1, 2, ..., N\) is a rough upper bound to the shadow price of inventory, which is the cost reduced when the system has one more unit “free” inventory held at warehouse. The parameter values are presented in Table A1. We test stationary, seasonal and increasing poisson demand over time periods: \(\mu_{i,t} = 5\) for the stationary demand, \(\mu_{i,t} = 5 \times 1.2^{t-1}\) for the increasing demand and set \(\mu_i = (5, 7, 10, 7, 5, 3, 1, 3, 5, 7, 10)\) for the seasonal demand. To explore the impact of the initial inventory imbalance on the lower bound performance, we set different initial states \((x_{0,1}, x_1)\). There are totally \(243 \times 3 \times 3 = 2187\) instances to test. The performance improvement of our lower bound against Chen and Zheng’s induced-penalty bound in the distribution system is shown in Table A2. Consistent with observations in the cash pooling system, our lower bound outperforms the induced-penalty bound especially for non-stationary systems.

**Table A1** Parameter values for the two-echelon distribution systems.

<table>
<thead>
<tr>
<th>(T)</th>
<th>(N)</th>
<th>(\alpha)</th>
<th>(c)</th>
<th>(h_0)</th>
<th>(L_i)</th>
<th>(h_i)</th>
<th>(b_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>{2, 4, 8}</td>
<td>{0.9, 0.95, 0.99}</td>
<td>{0.2, 0.4, 0.6}</td>
<td>0.2</td>
<td>1</td>
<td>{0.1, 0.2, 0.4}</td>
<td>{2, 3.5, 5}</td>
</tr>
</tbody>
</table>
Table A2: The overall performance improvement of our lower bound against the induced-penalty bound for two-echelon distribution systems.

<table>
<thead>
<tr>
<th>N</th>
<th>((x_0, x_{i,1}, x_{j,1}))</th>
<th>Stationary</th>
<th>Seasonal</th>
<th>Increasing</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(i = 1, \ldots, N/2; j = N/2 + 1, \ldots, N)</td>
<td>Avg. (Max., Std.) %</td>
<td>Avg. (Max., Std.) %</td>
<td>Avg. (Max., Std.) %</td>
</tr>
<tr>
<td>2</td>
<td>(14, 7, 7)</td>
<td>0.32 (0.55 , 0.16)</td>
<td>0.79 (1.49 , 0.34)</td>
<td>1.36 (3.03 , 0.29)</td>
</tr>
<tr>
<td></td>
<td>(14, 4, 10)</td>
<td>0.72 (1.47 , 0.23)</td>
<td>1.40 (2.69 , 0.40)</td>
<td>2.01 (4.42 , 0.40)</td>
</tr>
<tr>
<td></td>
<td>(14, 1, 13)</td>
<td>1.11 (2.21 , 0.30)</td>
<td>2.31 (4.41 , 0.64)</td>
<td>3.29 (6.50 , 0.51)</td>
</tr>
<tr>
<td>4</td>
<td>(14, 7, 7)</td>
<td>0.64 (1.30 , 0.20)</td>
<td>1.08 (2.12 , 0.34)</td>
<td>2.31 (4.50 , 0.50)</td>
</tr>
<tr>
<td></td>
<td>(14, 4, 10)</td>
<td>1.19 (2.41 , 0.35)</td>
<td>1.83 (3.47 , 0.56)</td>
<td>3.24 (6.70 , 0.60)</td>
</tr>
<tr>
<td></td>
<td>(14, 1, 13)</td>
<td>2.09 (4.01 , 0.51)</td>
<td>3.11 (5.90 , 0.60)</td>
<td>4.17 (8.57 , 0.70)</td>
</tr>
<tr>
<td>8</td>
<td>(14, 7, 7)</td>
<td>1.51 (3.32 , 0.42)</td>
<td>2.06 (3.81 , 0.50)</td>
<td>3.07 (6.00 , 0.70)</td>
</tr>
<tr>
<td></td>
<td>(14, 4, 10)</td>
<td>2.78 (5.71 , 0.50)</td>
<td>3.25 (5.91 , 0.63)</td>
<td>4.03 (7.41 , 0.82)</td>
</tr>
<tr>
<td></td>
<td>(14, 1, 13)</td>
<td>4.19 (7.16 , 0.65)</td>
<td>4.77 (8.01 , 0.72)</td>
<td>5.18 (8.49 , 1.01)</td>
</tr>
</tbody>
</table>

B. Proofs

Proof of Lemma 1. We prove the result by induction. As \(V_{T+1}(w_{T+1}, x_{T+1}) = 0\), the result trivially holds for period \(T + 1\). Now, we assume that the result is true for period \(t + 1\), i.e., \(V_{t+1}(w_{t+1}, x_{t+1})\) is jointly convex in \((w_{t+1}, x_{t+1})\). We next show that \(V_t(w_t, x_t)\) is also jointly convex in \((w_t, x_t)\).

As convexity can be preserved by composition with affine functions, \(V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)\) is jointly convex in \((r_t, y_t)\). Due to the preservation of convexity under expectation, \(E[V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)]\) is also jointly convex in \((r_t, y_t)\). As \(F(x) = \beta_t x^+ + \beta_0 x^-\) is a convex function, so is \(F(r_t - w_t)\). Consequently, one can readily prove that the single-period cost function \(G_t(w_t, r_t, y_t)\) is joint convex in \((w_t, r_t, y_t)\). Therefore, the objective function \(G_t(w_t, r_t, y_t) + E[V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)]\) in (12) is jointly convex in \((w_t, r_t, y_t)\) in the constraint set \(S_t(x_t)\). By Proposition 2.2.15 of Simchi-Levi et al. (2004), \(V_t(w_t, x_t)\) is jointly convex in \((w_t, x_t)\), which completes the induction. □

Proof of Lemma 2. Denote by \((r_t^*, y_t^*)\) and \((r_t^*, y_t^*\Delta)\) the optimal solutions of problem (12) with initial states \((w_t, x_t)\) and \((w_t + \Delta w, x_t)\), respectively. Let \(J_t(w_t, r_t, y_t) = G_t(w_t, r_t, y_t) + \alpha E[V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)]\). To derive the upper bound of \(\frac{\partial}{\partial w_t} V_t(w_t, x_t)\), we show that

\[
\frac{V_t(w_t + \Delta w, x_t) - V_t(w_t, x_t)}{\Delta w} = \frac{J_t(w_t + \Delta w, r_t^*, y_t^*) - J_t(w_t, r_t^*, y_t^*)}{\Delta w}
\leq \frac{J_t(w_t + \Delta w, r_t^*, y_t^*) - J_t(w_t, r_t^*, y_t^*)}{\Delta w} + \beta_t (r_t^* - w_t - \Delta w)^+ + \beta_0 (r_t^* - w_t - \Delta w)^-
\leq \beta_0,
\]

\[
\frac{\partial}{\partial w_t} V_t(w_t, x_t) \leq \beta_0,
\]
where the first inequality follows from the fact that \((r_t^*, y_t^*)\) is also a feasible solution of problem (12) with the initial state \((w_t + \Delta w, x_t)\).

Similarly, we can show that
\[
\frac{V_t(w_t + \Delta w, x_t) - V_t(w_t, x_t)}{\Delta w} = J_t(w_t + \Delta w, r_t^*, y_t^*) - J_t(w_t, r_t^*, y_t^*)
\]
\[
\geq \frac{J_t(w_t + \Delta w, r_t^*, y_t^*) - J_t(w_t, r_t^*, y_t^*)}{\Delta w}
\]
\[
= \frac{\beta_t(r_t^* - w_t - \Delta w)^+ + \beta_D(r_t^* - w_t - \Delta w)^-}{\Delta w}
\]
\[
- \frac{\beta_t(r_t^* - w_t)^+ + \beta_D(r_t^* - w_t)^-}{\Delta w}
\]
\[
\geq -\beta_t,
\]

where the first inequality follows from the fact that \((r_t^*, y_t^*)\) is also a feasible solution of problem (12) with the initial state \((w_t, x_t)\).

Therefore, the results hold. □

**Proof of Theorem 1.** We prove these results based on the KKT conditions (15)-(20) and Lemma 2.

(i) If \(r_t^* > w_t\), the condition (15), together with Lemma 2, implies that
\[
\eta + \beta_t - \alpha \beta_t \leq \lambda_t^* = \eta + \beta_t + \alpha \frac{\partial}{\partial r_t} \mathbb{E}[V_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)] \leq \eta + \beta_t + \alpha \beta_D.
\]

It immediately follows that \(\lambda_t^* \geq \eta + \beta_t - \alpha \beta_t > 0\). The condition (19) implies that \(r_t^* = c^T y_t^*\). As the initial system working capital is larger than the initial inventory value, i.e., \(w_t \geq c^T x_t\), \(c^T y_t^* = r_t^* > w_t \geq c^T x_t\). That is, \(c^T y_t^* > c^T x_t\) which implies that either \(y_{1,t} > x_{1,t}\) or \(y_{2,t} > x_{2,t}\). Consequently, it follows from the condition (20) that either \(\mu_{1,t}^* = 0\) or \(\mu_{2,t}^* = 0\), i.e., \(\prod_{i=1}^{2} \mu_{i,t}^* = 0\). Hence, the result (i) holds.

(ii) If \(r_t^* = w_t\), the condition (18) and Lemma 2 imply that \(\eta - \beta_D - \alpha \beta_t \leq \lambda_t^* \leq \eta + \beta_t + \alpha \beta_D\). The lower bound is nonpositive due to the assumption that \(\eta \leq \beta_D + \alpha \beta_t\). However, the Lagrange multipliers should be nonnegative. As a result, a tighter bound should be \(0 \leq \lambda_t^* \leq \eta + \beta_t + \alpha \beta_D\). The result \(\prod_{i=1}^{2} \mu_{i,t}^* \geq 0\) trivially holds, as the dual optimal multipliers \(\mu_{i,t}^*\) are nonnegative.

(iii) Similarly, if \(r_t^* < w_t\), it follows from the condition (16) and Lemma 2 that \(\eta - \beta_O - \alpha \beta_t \leq \lambda_t^* \leq \eta - \beta_O + \alpha \beta_D\). As the Lagrange multiplier should be nonnegative, the lower bound can be refined to be zero. □

**Proof of Proposition 1.** We first construct a new system by replacing the cash transaction cost \(\beta_t(r_t - w_t)^+ + \beta_O(r_t - w_t)^-\) in the original problem in (12) with \(\beta_O(w_t - r_t)\):

\[
\bar{V}_t(w_t, x_t) = \min_{(r_t, y_t) \in \mathcal{S}_t(x_t)} \left\{ \bar{G}_t(w_t, r_t, y_t) + \alpha \mathbb{E}[\bar{V}_{t+1}(r_t + (p - c)^T D_t, y_t - D_t)] \right\}, \tag{A2}
\]

where \(\mathcal{S}_t(x_t)\) is defined in (11) and

\[
\bar{G}_t(w_t, r_t, y_t) = \eta r_t + \beta_O(w_t - r_t) + \sum_{i=1}^{2} H_{i,t}(y_{i,t}), \tag{A3}
\]

where \(H_{i,t}(y_{i,t})\) are defined by (14).

Because \(\beta_t(r_t - w_t)^+ + \beta_O(r_t - w_t)^- \geq \beta_O(r_t - w_t)^- \geq \beta_O(w_t - r_t)\), this system is a lower bound to the original system. Specifically, \(\bar{V}_t(w_t, x_t) \leq V_t(w_t, x_t)\) for any \((w_t, x_t)\). We next analyze the optimal cash retention decision of this system. One can easily prove that \(\bar{V}_t(w_t, x_t)\) is increasing in \(w_t\) and the marginal cost
of cash is bounded above by \( \beta \), i.e., \( 0 \leq \frac{\partial \tilde{V}_t(w_t, x_t)}{\partial x_t} \leq \beta \). Define \( \tilde{J}_t(w_t, r_t, y_t) = \tilde{G}_t(w_t, r_t, y_t) + \alpha \mathbb{E}[\tilde{V}_{t+1}(r_t + (p - c)T D_t, y_t - D_t)] \). Then, we have
\[
\frac{\partial \tilde{J}_t(w_t, r_t, y_t)}{\partial r_t} = \eta - \beta \alpha + \frac{\partial \alpha \mathbb{E}[\tilde{V}_{t+1}(r_t + (p - c)T D_t, y_t - D_t)]}{\partial r_t} \\
\geq \eta - \beta \\
> 0,
\]
where the last inequality follows from the assumption \( \eta > \beta \).

It states that the objective function in (A2) is increasing in \( r_t \). Therefore, the optimal cash retention decision for the new system should satisfy the equation \( r_t = c^T y_t \). Then we can further simplify the problem in (A2) to
\[
\tilde{V}_t(w_t, x_t) = \min_{y_t \geq x_t} \left\{ \tilde{G}_t(w_t, y_t) + \alpha \mathbb{E}[\tilde{V}_{t+1}(c^T y_t + (p - c)T D_t, y_t - D_t)] \right\},
\]
where
\[
\tilde{G}_t(w_t, y_t) = \eta c^T y_t + \beta \alpha (w_t - c^T y_t) + \sum_{i=1}^{2} H_{t,i}(y_{t,i}).
\]

We next show that the problem in (A4) is decomposable. Define the following dynamic program
\[
\tilde{V}_{t,T+1}(x_{T+1}) = 0.
\]

We now prove by induction that \( \tilde{V}_t(w_t, x_t) \) can be expressed as \( \tilde{V}_t(w_t, x_t) = \beta \alpha w_t + \sum_{i=1}^{2} \tilde{V}_{i,t}(x_{i,t}) \). The result trivially holds for period \( T + 1 \), as \( \tilde{V}_{T+1}(w_{T+1}, x_{T+1}) = \beta \alpha w_{T+1} \). Assume that it is true for period \( t + 1 \), i.e., \( \tilde{V}_{t+1}(w_{t+1}, x_{t+1}) = \beta \alpha w_{t+1} + \sum_{i=1}^{2} \tilde{V}_{i,t+1}(x_{i,t+1}) \). Then, the problem in (A4) can be rewritten as
\[
\tilde{V}_t(w_t, x_t) = \min_{y_t \geq x_t} \left\{ \eta c^T y_t + \beta \alpha (w_t - c^T y_t) + \sum_{i=1}^{2} H_{t,i}(y_{t,i}) \right\} \\
+ \alpha \beta \alpha (c^T y_t + (p - c)T D_t) + \sum_{i=1}^{2} \mathbb{E}[\tilde{V}_{i,t+1}(y_{i,t} - D_{i,t})],
\]
which completes the induction.

It is well known that in the stationary setting, the optimal inventory policy for problem (A6) is a myopic base-stock policy with reorder points \( S_t \) defined by
\[
S_t = \arg \min \left\{ \eta - \beta \alpha + \alpha \beta c^T y_t + H_{t,i}(y_t) \right\}.
\]

Therefore, the optimal policy of the problem in (A4) is as follows: the inventory policy is a base-stock policy with base-stock level \( S_t \), while the corporate treasury transfers all excess cash to the investment account after inventory payment.

Finally, we implement the optimal policy of the problem in (A4) to the original problem and show that it can achieve the same expected costs. As a result, the optimal policy of the problem in (A4) is also optimal for the original problem, as the new problem is a lower bound to the original system.
Note that the original and new systems share the same inventory-related costs as they charge identical inventory costs and adopt the same inventory policy. The cost difference arises from the cash-related costs. The problem (A4) charges $\beta_O(w_t - r_t)$, whereas the original problem charges $\beta_1(r_t - w_t)^+ + \beta_O(r_t - w_t)^-$. However, the two cost schemes are equivalent if we can show $w_t \geq r_t$ for every period $t$, namely, the initial working capital of each period is sufficient for the inventory replenishment. We prove it by the sample path approach. Recall that $r_t = c_1S_1 + c_2S_2$. By the assumption that $w_1 \geq c_1S_1 + c_2S_2$, the result is true for period 1. Consider a particular demand path $\{d_{1,1}, d_{1,2}, d_{2,2}, \ldots, (d_{1,T}, d_{2,T})\}$. For period 2, the initial working capital is $w_2 = c^Ty_t + (p - c)^T d_t = c_1S_1 + c_2S_2 + (p - c)^T d_t \geq c_1S_1 + c_2S_2 = r_t$. Similarly, one can show that the result still holds for other subsequent periods. $\square$

**Proof of Theorem 2.** We first prove $V_t(w_t, x_t) \geq V_t(w_t, x_t | a_t)$ by induction. As $V_{T+1}(w_{T+1}, x_{T+1}) = V_{T+1}(w_{T+1}, x_{T+1} | a_{T+1}) = 0$, the result trivially holds for period $T + 1$. Assume that the result is true for period $t + 1$, i.e., $V_{t+1}(w_{t+1}, x_{t+1}) \geq V_{t+1}(w_{t+1}, x_{t+1} | a_{t+1})$. As $a_t$ is nonnegative, $G_t(w_t, r_t, y_t | a_t) \leq G_t(w_t, r_t, y_t)$. Hence, the objective function in (22) is smaller than that in (12). It immediately follows that $V_t(w_t, x_t) \geq V_t(w_t, x_t | a_t)$, as the two optimization problems have the same constraints. The proof of the result $V_t(w_t, x_t | a_t) \geq V_{H,t}(w_t | a_t) + \sum_{i=1}^2 V_{i,t}(x_t | a_t)$ corresponds to the cost allocation scheme stated by Chen and Zheng (1994) and is similar with the proof in Appendix B of their paper. Hence, we omit the proof here. $\square$

**Proof of Theorem 3.** We first formulate the Lagrange relaxation of the original problem as follows. Denote by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_T)$ the Lagrange multipliers associated with the cash allocation constraints over the whole horizon, and by $\lambda_t = (\lambda_t, \lambda_{t+1}, \ldots, \lambda_T)$ the subvector that records the multipliers from period $t$ to $T$. The Lagrange relaxation of the original problem by relaxing the cash allocation constraints can be expressed as

$$V_t^L(w_t, x_t | \lambda_t) = \min_{r_t \in \mathbb{R}, y_t \geq x_t} \left\{ G_t(w_t, r_t, y_t) - \lambda_t(r_t - c^T y_t) + \alpha \beta V_{t+1}^L(r_t + (p - c)^T D_t, y_t - D_t | \lambda_{t+1}) \right\}$$

with zero terminal value.

We next prove by induction that the Lagrange relaxation problem is a lower bound to our proposed bound as long as $\lambda = a$. The result trivially holds for period $T + 1$. We assume it holds in period $t + 1$, i.e., $V_{t+1}^L(w_{t+1}, x_{t+1} | \lambda_{t+1}) \leq V_{H,t+1}(w_{t+1} | a_{t+1}) + \sum_{i=1}^2 V_{i,t+1}(x_{t+1} | a_{t+1})$ when $\lambda_{t+1} = a_{t+1}$. As a result, we can obtain

$$V_t^L(w_t, x_t | \lambda_t) = \min_{r_t \in \mathbb{R}, y_t \geq x_t} \left\{ G_t(w_t, r_t, y_t) - \lambda_t(r_t - c^T y_t) + \alpha \beta V_{t+1}^L(r_t + (p - c)^T D_t, y_t - D_t | \lambda_{t+1}) \right\}$$

$$\leq \min_{r_t \in \mathbb{R}, y_t \geq x_t} \left\{ G_t(w_t, r_t, y_t) - a_t(r_t - c^T y_t) + \alpha \beta V_{H,t+1}^L(r_t + (p - c)^T D_t | a_{t+1}) \right\}$$

$$+ \sum_{i=1}^2 V_{i,t+1}(y_t - D_t | a_{t+1})$$

$$= \min_{r_t \in \mathbb{R} | y_t \geq x_t} \left\{ (\eta - a_t)r_t + F(r_t - w_t) + \alpha \beta V_{H,t+1}^L(r_t + (p - c)^T D_t | a_{t+1}) \right\}$$

$$+ \sum_{i=1}^2 \left[ \min_{y_t \geq x_t, t} \left\{ H_i(t, y_t) + a_t c_i y_t + \alpha \beta V_{i,t+1}^L(y_t, y_t - D_t | a_{t+1}) \right\} \right]$$

$$\leq \min_{r_t \in \mathbb{R} | y_t \geq x_t} \left\{ (\eta - a_t)r_t + F(r_t - w_t) + \alpha \beta V_{H,t+1}^L(r_t + (p - c)^T D_t | a_{t+1}) + \min_y \sum_{i=1}^2 \sum_{t \geq y} \sum_{i=1}^2 V_{i,t}(y_t | a_t) \right\}$$
\[ + 2 \sum_{i=1}^{2} \min_{y_{i,t} \geq s_{i,t}} \left\{ H_{i,t}(y_{i,t}) + a_{i} c_{i} y_{i,t} + \alpha E[V_{i,t+1}(y_{i,t} - D_{i,t}|a_{i+1})] \right\}, \]

\[ = V_{H,t}(w_t|a_t) + 2 \sum_{i=1}^{2} \min_{y_{i,t} \geq s_{i,t}} \left\{ H_{i,t}(y_{i,t}) + a_{i} c_{i} y_{i,t} - \Gamma_{i,t}(y_{i,t}|a_{i}) + \alpha E[V_{i,t+1}(y_{i,t} - D_{i,t}|a_{i+1})] \right\}, \]

\[ = V_{H,t}(w_t|a_t) + 2 \sum_{i=1}^{2} V_{i,t}(x_{i,t}|a_{i}), \quad (A8) \]

where the first inequality follows from the induction assumption and the second from the non-negativity of \( \Gamma_{i,t}(y_{i,t}|a_{i}) \) which are defined in (26). That is, the result is also true for period \( t \) when \( \lambda_t = a_t \) and hence the induction completes.

As the original problem is convex and satisfies the Slater’s condition, there is no gap between the dual problem and the original problem, i.e., \( V_{i,t}(w_t, x_t|\lambda_t^*) = V_i(w_t, x_t) \) where \( \lambda_t^* \) is the dual optimal multipliers. By (A8), \( V_{H,t}(w_t|\lambda_t^*) + \sum_{i=1}^{2} V_{i,t}(x_{i,t}|\lambda_t^*) \geq V_{i,t}(w_t, x_t|\lambda_t^*) = V_i(w_t, x_t) \). However, Theorem 2 implies that \( V_i(w_t, x_t) \geq V_{H,t}(w_t|\lambda_t^*) + \sum_{i=1}^{2} V_{i,t}(x_{i,t}|\lambda_t^*) \). Therefore, we have \( V_{i,t}(w_t, x_t|\lambda_t^*) = V_{H,t}(w_t|\lambda_t^*) + \sum_{i=1}^{2} V_{i,t}(x_{i,t}|\lambda_t^*) = V_i(w_t, x_t). \) \( \square \)

**Proof of Theorem 4.** (i) Division \( i \) faces the following optimization problem

\[ V_{i,t}(x_{i,t}|a_t) = \min_{y_{i,t} \geq s_{i,t}} \left\{ J_{i,t}(y_{i,t}|a_t) - \Gamma_{i,t}(y_{i,t}|a_t) \right\}, \]

where \( J_{i,t}(y_{i,t}|a_t) \) is defined in (23). Note that the cost objective function is

\[ \left\{ \begin{array}{ll}
J_{i,t}(S_{i,t}(a_t)|a_t), & \text{if } y_{i,t} \leq S_{i,t}(a_t); \\
J_{i,t}(y_{i,t}|a_t), & \text{otherwise},
\end{array} \right. \]

where \( S_{i,t}(a_t) \) is defined in (24). It can be readily proven by induction that the problem is convex and a base-stock policy with the base-stock level \( S_{i,t}(a_t) \) is optimal.

We next prove the monotonicity of the base-stock level \( S_{i,t}(a_t) \) with respect to \( a_t \) for any \( \tau \geq t \). By the definition of \( \Gamma_{i,t}(y_{i,t}|a_t) \) in (26), \( V_{i,t}(x_{i,t}|a_t) \) can be rewritten as

\[ V_{i,t}(x_{i,t}|a_t) = \min_{y_{i,t} \geq s_{i,t}} \left\{ J_{i,t}(y_{i,t}|a_t) \right\}. \]

So we just need to prove \( J_{i,t}(y_{i,t}|a_t) \) is supermodular in \( (y_{i,t}, a_{\tau}) \) for all \( \tau \geq t \). The result trivially holds for period \( T+1 \). Assume that \( V_{i,t+1}(x_{i,t+1}|a_{t+1}) \) is supermodular in \( (x_{i,t+1}, a_{\tau}) \) for \( \tau \geq t+1 \). Because supermodularity can be preserved under addition and positive scalar multiplication, it follows that the expectation \( \alpha E[V_{i,t+1}(y_{i,t} - D_{i,t}|a_{t+1})] \) is supermodular in \( (y_{i,t}, a_{\tau}) \) for \( \tau \geq t+1 \). One also can easily verify that the term \( H_{i,t}(y_{i,t}) + a_{i} c_{i} y_{i,t} \) is supermodular in \( (y_{i,t}, a_{\tau}) \). Therefore, \( J_{i,t}(y_{i,t}|a_t) \) is supermodular in \( (y_{i,t}, a_{\tau}) \) for \( \tau \geq t \). To complete the induction, we next prove that \( V_{i,t}(x_{i,t}|a_t) \) is supermodular in \( (x_{i,t}, a_{\tau}) \) for all \( \tau \geq t \).

We can derive the partial derivative of \( V_{i,t}(x_{i,t}|a_t) \) with respect to \( x_{i,t} \)

\[ \frac{\partial V_{i,t}(x_{i,t}|a_t)}{\partial x_{i,t}} = \left\{ \begin{array}{ll}
\frac{\partial J_{i,t}(x_{i,t}|a_t)}{\partial x_{i,t}}, & \text{0, if } x_{i,t} \leq S_{i,t}(a_t); \\
0, & \text{otherwise},
\end{array} \right. \quad (A9) \]

To prove the supermodularity, we need to verify that \( \frac{\partial V_{i,t}(x_{i,t}|a_t)}{\partial x_{i,t}} \) is increasing in \( a_{\tau} \) for \( \tau \geq t \). Given a period \( \tau \geq t \), let \( a_{t}' = (a_t, a_{t+1}, ..., a_{\tau}^') \) and \( a_{t}'' = (a_t, a_{t+1}, ..., a_{\tau}^'') \) such that \( 0 \leq a_{t}' \leq a_{t}'' \). As \( J_{i,t}(y_{i,t}|a_t) \) is supermodular in \( (y_{i,t}, a_{\tau}) \), the base-stock level \( S_{i,t}(a_t) \) is decreasing in \( a_{\tau} \), i.e., \( S_{i,t}(a_{t}') \geq S_{i,t}(a_{t}'') \). We consider the following three cases.
Case 1. If \(x_{i,t} \leq S_{i,t}(a_t'')\), then it follows from (A9) that \(\frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t' = a_t''} = \frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t'} = 0\).

Case 2. If \(S_{i,t}(a_t'') < x_{i,t} \leq S_{i,t}(a_t')\), then \(\frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t'} = 0\) and \(\frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t''} = \frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t'}\).

The convexity of \(J_{i,t}(\cdot | a_t)\) implies \(\frac{\partial^2 J_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}^2} |_{a_t = a_t'} \geq 0\). Therefore \(\frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t'} \leq \frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t''}\).

Case 3. If \(S_{i,t}(a_t') < x_{i,t} \leq S_{i,t}(a_t'')\), then \(\frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t'} = \frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t''}\).

The supermodularity of \(J_{i,t}(\cdot | a_t)\) implies \(\frac{\partial^2 J_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}^2} |_{a_t = a_t'} \leq \frac{\partial^2 J_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}^2} |_{a_t = a_t''}\). Therefore \(\frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t'} \leq \frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}} |_{a_t = a_t''}\).

Therefore, \(\frac{\partial V_{i,t}(x_{i,t} | a_t)}{\partial x_{i,t}}\) is increasing in \(a_t\), and the result holds for any \(\tau \geq t\). Hence \(V_{i,t}(x_{i,t} | a_t)\) is supermodular in \((x_{i,t}, a_t)\) for \(\tau \geq t\). Complete the induction. As a global minimizer of a supermodular function, the base-stock level \(S_{i,t}(a_t)\) is nonincreasing in the value of \(a_t\) for \(\tau \geq t\) given any period \(t\).

(ii) The cash subsystem \(H\) solves the following optimization problem

\[
V_{H,t}(w_t | a_t) = \min_{\eta \geq c^T y_t} \left\{ (\eta - a_t) r_t + F(r_t - w_t) + \sum_{i=1}^{2} \Gamma_{i,t}(y_{i,t} | a_t) + \alpha E[V_{H,t+1}(r_t + (p - e)^T D_t | a_{t+1})] \right\}.
\]

It is easy to verify that this dynamic optimization is a convex problem, and the optimal cash retention policy is a two-threshold policy with the upper and lower thresholds defined in (27) and (28). We refer the reader to Luo and Shang (2015) for the detailed proof of the policy structure. \(\square\)

References.


